Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 95, pp. 1–15.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.95

UNIFORM ESTIMATES FOR ELLIPTIC EQUATIONS WITH CARATHEODORY NONLINEARITIES AT THE INTERIOR AND ON THE BOUNDARY

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Abstract. We establish an explicit uniform a priori estimate for weak solutions to slightly subcritical elliptic problems with nonlinearities simultaneously at the interior and on the boundary. Our explicit $L^\infty(\Omega)$ a priori estimates are in terms of powers of their $H^1(\Omega)$ norms. To prove our result, we combine a De Giorgi-Nash-Moser iteration scheme together with elliptic regularity and the Gagliardo-Nirenberg interpolation inequality.

1. Introduction

Let us consider the nonlinear boundary value problem of semilinear eliptic equations

$$\begin{aligned} -\Delta u + u &= f(x,u), \quad x \in \Omega, \\ \frac{\partial u}{\partial \eta} &= f_B(x,u), \quad x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $(N > 2)$, is an open, connected, bounded domain with C^2 boundary, $\partial/\partial\eta = \eta \cdot \nabla$ is the (unit) outer normal derivative, and the functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$, and $f_B : \partial\Omega \times \mathbb{R} \to \mathbb{R}$, are both slightly subcritical Carathéodory functions. In (H1)–(H4) below, we give the precise statement of the hypotheses on the nonlinearities at the interior, and on the boundary.

Our goal is to establish explicit $L^\infty(\Omega)$ *a priori* estimates for weak solutions to (1.1), in terms of powers of their $H^1(\Omega)$ norms (see Theorem 2.2). Our estimates are independent of the sign of the solutions. Consequently, any sequence of solutions to (1.1), uniformly bounded in the $H^1(\Omega)$ norm, is also uniformly bounded in the $L^\infty(\Omega)$ norm.

Our techniques are based on an iterative process due to Moser, in the elliptic regularity theory, and in the Gagliardo-Nirenberg interpolation inequality.

For the homogeneous Dirichlet boundary conditions, by a Moser's type procedure, it is well known that weak solutions to a subcritical or even critical elliptic problem are in $L^q(\Omega)$ for all $1 < q < \infty$ (see [11, Lemma 1], see also [4, Section 2.2], [15, Lemma B.3]. Moreover, by elliptic regularity, the solutions are in $L^\infty(\Omega)$.

Moser's results can be extended to the case of nonlinear boundary conditions, and also to a general quasilinear problem, which includes in particular (1.1), see, for instance, [9, Theorem 3.1]. In [9] the authors state that weak solutions to some quasilinear problem are in $L^\infty(\Omega) \cap L^\infty(\partial\Omega)$. By elliptic regularity, weak solutions to (1.1) are in fact more regular, and in particular, they are uniformly continuous functions. Indeed, the elliptic regularity theory, applied to weak solutions of a subcritical or even critical problem implies that they are in $C(\overline{\Omega})$, see estimate (5.2) in Theorem 5.1. So, in that case,

$$\|u\|_{L^\infty(\partial\Omega)} \le \|u\|_{C(\overline{\Omega})} = \|u\|_{L^\infty(\Omega)}. \tag{1.2}$$

2020 *Mathematics Subject Classification.* 35B45, 35J66, 35B33, 35J75, 35J25.
Key words and phrases. A priori estimates; slightly subcritical non-linearities; L^∞ a priori bounds; nonlinear boundary conditions.
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Submitted January 24, 2025. Published October 13, 2025.

The type of $L^\infty(\Omega)$ estimates given by (2.12) are known for slightly subcritical nonlinearities in the homogeneous Dirichlet problem with the Laplacian operator, see [13, Theorem 1.5], with the p-Laplacian operator, see [14, Theorem 1.6], and also with a linear problem at the interior joint with nonlinear boundary conditions on the boundary of power type, see [3].

In this article, we analyze the combined effect of both nonlinearities simultaneously. We establish the explicit estimates provided by Theorem 2.2, where both nonlinearities in the interior and on the boundary are slightly subcritical, not necessarily of power type.

This article is organized in the following way. Section 2 contains the statement of our main result, Theorem 2.2; we also give an application to finite energy solutions. The proof of Theorem 2.2 is achieved in Section 3. By the sake of completeness, we include two appendices. In Appendix 4, we recall the regularity of weak solution to the linear problem with non homogeneous data both at the interior and on the boundary, see Theorem 4.1. Appendix 5 deals with further regularity of weak solutions to (1.1), see Theorem 5.1.

2. Main result

For $p > 1$, we define the trace operator $\Gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, in the following way

(1) $\Gamma u = u|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$,
(2) $\|\Gamma u\|_{L^p(\partial\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}$,

where $C = C(p, |\Omega|)$ is a constant and $\partial\Omega$ is C^2. From the surjectivity and the continuity of the trace operator, we obtain

$$\Gamma : W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega) \hookrightarrow L^q(\partial\Omega), \quad \text{for} \quad 1 \le q \le \frac{(N-1)p}{N-p},$$

and

$$\|\Gamma u\|_{L^q(\partial\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}, \quad \text{for some } C > 0,$$

this operator is continuous for $1 \le q \le \frac{(N-1)p}{N-p}$, and compact for $1 \le q < \frac{(N-1)p}{N-p}$ (see [6, Theorem 6.4.1] and [2, Lemma 9.9]).

Throughout this article, we use the Sobolev embedding

$$H^1(\Omega) \hookrightarrow L^{2^*}(\Omega), \tag{2.1}$$

and the continuity of the trace operator

$$H^1(\Omega) \hookrightarrow L^{2_*}(\partial\Omega),$$

where

$$2^* := \frac{2N}{N-2} \quad \text{and} \quad 2_* := \frac{2(N-1)}{N-2} = \frac{(N-1)}{N}\, 2^*, \tag{2.2}$$

are the critical Sobolev exponent and the critical exponent in the sense of the trace, respectively.

For $1 < p,\, p_B \le \infty$, we denote

$$2^*_{N/p} := \frac{2^*}{p'} = 2^*\Big(1 - \frac{1}{p}\Big) \quad \text{and} \quad 2_{*,N/p_B} := \frac{2_*}{p'_B} = 2_*\Big(1 - \frac{1}{p_B}\Big), \tag{2.3}$$

where p' is the conjugate exponent of p, that is $\frac{1}{p} + \frac{1}{p'} = 1$.

For the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$, we assume the following hypothesis at the interior:

(H1) f is a *Carathéodory* function:
  (a) $f(\cdot, t)$ is measurable for each $t \in \mathbb{R}$;
  (b) $f(x, \cdot)$, is continuous for each $x \in \Omega$;

(H2) f is *slightly subcritical (at infinity)*, that is,

$$|f(x,t)| \le |a(x)|\tilde{f}\big(|t|\big), \tag{2.4}$$

with $a(x) \in L^r(\Omega)$ for $r > \frac{N}{2}$, $\tilde{f} : [0, +\infty) \to [0, +\infty)$ is continuous, non-decreasing, $\tilde{f}(t) > 0$ for $t > 0$, and such that

$$\lim_{t \to +\infty} \frac{\tilde{f}(t)}{t^{2^*_{N/r}-1}} = 0. \tag{2.5}$$

Likewise, for the nonlinearity $f_B : \partial\Omega \times \mathbb{R} \to \mathbb{R}$, we assume the following hypothesis on the boundary:

(H3) f_B is a *Carathéodory* function:
  (a) $f_B(\cdot, t)$ is measurable for each $t \in \mathbb{R}$;
  (b) $f_B(x, \cdot)$ is continuous for each $x \in \partial\Omega$.

