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Defuzzification of intuitionistic fuzzy thermal diffusivity in heat equation: Comparative analysis of methods

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Abstract: This study aims to enhance the modeling of heat transfer under uncertainty by representing thermal diffusivity as an intuitionistic triangular fuzzy number (ITFN) and comparing various defuzzification methods. The methodology involves fuzzifying the nominal thermal diffusivity of copper (approximately 1.11×10^{-4} m²/s) into an ITFN, followed by the application of three defuzzification techniques—weighted average, score function-based, and centroid methods—to derive crisp values. Numerical simulations of the heat equation were conducted using these DE fuzzified values to assess their impact on temperature distribution predictions. Findings indicate that the weighted average and centroid methods yield nearly identical values (1.113×10^{-4} m²/s), whereas the score function-based method produces a slightly higher value of 1.158×10^{-4} m²/s. Although differences in predicted temperature profiles are minimal for copper, the study highlights that for materials with greater variability, the choice of defuzzification method may significantly influence simulation outcomes. These results suggest that employing ITFNs in heat transfer modeling provides a robust framework for capturing material uncertainties, thereby improving the reliability of engineering analyses in uncertainty-sensitive applications.
Keywords: *Defuzzification methods, Heat equation, Intuitionistic fuzzy numbers, Thermal diffusivity.*

1. Introduction

In Bertone, et al. [1] it was rigorously demonstrated that fuzzy solutions—constructed from analytical solutions of partial differential equations (PDEs)—exist via the application of the Zadeh Extension Principle. This foundational result has paved the way for incorporating fuzzy logic into various areas of applied mathematics, particularly in the modeling of systems where uncertainty plays a critical role. Heat conduction, a phenomenon central to thermal engineering [2] material science [3] and applied mathematics [4] is traditionally modeled by the classical heat equation. This equation presupposes a precise, deterministic thermal diffusivity. However, real-world scenarios are seldom ideal; uncertainties arising from material heterogeneities, measurement errors, and environmental fluctuations necessitate a more robust representation of such parameters [5].

Fuzzy set theory, as introduced by Zadeh [6] and further extended by Bellman and Zadeh [7] has provided the conceptual basis for representing uncertainty. More recent studies have employed the Zadeh Extension Principle to construct fuzzy analytical solutions to PDEs [1] thereby establishing a rigorous framework for uncertainty quantification in heat transfer problems. In this context, intuitionistic fuzzy numbers (IFNs) have emerged as an advanced tool by incorporating both membership and non-membership degrees, offering a more nuanced description of uncertainty [8]. The

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integration of fuzzy methodologies in various computational domains, including machine learning, has demonstrated their efficacy in handling uncertainty and improving model robustness [9-14].

The application of Fuzzy Intuitionistic Fuzzy Numbers (IFNs) in thermal diffusivity modeling remains relatively rare compared to their use in other fields, such as control systems and signal processing [15-18]. Most existing studies focus on conventional fuzzy models, while IFN-based approaches are only beginning to be explored in the context of heat transfer. The scarcity of research in this area highlights the need for further investigation into the applicability and advantages of IFNs in thermal modeling.

While Bertone, et al. [1] laid a robust theoretical foundation for constructing fuzzy solutions to partial differential equations using Zadeh's extension principle, their work primarily focused on converting analytic solutions into fuzzy ones through conventional fuzzy numbers. In contrast, our research extends this framework by employing intuitionistic fuzzy numbers (IFNs) to fuzzify the analytic solution of the heat equation. This approach not only mirrors the methodology of Bertone, et al. [1] but also introduces a significant innovation—by incorporating both membership and non-membership degrees, IFNs offer a richer and more nuanced representation of uncertainty. Such a dual representation is particularly advantageous in accurately modeling thermal diffusivity in situations where experimental data are sparse or highly variable. Thus, our work fills a critical gap by demonstrating how the analytic fuzzy solution, when constructed with IFNs, can enhance the predictive accuracy and reliability of thermal simulations in engineering applications.