(H4) f_B is *slightly subcritical (at infinity)*, that is:
$$|f_B(x,t)| \le |a_B(x)|\tilde{f}_B(|t|), \tag{2.6}$$
with $a_B(x) \in L^{r_B}(\partial\Omega)$ for $r_B > N-1$, and $\tilde{f}_B : [0,+\infty) \to [0,+\infty)$ is continuous, non-decreasing, $\tilde{f}_B(t) > 0$ for $t > 0$, and such that
$$\lim_{t\to+\infty} \frac{\tilde{f}_B(t)}{t^{2_{*,N/r_B}-1}} = 0. \tag{2.7}$$

We say that $u \in H^1(\Omega)$ is a *weak solution* to (1.1) if $f(\cdot,u) \in L^{(2^*)'}(\Omega)$, and $f_B(\cdot,u) \in L^{(2_*)'}(\partial\Omega)$ are such that for all $\psi \in H^1(\Omega)$,
$$\int_\Omega \nabla u \nabla\psi\, d + \int_\Omega u\psi\, d = \int_\Omega f(x,u)\psi\, dx + \int_{\partial\Omega} f_B(x,u)\psi\, dS\,,$$
being $(2^*)' = \frac{2N}{N+2}$ and $(2_*)' = \frac{2(N-1)}{N}$ the conjugate exponents of 2^* and 2_*, respectively.

Remark 2.1. (i) Let $u \in H^1(\Omega)$. By Sobolev embeddings, for f and f_B slightly subcritical, we have
$$\tilde{f}(|u|) \in L^{\frac{2^*}{2^*_{N/r}-1}}(\Omega), \quad \text{where} \quad \frac{2^*_{N/r}-1}{2^*} = \frac{1}{2} + \frac{1}{N} - \frac{1}{r},$$
$$\tilde{f}_B(|u|) \in L^{\frac{2_*}{2_{*,N/r_B}-1}}(\partial\Omega), \quad \text{where} \quad \frac{2_{*,N/r_B}-1}{2_*} = \frac{N}{2(N-1)} - \frac{1}{r_B}.$$
Hence,
$$f(\cdot,u) \in L^{(2^*)'}(\Omega) \quad \text{and} \quad f_B(\cdot,u) \in L^{(2_*)'}(\partial\Omega).$$
(ii) We can always choose $\tilde{f}$ and $\tilde{f}_B$ such that $\tilde{f}(t) > 0$ and $\tilde{f}_B(t) > 0$ for $t > 0$. Note that redefining both functions, $\tilde{f}(t)$ and $\tilde{f}_B(t)$, as $\max_{[0,t]} \tilde{f}$ and $\max_{[0,t]} \tilde{f}_B$, respectively, we can always choose $\tilde{f}(t)$ and $\tilde{f}_B(t)$ as non-decreasing functions for $t > 0$.

Now, let us define two new functions,
$$h(t) := \frac{t^{2^*_{N/r}-1}}{\tilde{f}(t)} \quad \text{and} \quad h_B(t) := \frac{t^{2_{*,N/r_B}-1}}{\tilde{f}_B(t)} \quad \text{for } t > 0. \tag{2.8}$$
Since the nonlinearities f and f_B are both slightly subcritical, it follows that
$$h(t) \to \infty \quad \text{and} \quad h_B(t) \to \infty \quad \text{as } t \to \infty. \tag{2.9}$$
Let h_m be defined as the minimum of h and a certain power of h_B, specifically
$$h_m(t) := \min\big\{h(t),\, h_B^{\frac{2^*_{N/r}-1}{2_{*,N/r_B}-1}}(t)\big\}, \tag{2.10}$$
with h and h_B defined in (2.8).

We will denote as a_M the maximum of the corresponding norms of $a \in L^r(\Omega)$ and of $a_B \in L^{r_B}(\partial\Omega)$, that is
$$a_M := \max\{\|a\|_{L^r(\Omega)}, \|a_B\|_{L^{r_B}(\partial\Omega)}\}. \tag{2.11}$$
The next Theorem provies estimates for $h_m(\|u\|_{L^\infty(\Omega)})$ in terms of their $H^1(\Omega)$ norms.

Theorem 2.2. *Assume* (H1)–(H4) *hold and u is a weak solution to* (1.1). *Then, for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ depending of ε, N, $|\Omega|$ and $|\partial\Omega|$, but independent of u, such that*
$$h_m(\|u\|_{L^\infty(\Omega)}) \le C_\varepsilon a_M^{A+\varepsilon}\Big(1 + \|u\|_{H^1(\Omega)}^{(2^*_{N/r}-2)(A+\varepsilon)}\Big), \tag{2.12}$$

where h_m is defined by (2.10), a_M by (2.11), and

$$A := \begin{cases} \frac{\frac{1}{2}-\frac{N-r}{Nr}}{\frac{1}{2}-\frac{N-1}{Nr_B}} & \text{if either } r \geq N, \text{ or } N/2 < r < N \text{ and } r^* \geq \frac{Nr_B}{N-1}, \\ 1 & \text{if } N/2 < r < N \text{ and } r^* \leq \frac{Nr_B}{N-1}. \end{cases} \tag{2.13}$$

Remark 2.3. Since (1.2), we have

$$h_m(\|u\|_{C(\overline{\Omega})}) \leq C_\varepsilon a_M^{A+\varepsilon}\Big(1+\|u\|_{H^1(\Omega)}^{(2^*_{N/r}-2)(A+\varepsilon)}\Big).$$

Remark 2.4. From the definitions of h and h_B given in (2.8), we note that

$$h(t) = \frac{t^{2^*_{N/r}-1}}{\tilde{f}(t)} \quad \text{and} \quad h_B^{\frac{2^*_{N/r}-1}{2_{*,N/r_B}-1}}(t) = \Big(\frac{t^{2_{*,N/r_B}-1}}{\tilde{f}_B(t)}\Big)^{\frac{2^*_{N/r}-1}{2_{*,N/r}-1}}.$$

Thus,

$$h_m(t) = \min\Big\{\frac{t^{2^*_{N/r}-1}}{\tilde{f}(t)}, \frac{t^{2^*_{N/r}-1}}{\tilde{f}_B^{\frac{2^*_{N/r}-1}{2_{*,N/r}-1}}(t)}\Big\}.$$

We apply our result to finite energy solutions of subcritical problems satisfying Ambrosetti-Rabinowitz condition. A sequence $\{u_n\} \subset H^1(\Omega)$ of weak solutions to (1.1) has *uniformly bounded energy* if there exists a constant $c_0 > 0$, such that $J[u_n] \leq c_0$, where J is the associated energy functional defined by

$$J[u] := \frac{1}{2}\int_\Omega \Big(|\nabla u|^2 + u^2\Big) - \int_\Omega F(x,u)\,dx - \int_{\partial\Omega} F_B(x,u)\,d\sigma_x$$

with $F(x,t) := \int_0^t f(x,s)\,ds$, and $F_B(x,t) := \int_0^t f_B(x,s)\,ds$.

The *Ambrosetti–Rabinowitz condition* holds if there exist two constants $\theta > 2$, and $s_0 > 0$ such that

$$\begin{aligned} \theta F(x,s) &\leq s f(x,s), \quad \forall x \in \Omega,\ \forall |s| > s_0, \\ \theta F_B(x,s) &\leq s f_B(x,s), \quad \forall x \in \partial\Omega,\ \forall |s| > s_0. \end{aligned} \tag{2.14}$$

Assuming that (H1)–(H4) and (2.14) hold, a sequence of solutions to (1.1) is uniformly $L^\infty(\Omega)$ *a priori* bounded if and only if it has uniformly bounded energy. It can be proved using the same arguments as in [3, theorem 5.1].

3. $L^\infty(\Omega)$ a priori explicit estimates

Our method combines elliptic regularity with the Gagliardo-Nirenberg interpolation inequality. Let u be an arbitrary solution to (1.1). First, we find estimates of the nonlinearities in terms of products of the $H^1(\Omega)$-norm of u and their $L^\infty(\Omega)$-norm. With it, using elliptic regularity (see Theorem (4.1)), we obtain estimates of the $W^{1,m}(\Omega)$-norm, with $m > N$, of the solutions to (1.1). Finally, applying the Gagliardo-Nirenberg interpolation inequality, (see [12]), we obtain an explicit estimate of the $L^\infty(\Omega)$-norm of u in terms of the $H^1(\Omega)$ norm of u.

Proof of Theorem 2.2. Let $u \in H^1(\Omega)$ be a weak solution to (1.1). By Theorem 5.1, $u \in H^1(\Omega) \cap L^\infty(\Omega)$.

Firstly, we will estimate both nonlinearities (the interior and the boundary nonlinearities) in terms of the $H^1(\Omega)$-norm and the $L^\infty(\Omega)$-norm of u.

Step 1. $W^{1,m}(\Omega)$ estimates for $m > N$. By hypothesis, $\tilde{f}$ and $\tilde{f}_B$ are both increasing. By (1.2) we denote

$$\begin{aligned} M &:= \tilde{f}(\|u\|_{L^\infty(\Omega)}) = \max_{[0,\|u\|_{L^\infty(\Omega)}]} \tilde{f}, \\ M_B &:= \tilde{f}_B(\|u\|_{L^\infty(\Omega)}) = \max_{[0,\|u\|_{L^\infty(\Omega)}]} \tilde{f}_B. \end{aligned} \tag{3.1}$$

Along this proof, we will use the obvious fact that for any $\gamma > 0$, there exist two constants C_1 and C_2, only dependent on γ, such that

$$C_1(1+x^\gamma) \leq (1+x)^\gamma \leq C_2(1+x^\gamma), \quad \text{for all } x \geq 0. \tag{3.2}$$

Throughout this proof, C denotes several constants independent of u.

By the growth condition (2.4) and the definition given in (3.1), we have that

$$\int_\Omega |f(x,u)|^q dx \le \int_\Omega |a(x)|^q \tilde f(|u|)^{q-t+t}\,dx \le CM^{q-t}\int_\Omega |a(x)|^q \tilde f(|u|)^t\,dx, \tag{3.3}$$

for all $t<q$, and all

$$q\in \big(\frac{N}{2}, \min\{r,N\}\big). \tag{3.4}$$

Using Hölder's inequality, for all $1<s<\infty$, we can write

$$\int_\Omega |a(x)|^q \tilde f(|u|)^t\,dx \le \Big(\int_\Omega |a(x)|^{qs} dx\Big)^{1/s}\Big(\int_\Omega \tilde f(|u|)^{ts'} dx\Big)^{1/s'}, \tag{3.5}$$

where s' is such that $\frac1s+\frac1{s'}=1$. Choosing s and $t<q$, so that $qs=r$ and $ts'=\frac{2^*}{2^*_{N/r}-1}$, we have

$$\begin{aligned} t &:= \frac{2^*}{2^*_{N/r}-1}\Big(1-\frac{q}{r}\Big)<q\\ &\iff \frac1q-\frac1r<\frac{2^*_{N/r}-1}{2^*}=1-\frac1r-\frac12+\frac1N\\ &\iff q>\frac{2N}{N+2}, \end{aligned} \tag{3.6}$$

since $q>\frac N2>\frac{2N}{N+2}$.

On the other hand, by subcriticality, see (2.5), and the Sobolev embeddings, see (2.1), we have

$$\begin{aligned} \int_\Omega |\tilde f(|u|)|^{\frac{2^*}{2^*_{N/r}-1}}dx &\le C\int_\Omega \Big(1+|u|^{2^*}\Big)\,dx\\ &\le C\Big(1+\|u\|^{2^*}_{L^{2^*}(\Omega)}\Big) \qquad (3.7)\\ &\le C\Big(1+\|u\|^{2^*}_{H^1(\Omega)}\Big)\,. \qquad (3.8) \end{aligned}$$

Substituting (3.8) in the second factor on the right-hand side of (3.5),

$$\int_\Omega |a(x)|^q \tilde f(|u|)^t\,dx \le C\Big(\int_\Omega |a(x)|^{qs} dx\Big)^{1/s}\Big(1+\|u\|^{2^*}_{H^1(\Omega)}\Big)^{1/s'}. \tag{3.9}$$

Finally, substituting (3.9) in (3.3) and since $1/(qs')=1/q-1/r$, we obtain

$$\Big(\int_\Omega |f(x,u)|^q dx\Big)^{1/q} \le CM^{1-\frac tq}\|a\|_{L^r(\Omega)}\Big(1+\|u\|^{2^*(\frac1q-\frac1r)}_{H^1(\Omega)}\Big). \tag{3.10}$$