2. Theoretical Background

Definition 1. Nguyen [19] *The Zadeh extension principle of a function $f: X \to Z$ where X and Z are non-empty metric spaces, defines, $\hat{f}$ as a function that, when applied to a fuzzy set $D \subset X$, produces a fuzzy set $\hat{f}(D)$ in Z, whose membership function is given by*

$$\mu_{\hat{f}(D)}(Z) = \begin{cases} \sup\limits_{\{x: f(x)=z\}} \mu_D(x) \ if \ \{x: f(x) = z\} \neq \emptyset, \\ 0, others \end{cases}$$

where $\hat{f}(D) = f(D)$ if D is a classical set in X

As a consequence of Zadeh's extension principle, the following proposition is obtained:

Proposition 2. De Barros, et al. [20] and Goo and Park [21] *Let X and be non-empty metric spaces, let D be a fuzzy set in X and let $f: X \to Z$ be continuous. Then, for each $0 \leq \alpha \leq 1$. $\left[\hat{f}(D)\right]^{\alpha} = f([D]^{\alpha})$.*

Definition 3. Liu, et al. [22] *An intuitionistic fuzzy number $\tilde{A}$ on the universe X is defined as a set:*

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) \mid x \in X\}$$

subject to the constraint:

$$0 \leq \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1, \forall x \in X.$$

Where:

- $\mu_{\tilde{A}}(x)$ *is the membership function, indicating the degree to which x belongs to $\tilde{A}$.*
- $\nu_{\tilde{A}}(x)$ *is the non-membership function, indicating the degree to which x does not belong to $\tilde{A}$.*
- *The uncertainty function is defined as:*

$$\pi_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x).$$

This represents the degree of hesitation or uncertainty regarding the membership of x in $\tilde{A}$.

3. Solution of Heat Transfer Equation

3.1. Classical Heat Transfer Equation

The classical one-dimensional heat transfer equation is given by

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$$\begin{cases} \varphi_0 c_t \dfrac{\partial u}{\partial t} = \dfrac{k_0}{\mu} \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right), (x,t) \in (0,L) \times (0,t_0), \\ u(x,0) = u_0(x) = \sin(\pi x), x \in (0,L), \\ u(0,t) = u(1,t) = 0 \text{ for } t \in (0,t_0) \end{cases}$$

Where:

- φ_0 and c_t are material-specific parameters (initial thermal capacity and specific heat),
- k_0 is the thermal conductivity coefficient,
- μ is the mass density,
- $u(x,t)$ represents the temperature at position x and time t,
- The initial temperature distribution $u_0(y) = \sin(x\pi)$.

The analytical solution for the heat equation is obtained using the method of separation of variables and is expressed as:

$$u_d(x,t) = \int_0^L u_0(y) \sum_{n=1}^{+\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right) \exp\left(-\frac{dn^2\pi^2}{L^2} t\right) dy$$

where $d = 1.15 \times 10^{-4}\, m/s^2$ represents the heat diffusion coefficient. Given that u the solution becomes:

$$u_d(x,t) = \int_0^L \sin(x\pi) \sum_{n=1}^{+\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right) \exp\left(-\frac{dn^2\pi^2}{L^2} t\right) dy$$

The series solution above can be approximated numerically. The computed result for $u(x,t)$ over the interval $(0,L) \times (0,t_0)$ is visualized in Figure 1. This figure represents the temperature distribution over time, demonstrating the typical diffusion process where heat gradually spreads out until it reaches a stable state.