Likewise, by the condition (2.6) and the subcriticality (2.7), we obtain

$$\begin{aligned} \int_{\partial\Omega} |f_B(x,u)|^{q_B}\,dS &\le \int_{\partial\Omega} |a_B(x)|^{q_B}\tilde f_B(|u|)^{q_B-t_B+t_B}\,dS\\ &\le CM_B^{q_B-t_B}\int_{\partial\Omega}|a_B(x)|^{q_B}\tilde f_B(|u|)^{t_B}\,dS, \end{aligned} \tag{3.11}$$

for all $t_B<q_B$, and all

$$q_B\in (N-1, r_B). \tag{3.12}$$

Using Hölder's inequality, for all $1<s_B<\infty$, we obtain

$$\int_{\partial\Omega} |a_B(x)|^{q_B}\tilde f_B(|u|)^{t_B}\,dS \le \Big(\int_{\partial\Omega}|a_B(x)|^{q_Bs_B}dS\Big)^{1/s_B}\Big(\int_{\partial\Omega}\tilde f_B(|u|)^{t_Bs'_B}dS\Big)^{1/s'_B}, \tag{3.13}$$

where s'_B is such that $\frac{1}{s_B}+\frac{1}{s'_B}=1$. Choosing, as before, s_B, $t_B<q_B$, so that $q_Bs_B=r_B$, and $t_Bs'_B=\frac{2_*}{2_{*,N/r_B}-1}$; thus,

$$\begin{aligned} t_B &:= \frac{2_*}{2_{*,N/r_B}-1}\Big(1-\frac{q_B}{r_B}\Big)<q_B \\ \iff \frac{1}{q_B}-\frac{1}{r_B} &< \frac{2_{*,N/r_B}-1}{2_*}=1-\frac{1}{r_B}-\frac{N-2}{2(N-1)} \\ \iff \frac{1}{q_B} &< \frac{N}{2(N-1)} \\ \iff q_B &> \frac{2(N-1)}{N}, \end{aligned} \tag{3.14}$$

and the last inequality is satisfied since $q_B>N-1$ and $N>2$.

On the other hand, again by subcriticality, see (2.6) and (2.7), we have

$$\begin{aligned} \int_{\partial\Omega}|\tilde f_B(|u|)|^{\frac{2_*}{2_{*,N/r_B}-1}}dx &\le C\int_{\partial\Omega}\left(1+|u|^{2_*}\right)dS \\ &\le C\left(1+\|u\|^{2_*}_{L^{2_*}(\partial\Omega)}\right) \quad (3.15)\\ &\le C\left(1+\|u\|^{2_*}_{H^1(\Omega)}\right), \quad (3.16) \end{aligned}$$

Since $t_Bs'_B=2_*/(2_{*,N/r_B}-1)$, and substituting (3.16) in the second factor on the right-hand side of (3.13),

$$\int_{\partial\Omega}|a_B(x)|^{q_B}\tilde f_B(|u|)^{t_B}\,dS\le C\Big(\int_{\partial\Omega}|a_B(x)|^{q_Bs_B}dS\Big)^{1/s_B}\left(1+\|u\|^{2_*}_{H^1(\Omega)}\right)^{1/s'_B}, \tag{3.17}$$

Finally, substituting (3.17) into (3.11), and since $1/(q_Bs'_B)=1/q_B-1/r_B$, we obtain

$$\Big(\int_{\partial\Omega}|f_B(x,u)|^{q_B}dx\Big)^{1/q_B}\le CM_B^{1-\frac{t_B}{q_B}}\|a_B\|_{L^{r_B}(\partial\Omega)}\Big(1+\|u\|^{2_*(\frac{1}{q_B}-\frac{1}{r_B})}_{H^1(\Omega)}\Big). \tag{3.18}$$

Now, using elliptic regularity, we estimate the norm $\|u\|_{W^{1,m}(\Omega)}$ in terms of the corresponding norms of the nonlinearities, see Theorem 4.1, Equation (4.2). Specifically, using (3.10) and (3.18), we obtain

$$\begin{aligned} \|u\|_{W^{1,m}(\Omega)}\le C\Big[&M^{1-\frac{t}{q}}\|a\|_{L^r(\Omega)}\Big(1+\|u\|^{2^*(\frac{1}{q}-\frac{1}{r})}_{H^1(\Omega)}\Big) \\ &+M_B^{1-\frac{t_B}{q_B}}\|a_B\|_{L^{r_B}(\partial\Omega)}\Big(1+\|u\|^{2_*\left(\frac{1}{q_B}-\frac{1}{r_B}\right)}_{H^1(\Omega)}\Big)\Big], \end{aligned} \tag{3.19}$$

where $m=\min\{q^*,\frac{Nq_B}{N-1}\}$ ($q^*:=\frac{Nq}{N-q}$), whenever $1\le q<N$, see Theorem 4.1. Fixing

$$q_B:=\frac{(N-1)q^*}{N}\implies m=q^*=\frac{Nq_B}{N-1}>N, \tag{3.20}$$

(in the forthcoming Remark 3.1, we explain the necessity of the election for q_B), moreover, we have the following equivalences

$$q_B:=\frac{(N-1)q^*}{N}\iff\frac{2_*}{q_B}=\frac{2^*}{q^*}\iff 2_{*,N/q_B}=2^*_{N/q}. \tag{3.21}$$

Indeed, we only have to notice that, using the definitions (2.2), (2.3) and (3.20), we can conclude that

$$2_{*,N/q_B}=2_*-\frac{2_*}{q_B}=2_*+\frac{2^*}{N}-\frac{2^*}{q}=2^*-\frac{2^*}{q}=2^*_{N/q}.$$

With that election of q_B, we also need to restrict q in order to satisfy (3.12). Specifically

$$q\in\Big(\frac{N}{2},\min\big\{r,\frac{Nr_B}{N-1+r_B}\big\}\Big). \tag{3.22}$$

Note that, because of the definition of q_B, see (3.20), and their restriction, (3.12), the following inequality has to be satisfied

$$N-1<\frac{(N-1)q^*}{N}=q_B<r_B.$$

By (3.4), we obtain that $q^*>N$ so $\frac{(N-1)q^*}{N}>N-1$. Thus, we only need to check that

$$q^*<\frac{Nr_B}{N-1}\iff\frac{1}{q}-\frac{1}{N}>\frac{N-1}{Nr_B}\iff\frac{1}{q}>\frac{N-1}{Nr_B}+\frac{1}{N}\iff q<\frac{Nr_B}{N-1+r_B},$$

from which, using (3.4), and that

$$\frac{Nr_B}{N-1+r_B}<N, \tag{3.23}$$

we conclude (3.22).

Step 2. Gagliardo-Nirenberg interpolation inequality. The Gagliardo-Nirenberg interpolation inequality (see [12]), implies that there exists a constant $C=C(N,q,|\Omega|)$, such that

$$\|u\|_{L^\infty(\Omega)}\le C\|u\|^{\sigma}_{W^{1,q^*}(\Omega)}\|u\|^{1-\sigma}_{L^{2^*}(\Omega)}, \tag{3.24}$$

where

$$\frac{1}{\sigma}=1+2^*\Big(\frac{2}{N}-\frac{1}{q}\Big). \tag{3.25}$$

From (3.21), by the definition of $2^*_{N/q}$, see (2.3), it is easy to check that

$$\frac{1}{\sigma}=1+2^*\Big[\Big(\frac{2-N}{N}\Big)+\Big(1-\frac{1}{q}\Big)\Big]=2^*_{N/q}-1. \tag{3.26}$$

Substituting the estimate of $\|u\|_{W^{1,m}(\Omega)}$, see 3.19, and using (3.2) in the inequality (3.24), we obtain

$$\begin{aligned}\|u\|_{L^\infty(\Omega)}&\le C\Big[M^{1-\frac{t}{q}}\|a\|_{L^r(\Omega)}\Big(1+\|u\|^{2^*(\frac{1}{q}-\frac{1}{r})}_{H^1(\Omega)}\Big)\\&\quad+M_B^{1-\frac{t_B}{q_B}}\|a_B\|_{L^{r_B}(\partial\Omega)}\Big(1+\|u\|^{2_*(\frac{1}{q_B}-\frac{1}{r_B})}_{H^1(\Omega)}\Big)\Big]^{\sigma}\|u\|^{(1-\sigma)}_{L^{2^*}(\Omega)}\\&\le C\Big[M^{(1-\frac{t}{q})\sigma}\|a\|^{\sigma}_{L^r(\Omega)}\Big(1+\|u\|^{2^*(\frac{1}{q}-\frac{1}{r})\sigma}_{H^1(\Omega)}\Big)\\&\quad+M_B^{(1-\frac{t_B}{q_B})\sigma}\|a_B\|^{\sigma}_{L^{r_B}(\partial\Omega)}\Big(1+\|u\|^{2_*(\frac{1}{q_B}-\frac{1}{r_B})\sigma}_{H^1(\Omega)}\Big)\Big]\|u\|^{(1-\sigma)}_{L^{2^*}(\Omega)},\end{aligned} \tag{3.27}$$

We now look closely at the exponents of $\|u\|_{L^\infty(\Omega)}$ in the right-hand side, in order to achieve our estimates. Taking into account the definitions of M and M_B, see (3.1), that f and f_B are non-decreasing, and the definitions of the functions h and h_B, see (2.8), we can write the following relation between them,

$$M=\frac{\|u\|^{2^*_{N/r}-1}_{L^\infty(\Omega)}}{h(\|u\|_{L^\infty(\Omega)})}\quad\text{and}\quad M_B=\frac{\|u\|^{2_{*,N/r_B}-1}_{L^\infty(\Omega)}}{h_B(\|u\|_{L^\infty(\Omega)})}. \tag{3.28}$$

Moreover, using the definitions of t, see (3.6), and of $2^*_{N/p}$, see (2.3), we obtain

$$1-\frac{t}{q}=1-\frac{2^*}{2^*_{N/r}-1}\Big(\frac{1}{q}-\frac{1}{r}\Big)=\frac{2^*_{N/q}-1}{2^*_{N/r}-1}. \tag{3.29}$$

Thus, because of the expression (3.29), we deduce

$$\Big(2^*_{N/r}-1\Big)\Big(1-\frac{t}{q}\Big)=(2_{*,N/q}-1),$$

and because of the definition of σ, see (3.26),

$$\Big(2^*_{N/r}-1\Big)\Big(1-\frac{t}{q}\Big)\sigma=1. \tag{3.30}$$

Similarly, from the definitions of t_B, see (3.14), and of $2_{*,N/q_B}$, see (2.3), we obtain

$$1-\frac{t_B}{q_B}=1-\frac{2_*}{2_{*,N/r_B}-1}\Big(\frac{1}{q_B}-\frac{1}{r_B}\Big)=\frac{2_{*,N/q_B}-1}{2_{*,N/r_B}-1}. \tag{3.31}$$

Likewise, since (3.31), the definition of σ, see (3.26), and the equivalences (3.21), we obtain

$$\left(2_{*,N/r_B}-1\right)\Big(1-\frac{t_B}{q_B}\Big)\sigma=\frac{2_{*,N/q_B}-1}{2_{*,N/q}-1}=1. \tag{3.32}$$

Now, we divide both sides of the inequality (3.27) by $\|u\|_{L^\infty(\Omega)}$. Using the definitions of M and M_B, also the two expressions concerning σ; (3.30), (3.32), and the definition of a_M, see (2.11), we obtain

$$1\le Ca_M^\sigma\Bigg(\frac{\left(1+\|u\|_{H^1(\Omega)}^{2^*(\frac1q-\frac1r)\sigma}\right)}{h^{\frac{1}{2^*_{N/r}-1}}(\|u\|_{L^\infty(\Omega)})}+\frac{\left(1+\|u\|_{H^1(\Omega)}^{2_*(\frac{1}{q_B}-\frac{1}{r_B})\sigma}\right)}{h_B^{\frac{1}{2_{*,N/r_B}-1}}(\|u\|_{L^\infty(\Omega)})}\Bigg)\|u\|_{L^{2^*}(\Omega)}^{(1-\sigma)}. \tag{3.33}$$

The definition of h_m (see (2.10)), implies that

$$\frac{1}{h_m^{\frac{1}{2^*_{N/r}-1}}(\|u\|_{L^\infty(\Omega)})}=\max\Bigg\{\frac{1}{h^{\frac{1}{2^*_{N/r}-1}}(\|u\|_{L^\infty(\Omega)})},\frac{1}{h_B^{\frac{1}{2_{*,N/r_B}-1}}(\|u\|_{L^\infty(\Omega)})}\Bigg\}.$$

So, substituting this maximum in the inequality (3.33), we obtain

$$h_m^{\frac{1}{2^*_{N/r}-1}}(\|u\|_{L^\infty(\Omega)})\le Ca_M^\sigma\Big(1+\|u\|_{H^1(\Omega)}^{2^*(\frac1q-\frac1r)\sigma}+\|u\|_{H^1(\Omega)}^{2_*(\frac{1}{q_B}-\frac{1}{r_B})\sigma}\Big)\|u\|_{L^{2^*}(\Omega)}^{(1-\sigma)}. \tag{3.34}$$

The right-hand side in the above inequality is bounded above by a term with the largest exponent of both addends. Let us denote this maximum by

$$E_M:=\max\big\{2^*(\frac1q-\frac1r),2_*(\frac{1}{q_B}-\frac{1}{r_B})\big\}. \tag{3.35}$$

From inequality (3.34), definition (3.35) and Sobolev's embedding, we obtain

$$h_m(\|u\|_{L^\infty(\Omega)})\le Ca_M^\theta\Big(1+\|u\|_{H^1(\Omega)}^\beta\Big), \tag{3.36}$$

where

$$\theta:=\Big(2^*_{N/r}-1\Big)\,\sigma=\frac{2^*_{N/r}-1}{2^*_{N/q}-1}, \tag{3.37}$$

$$\beta:=\Big(E_M+\frac{1-\sigma}{\sigma}\Big)\theta. \tag{3.38}$$

Now, we look closely at the definition of E_M. Firstly by definitions of 2^* and of 2_*, see (2.3), secondly by election of q_B, see (3.20), and finally rearranging terms, we observe that

$$\begin{aligned}&2^*\Big(\frac1q-\frac1r\Big)\ge 2_*\Big(\frac{1}{q_B}-\frac{1}{r_B}\Big)\\ \iff\ &\frac1q\mp\frac1N-\frac1r\ge\frac{N-1}{N}\Big(\frac{1}{q_B}-\frac{1}{r_B}\Big)\\ \iff\ &\frac1r-\frac1N\le\frac{N-1}{Nr_B}.\end{aligned} \tag{3.39}$$