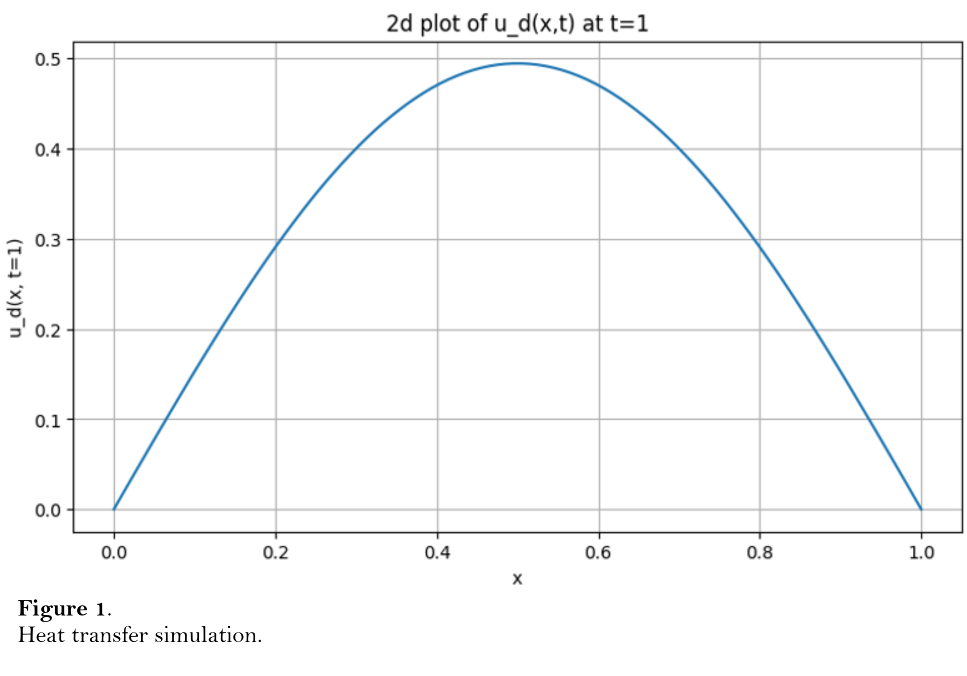

Figure 1.
Heat transfer simulation.

3.2. Fuzzy solution

The fuzzy solution is obtained by considering the coefficient in the partial differential equation as a fuzzy number D with level zero, denoted as $[D]^0$. Here, (x,t) represents the domain of the partial differential equation, and the function fungsi $S_{(x,t)}: [D]^0 \to \mathbb{R}$ provides the analytic solution value $S_{(x,t)}(d)$ at point (x,t) for each parameter $d \in [D]^0$ [1].

Let $\hat{S}_{(x,t)}(D)$ be the extension of $S_{(x,t)}(d)$ for $d \in [D]^0$ using Zadeh's extension principle. The fuzzy solution is then defined as: $\bigcup_{(x,t)} \hat{S}_{(x,t)}(D)$,This represents the fuzzification of the analytic solution through Zadeh's extension. Subsequently, D is assumed to be a triangular fuzzy number [1].

In this approach, the coefficient is not only treated as a triangular fuzzy number but also further extended and modeled as an intuitionistic triangular fuzzy number. Mathematically, an intuitionistic triangular fuzzy number is represented as a tuple:

$$D = ((a,b,c), \mu_D, \nu_D)$$

Where:

- (a,b,c) are the parameters of the triangular fuzzy number, with $a \leq b \leq c$.
- $\mu_D: [a,c] \to [0,1]$ is the membership function.
- $\nu_D: [a,c] \to [0,1]$ is the non-membership function, satisfying $0 \leq \mu_D(x) + \nu_D(x) \leq 1$ for all $x \in [a,c]$.

To apply fuzzification in a real-world scenario, we consider copper as the material, where the thermal conductivity is modeled as an intuitionistic fuzzy triangular number. The nominal thermal conductivity diffusivity of copper at room temperature is approximately $d = 1.11 \times 10^{-4} m^2/s$, with slight variations depending on purity and temperature.

We define the triangular intuitionistic fuzzy number for copper's thermal conductivity as:
where:

- $1.05 \times 10^{-4} m^2/s$ (Lower bound)
- $1.11 \times 10^{-4} m^2/s$ (Modal value)
- $1.18 \times 10^{-4} m^2/s$ (Upper bound)
- The membership function is defined as: $\mu_d = 0.85$
- The non-membership function $\nu_d = 0.10$

intuitionistic fuzzy thermal diffusivity as.

$$d^* = (1.05, 1.11, 1.18; \mu_d = 0.85, \nu_d = 0.10)$$

This representation allows uncertainty in the thermal conductivity of copper to be incorporated into the heat transfer model, making the solution more robust under real-world conditions. Thus, the fuzzy solution of the heat equation is.

$$u_d(x,t) = \int_0^L \sin(x\pi) \sum_{n=1}^{+\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right) \exp\left(-\frac{d^* n^2 \pi^2}{L^2} t\right) dy$$

4. Defuzzification Methods

Defuzzification is the process of converting an intuitionistic fuzzy number into a crisp value for practical applications. In this section, we describe three widely used defuzzification techniques applied to the intuitionistic fuzzy thermal diffusivity.