If $r\ge N$, then $1/r-1/N\le 0$, and the last inequality holds. Moreover, if $N/2<r<N$, then the last inequality holds if and only if

$$(1/r-1/N)^{-1}=:r^*\ge Nr_B/(N-1).$$

Observe that

$$r^*\ge Nr_B/(N-1)\iff 2^*_{N/r}\ge 2_{*,N/r_B}.$$

On the contrary, the reverse inequality to (3.39), will be satisfied whenever $N/2 < r < N$, and $r^* \le Nr_B/(N-1)$. Hence

$$E_M = \begin{cases} 2^*\left(\frac{1}{q} - \frac{1}{r}\right) & \text{if } r \ge N \text{ or } N/2 < r < N \text{ and } r^* \ge \frac{Nr_B}{N-1}, \\ 2_*\left(\frac{1}{q_B} - \frac{1}{r_B}\right) & \text{if } N/2 < r < N \text{ and } r^* \le \frac{Nr_B}{N-1}. \end{cases} \tag{3.40}$$

Consequently, we have two cases in the search for the optimum exponents θ and β varying q, see (3.22),

Case (I): Either $r \ge N$, or $N/2 < r < N$ and $r^* \ge \frac{Nr_B}{N-1}$. Using the definition of β, (3.38), the first equality for E_M in (3.40), and the expression for σ, see (3.25), and for $2^*_{N/r}$, see (2.3), we have

$$\beta = \left[2^*\left(\frac{1}{q} - \frac{1}{r}\right) + \frac{1-\sigma}{\sigma}\right]\theta \tag{3.41}$$

$$= \left[2^*\left(\frac{1}{q} - \frac{1}{r}\right) + 2^*\left(\frac{2}{N} - \frac{1}{q}\right)\right]\theta = (2^*_{N/r} - 2)\theta. \tag{3.42}$$

The function $\theta : q \mapsto \theta(q)$, defined by (3.37) is decreasing. We look for the infimum for q in the interval (3.4).

Assume $r \ge N$. Since (3.22)–(3.23), we deduce that $q \in \left(\frac{N}{2}, \frac{Nr_B}{N-1+r_B}\right)$.

Assume $N/2 < r < N$ *and* $r^* \ge \frac{Nr_B}{N-1}$. Note that

$$r^* \ge \frac{Nr_B}{N-1} \iff \frac{1}{r} - \frac{1}{N} \le \frac{N-1}{Nr_B} \iff r \ge \frac{Nr_B}{N-1+r_B}, \tag{3.43}$$

and $q \in \left(\frac{N}{2}, \frac{Nr_B}{N-1+r_B}\right)$.

Hence, in case I,

$$\inf_{q \in (\frac{N}{2}, \frac{Nr_B}{N-1+r_B})} \theta(q) = \theta\Big(\frac{Nr_B}{N-1+r_B}\Big) = \frac{\frac{1}{2} - \frac{N-r}{Nr}}{\frac{1}{2} - \frac{N-1}{Nr_B}}. \tag{3.44}$$

Case (II): $N/2 < r < N$ and $r^* \le \frac{Nr_B}{N-1}$. Likewise, using the definition of β, (3.38), the second equality for E_M in (3.40), and the expressions for σ (3.25), for $2^*_{N/p}$, $2_{*,N/p_B}$ (2.3), and the equivalence (3.21), we have

$$\begin{aligned} \beta &= \Big[2_*\Big(\frac{1}{q_B} - \frac{1}{r_B}\Big) + \frac{1-\sigma}{\sigma}\Big]\theta \\ &= \Big[2_*\Big(\frac{1}{q_B} \mp 1 - \frac{1}{r_B}\Big) + 2^*\Big(\frac{2}{N} \mp 1 - \frac{1}{q}\Big)\Big]\theta \\ &= \Big[2_{*,N/r_B} - 2_{*,N/q_B} - 2 + 2^*_{N/q}\Big]\theta \\ &= [2_{*,N/r_B} - 2]\theta. \end{aligned} \tag{3.45}$$

For q satisfying (3.22), thanks to (3.43), we deduce that $q \in (\frac{N}{2}, r)$, hence

$$\inf_{q \in (\frac{N}{2}, r)} \theta(q) = \theta(r) = 1. \tag{3.46}$$

Finally, we introduce into the inequality (3.36), the infima of θ and β given by (3.44) and (3.42) respectively in case I, and by (3.46) and (3.45), in case II. Since these infima are not attained in the set where q belongs, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$h_m(\|u\|_{L^\infty(\Omega)}) \le C_\varepsilon a_M^{A+\varepsilon}\Big(1 + \|u\|_{H^1(\Omega)}^{(2^*_{N/r}-2)(A+\varepsilon)}\Big),$$

where A is defined in (2.13), and $C_\varepsilon = C_\varepsilon(\varepsilon, N, |\Omega|, |\partial\Omega|)$ and it is independent of u. □

In the following Remark, we state the necessity of the election for q_B, see (3.20).

Remark 3.1. Assume that (3.20) does not hold and, to fix ideas, that

$$q_B < \frac{(N-1)q^*}{N} \implies m = \frac{Nq_B}{N-1} > N.$$

We also have the equivalence

$$q_B < \frac{(N-1)q^*}{N} \iff \frac{2^*}{q^*} < \frac{2_*}{q_B} \iff 2_{*,N/q_B} < 2^*_{N/q}.$$

Indeed, the first equivalence is obvious. With respect to the second one, notice that, due to the definitions of $2^*_{N/q}$ and of $2_{*,N/q_B}$, see (2.3), we can conclude that

$$2_{*,N/q_B} = 2_* - \frac{2_*}{q_B} < 2_* - \frac{2^*}{q^*} = 2^* - \frac{2^*}{q} = 2^*_{N/q}.$$

Now, in the Gagliardo-Nirenberg interpolation inequality, see (3.24), the parameter σ is given by

$$\begin{aligned}\frac{1}{\sigma} &= 1 + 2^*\Big(\frac{1}{N} - \frac{1}{m}\Big)\\ &= 1 + 2^*\Big(\frac{1}{N} \mp 1 - \frac{N-1}{Nq_B}\Big)\\ &= 2_{*,N/q_B} - 1\\ &< 2^*_{N/q} - 1.\end{aligned}$$

And the expression (3.30) becomes

$$\Big(2^*_{N/r} - 1\Big)\Big(1 - \frac{t}{q}\Big)\sigma = \frac{2_{*,N/q} - 1}{2_{*,N/q_B} - 1} > 1.$$

The above inequality implies that the exponent of $\|u\|_{L^\infty(\Omega)}$ in the right-hand side will dominate 1, which is the exponent of $\|u\|_{L^\infty(\Omega)}$ in the LHS, and the bounds can not be reached.

Likewise, if $q_B > \frac{(N-1)q^*}{N}$, then $m = q^*$ and it can be proved that

$$\frac{1}{\sigma} = 1 + 2^*\Big(\frac{2}{N} \mp 1 - \frac{1}{q}\Big) = 2^*_{N/q} - 1 < 2_{*,N/q_B} - 1,$$

and so

$$\big(2_{*,N/r_B} - 1\big)\Big(1 - \frac{t_B}{q_B}\Big)\sigma = (2_{*,N/q_B} - 1)\sigma = \frac{2_{*,N/q_B} - 1}{2^*_{N/q} - 1} > 1,$$

concluding that necessarily, q_B has to be chosen as in (3.20).

Throughout that proof, we have explicit estimates of $h_m(\|u\|_{L^\infty(\Omega)})$ expressed in their $L^{2^*}(\Omega)$ norm and $L^{2_*}(\partial\Omega)$ norm (see (3.7) and (3.15)). Previously, we unify those estimates in their $H^1(\Omega)$ norm to simplify the expression. In the next Corollary, we split those estimates in terms of the $L^{2^*}(\Omega)$ norm and the $L^{2_*}(\partial\Omega)$ norm.

Corollary 3.2. *Assume that the hypotheses of Theorem 2.2 hold. Then, for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ depending of ε, N, $|\Omega|$ and $|\partial\Omega|$, but independent of u, such that*

$$h_m(\|u\|_{L^\infty(\Omega)}) \le C a_M^{A+\varepsilon}\Big(1 + \|u\|^{A_1+\varepsilon}_{L^{2^*}(\Omega)} + \|u\|^{A_2+\varepsilon}_{L^{2_*}(\partial\Omega)}\|u\|^{A_3+\varepsilon}_{L^{2^*}(\Omega)}\Big),$$

where A is defined in (2.13)*, with*

$$A_1 := (2^*_{N/r} - 2)A, \quad A_2 := 0, \quad A_3 := (2_{*,N/r_B} - 2)A,$$

if either $r \ge N$ or $N/2 < r < N$ and $r^* \ge \frac{Nr_B}{N-1}$; and

$$A_1 := 2^*_{N/r} - 2, \quad A_2 := 2_{*,N/r_B} - 2^*_{N/r}, \quad A_3 := 2^*_{N/r} - 2,$$

if $N/2 < r < N$ and $r^* \le \frac{Nr_B}{N-1}$.

Proof. The proof is similar to the proof of the Theorem (2.2).

Step 1. $W^{1,m}(\Omega)$ estimates for $m > N$. Substituting (3.7) in the second factor on the right-hand side of (3.5), and this is (3.3), we obtain

$$\Big(\int_{\Omega}|f(x,u)|^q dx\Big)^{1/q} \le CM^{1-\frac{t}{q}}\|a\|_{L^r(\Omega)}\Big(1+\|u\|_{L^{2^*}(\Omega)}^{2^*(\frac{1}{q}-\frac{1}{r})}\Big), \tag{3.47}$$

with M defined in (3.1), t in (3.6) and q in (3.4), respectively. See the analogy with (3.10).

On the other hand, replacing (3.15) in the second factor on the right-hand side of (3.13), and this in (3.11), we obtain

$$\Big(\int_{\partial\Omega}|f_B(x,u)|^{q_B} dx\Big)^{1/q_B} \le CM_B^{1-\frac{t_B}{q_B}}\|a_B\|_{L^{r_B}(\partial\Omega)}\Big(1+\|u\|_{L^{2_*}(\partial\Omega)}^{2_*(\frac{1}{q_B}-\frac{1}{r_B})}\Big), \tag{3.48}$$

with M_B defined in (3.1), t_B in (3.14), and q_B in (3.12) respectively. See the analogy with (3.18).

By elliptic regularity, we estimate the norm $\|u\|_{W^{1,m}(\Omega)}$ in terms of (3.47) and (3.48), see Theorem 4.1, obtaining

$$\begin{aligned}&\|u\|_{W^{1,m}(\Omega)}\\ &\le C\Big[M^{1-\frac{t}{q}}\|a\|_{L^r(\Omega)}\Big(1+\|u\|_{L^{2^*}(\Omega)}^{2^*(\frac{1}{q}-\frac{1}{r})}\Big)+M_B^{1-\frac{t_B}{q_B}}\|a_B\|_{L^{r_B}(\partial\Omega)}\Big(1+\|u\|_{L^{2_*}(\partial\Omega)}^{2_*(\frac{1}{q_B}-\frac{1}{r_B})}\Big)\Big],\end{aligned} \tag{3.49}$$

with $m > N$. See also the analogy with (3.19).

Step 2. Gagliardo-Nirenberg interpolation inequality. Substituting (3.49) in the Gagliardo-Nirenberg inequality (3.24) and using the inequality (3.2), we obtain

$$\begin{aligned}\|u\|_{L^\infty(\Omega)} \le C\Big[&M^{(1-\frac{t}{q})\sigma}\|a\|_{L^r(\Omega)}^{\sigma}\Big(1+\|u\|_{L^{2^*}(\Omega)}^{2^*(\frac{1}{q}-\frac{1}{r})\sigma}\Big)\\ &+M_B^{(1-\frac{t_B}{q_B})\sigma}\|a_B\|_{L^{r_B}(\partial\Omega)}^{\sigma}\Big(1+\|u\|_{L^{2_*}(\partial\Omega)}^{2_*(\frac{1}{q_B}-\frac{1}{r_B})\sigma}\Big)\Big]\|u\|_{L^{2^*}(\Omega)}^{(1-\sigma)},\end{aligned} \tag{3.50}$$

Using the definitions of M and M_B, see (3.28), using also (3.30), (3.32), the definition of a_M (see (2.11)), and dividing both sides of the inequality (3.50) by $\|u\|_{L^\infty(\Omega)}$, we obtain

$$1 \le Ca_M^{\sigma}\Bigg(\frac{\Big(1+\|u\|_{L^{2^*}(\Omega)}^{2^*(\frac{1}{q}-\frac{1}{r})\sigma}\Big)}{h^{\frac{1}{2^*_{N/r}-1}}(\|u\|_{L^\infty(\Omega)})}+\frac{\Big(1+\|u\|_{L^{2_*}(\partial\Omega)}^{2_*(\frac{1}{q_B}-\frac{1}{r_B})\sigma}\Big)}{h_B^{\frac{1}{2_{*,N/r_B}-1}}(\|u\|_{L^\infty(\Omega)})}\Bigg)\|u\|_{L^{2^*}(\Omega)}^{(1-\sigma)}.$$