4.1. Weighted Average Defuzzification

The weighted average method calculates a crisp value based on a weighting factor λ\lambda derived from the intuitionistic fuzzy membership and non-membership degrees. The formula is:

$$d^* = \frac{d_L + \lambda d_M + d_U}{\lambda + 2}, \lambda = \frac{\mu_d}{1 - v_d}.$$

Substituting the given values for thermal diffusivity:

$$\lambda = \frac{0.85}{0.90} = 0.944.$$

$$d^* = \frac{(1.05 + 0.944 \times 1.11 + 1.18) \times 10^{-4}}{2.944}.$$

$$d^* = 1.113 \times 10^{-4} m^2/s$$

This method ensures that the output value is weighted based on the relative importance of the modal value, making it robust for applications where the central tendency is emphasized.

4.2. Score Function-Based Defuzzification

The score function method utilizes the difference between the membership and non-membership degrees to determine a crisp value. The score function is defined as:

$$S(\tilde{d}) = \mu_d - v_d$$

For the given intuitionistic fuzzy thermal diffusivity:

$$S(\tilde{d}) = 0.85 - 0.10 = 0.75.$$

The defuzzified value is then calculated as:

$$d^* = d_M + \frac{S(\tilde{d})(dU - dL)}{2}.$$

Substituting values:

$$d^* = 1.11 \times 10^{-4} + 0.75 \times \frac{(1.18 - 1.05) \times 10^{-4}}{2}.$$

$$d^* = 1.158 \times 10^{-4} \, m^2/s.$$

This approach ensures that the DE fuzzified value accounts for both uncertainty and the central tendency of the data.

4.3. Centroid Defuzzification

The centroid method calculates the crisp value by averaging the three key values of the fuzzy number. It is defined as:

$$d^* = \frac{d_L + d_M + d_U}{3}.$$

Substituting the given values:

$$d^* = \frac{(1.05 + 1.11 + 1.18) \times 10^{-4}}{3}.$$

$$d^* = 1.113 \times 10^{-4} \, m^2/s.$$

This method provides a balanced approach by giving equal weight to all three values, making it suitable for cases where the intuitionistic fuzzy representation is symmetrically distributed.

4.4. Comparative Analysis of Defuzzification Methods

The results from the three methods indicate slight variations in the DE fuzzified values:

Table 1.
Comparison of defuzzification method.

No	Defuzzification method	d^* value
1	Weighted average defuzzification	$1.113 \times 10^{-4} m^2/s$
2	Score function-based defuzzification	$1.158 \times 10^{-4} \, m^2/s$
3	Centroid defuzzification	$1.113 \times 10^{-4} \, m^2/s$

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While the weighted average and centroid methods provide similar results, the score function-based method produces a slightly higher value due to its consideration of membership and non-membership differences. The choice of method should be based on the application context, where the centroid method is preferred for symmetric distributions, the weighted average method for practical approximations, and the score function-based method for uncertainty-sensitive applications. This result differs from the findings in Bertone, et al. [1] where the defuzzification results matched the analytical solution value. The use of intuitionistic fuzzy numbers leads to distinct solution values for the heat equation, highlighting the impact of fuzzification in thermal modeling.

5. Simulation Outcomes

To evaluate the effect of these defuzzied values on heat transfer predictions, a numerical simulation was conducted for the heat equation. Figure 1 shows the temperature distribution $u_d(x,t)$ at $t = 1$ for the three distinct values of d. Notably, the curves overlap almost perfectly, indicating that in the case of copper—where thermal properties exhibit relatively small variations—the impact of different defuzzied values is minimal. These modest discrepancies reflect the inherent robustness of copper's thermal behavior when subject to slight parameter fluctuations.

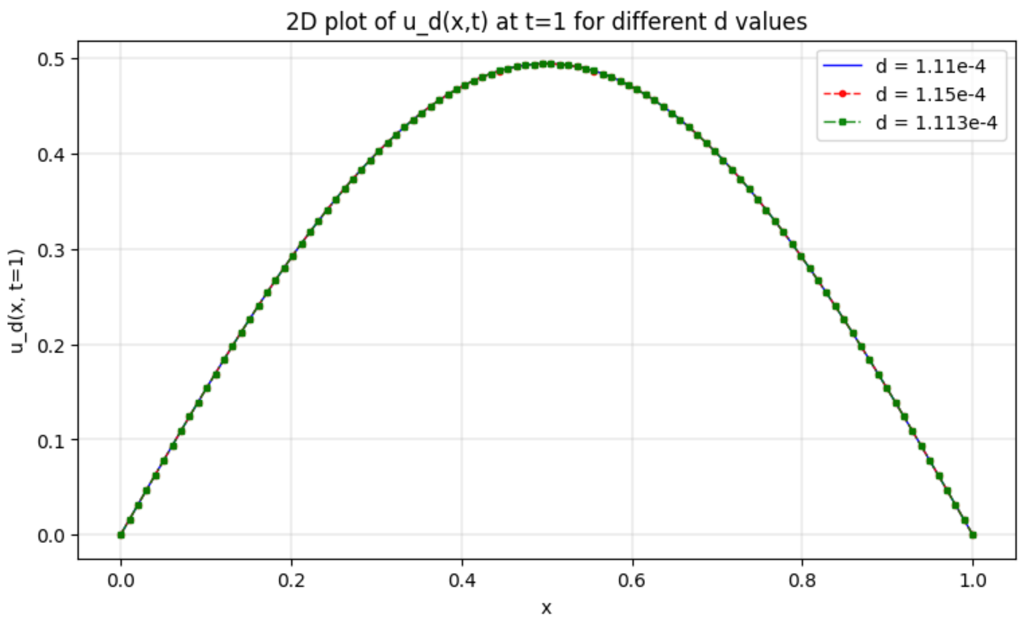

Figure 2.
2D plot of $u_d(x,t)$ at $t = 1$ for different d values. The three curves overlap, indicating minimal differencein predicted temperature distribution for copper.

Despite the similarity observed in this simulation, the results may differ considerably for materials with greater variability or higher sensitivity to temperature changes. In such cases, the choice of defuzzification method could significantly alter the predicted temperature field, potentially affecting design decisions in engineering applications. Consequently, although the present results suggest minimal differences for copper, they highlight the broader utility of intuitionistic fuzzy numbers and defuzzification techniques in capturing and propagating uncertainties for materials and systems where parameter variations are more pronounced.

6. Conclusion

This study explored the impact of different defuzzification methods on the numerical solution of the heat equation when the thermal diffusivity is modeled using intuitionistic fuzzy numbers. Three widely used techniques—weighted average, score function-based, and centroid defuzzification—were analyzed to assess their influence on the resulting crisp value of thermal diffusivity. The results demonstrated that while the weighted average and centroid methods produced similar DE fuzzified values, the score function-based method yielded a slightly higher value due to its explicit consideration of both membership and non-membership degrees.

These findings highlight the importance of selecting an appropriate defuzzification approach based on application-specific requirements. The centroid method is preferable for symmetric distributions, the weighted average method offers practical approximations, and the score function-based method is advantageous in scenarios requiring sensitivity to uncertainty. The differences in DE fuzzified values indicate that the choice of method can influence the computed temperature distribution in heat transfer problems. Future research can extend this analysis by incorporating time-dependent intuitionistic fuzzy parameters and evaluating the stability and convergence of numerical schemes under varying degrees of fuzziness.

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Transparency:
The authors confirm that the manuscript is an honest, accurate, and transparent account of the study; that no vital features of the study have been omitted; and that any discrepancies from the study as planned have been explained. This study followed all ethical practices during writing.

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