Then

$$h_m^{\frac{1}{2^*_{N/r}-1}}(\|u\|_{L^\infty(\Omega)}) \le Ca_M^{\sigma}\Big(2+\|u\|_{L^{2^*}(\Omega)}^{2^*(\frac{1}{q}-\frac{1}{r})\sigma}+\|u\|_{L^{2_*}(\partial\Omega)}^{2_*(\frac{1}{q_B}-\frac{1}{r_B})\sigma}\Big)\|u\|_{L^{2^*}(\Omega)}^{(1-\sigma)}.$$

where h_m is defined in (2.10). See the analogy with (3.34). Clearing, we obtain

$$h_m(\|u\|_{L^\infty(\Omega)}) \le Ca_M^{\sigma(2^*_{N/r}-1)}\Big(1+\|u\|_{L^{2^*}(\Omega)}^{2^*(\frac{1}{q}-\frac{1}{r})(2^*_{N/r}-1)\sigma}+\|u\|_{L^{2_*}(\partial\Omega)}^{2_*(\frac{1}{q_B}-\frac{1}{r_B})(2^*_{N/r}-1)\sigma}\Big)\|u\|_{L^{2^*}(\Omega)}^{(1-\sigma)(2^*_{N/r}-1)}.$$

Substituting in the exponents the parameter θ (see (3.37)), we obtain

$$h_m(\|u\|_{L^\infty(\Omega)}) \le Ca_M^{\theta}\Big(1+\|u\|_{L^{2^*}(\Omega)}^{[2^*(\frac{1}{q}-\frac{1}{r})+\frac{1-\sigma}{\sigma}]\theta}+\|u\|_{L^{2_*}(\partial\Omega)}^{2_*(\frac{1}{q_B}-\frac{1}{r_B})\theta}\|u\|_{L^{2^*}(\Omega)}^{\frac{1-\sigma}{\sigma}\theta}\Big). \tag{3.51}$$

Let us define the function $\theta_1 = \theta_1(q)$ as the first exponent inside the brackets. Using the definitions of $2^*_{N/q}$, see (2.3), and of σ, see (3.26), we obtain

$$\theta_1(q) := \Big[2^*\Big(\frac{1}{q}-\frac{1}{r}\Big)+\frac{1-\sigma}{\sigma}\Big]\theta(q) = (2^*_{N/r}-2)\theta(q),$$

note that this value is equal to β in case I of Theorem (2.2), see (3.41)–(3.42).

We define the function $\theta_2 = \theta_2(q)$ as the second exponent. By the definition of $2_{*,N/q_B}$, see (2.3), and the equivalence (3.21),

$$\theta_2(q) := 2_*\Big(\frac{1}{q_B}\mp 1-\frac{1}{r_B}\Big)\theta(q) = (2_{*,N/r_B}-2^*_{N/q})\theta(q).$$

We define the function $\theta_3 = \theta_3(q)$ as the third exponent. Using the expression (3.26) for σ,

$$\theta_3(q) := \big(\frac{1-\sigma}{\sigma}\big)\theta(q) = (2^*_{N/q} - 2)\theta(q).$$

As before, let q_B, θ be defined by (3.20), and (3.37) respectively. The function

$$(\theta_1 + \theta_2 + \theta_3)(q) = (2^*_{N/r} + 2_{*,N/rB} - 4)\theta(q),$$

is decreasing, and we look for their infimum for q in the interval (3.22). Thus, as before, we consider the previous two cases.

Case (I) Either $r \ge N$, or $N/2 < r < N$ and $r^* \ge \frac{Nr_B}{N-1}$. In this case, $q \in (\frac{N}{2}, \frac{Nr_B}{N-1+r_B})$. For A defined in (2.13), the exponents are

$$A_1' := \theta_1\Big(\frac{Nr_B}{N-1+r_B}\Big) = (2^*_{N/r} - 2)A,$$

$$A_2' := \theta_2\Big(\frac{Nr_B}{N-1+r_B}\Big) = \frac{2}{N-2}\Big(\frac{N-1+r_B}{r_B} - 1 - \frac{N-1}{r_B}\Big)A = 0,$$

and

$$\begin{aligned} A_3' := \theta_3\Big(\frac{Nr_B}{N-1+r_B}\Big) &= \Big(2^*\Big(1 - \frac{N-1+r_B}{Nr_B}\Big) - 2\Big)A \\ &= \Big(\frac{2(N-2)}{N-2}\Big(\frac{r_B-1}{r_B}\Big) - 2\Big)A \\ &= (2_{*,N/r_B} - 2)A. \end{aligned}$$

Hence, inequality (3.51) can be rewritten as

$$h_m(\|u\|_{L^\infty(\Omega)}) \le Ca_M^{A+\varepsilon}\Big(1 + \|u\|_{L^{2^*}(\Omega)}^{(2^*_{N/r}-2)A+\varepsilon} + \|u\|^{\varepsilon}_{L^{2_*}(\partial\Omega)}\|u\|_{L^{2^*}(\Omega)}^{(2_{*,N/r_B}-2)A+\varepsilon}\Big),$$

where A is defined in (2.13).

Case (II) $N/2 < r < N$ and $r^* \le \frac{Nr_B}{N-1}$. In that case, $q \in (\frac{N}{2}, r)$ (see (3.22)). The exponents are

$$A_1'' := \theta_1\,(r) = 2^*_{N/r} - 2,$$

$$A_2'' := \theta_2(r) = \frac{2}{N-2}\Big(\frac{N}{r} - 1 \mp N - \frac{N-1}{r_B}\Big) = 2_{*,N/r_B} - 2^*_{N/r},$$

$$A_3'' := \theta_3(r) = 2^*_{N/r} - 2.$$

Therefore, inequality (3.51) is rewritten as

$$h_m(\|u\|_{L^\infty(\Omega)}) \le C_\varepsilon a_M^{1+\varepsilon}\Big(1 + \|u\|_{L^{2^*}(\Omega)}^{2^*_{N/r}-2+\varepsilon} + \|u\|_{L^{2_*}(\partial\Omega)}^{2_{*,N/r_B}-2^*_{N/r}+\varepsilon}\|u\|_{L^{2^*}(\Omega)}^{2^*_{N/r}-2+\varepsilon}\Big). \qquad \square$$

The next corollary proves that any sequence $\{u_k\} \subset H^1(\Omega)$ of weak solution to (1.1), uniformly bounded in the $L^{2^*}(\Omega)$ norm and in the $L^{2_*}(\partial\Omega)$ norm, is also uniformly bounded in the $C(\overline{\Omega})$-norm.

Corollary 3.3. *Assume* (H1)–(H4) *hold and let* $\{u_k\} \subset H^1(\Omega)$ *be a sequence of weak solutions to* (1.1) *satisfying that, there exists* $C_0 > 0$*, such that*

$$\|u_k\|_{L^{2^*}(\Omega)} \le C_0 \quad \textit{and} \quad \|u_k\|_{L^{2_*}(\partial\Omega)} \le C_0.$$

Then, there exists $C > 0$ *such that,*

$$\|u_k\|_{C(\overline{\Omega})} \le C.$$

Proof. We proceed by contradiction, assuming that $\|u_k\|_{L^\infty(\Omega)} \to \infty$. By the Theorem (2.2) and the remark 3.1, we obtain

$$h_m(\|u_k\|_{C(\overline{\Omega})}) \le C, \quad \text{for } C > 0, \tag{3.52}$$

where, h_m is defined in (2.10). Using (2.9), we deduce that $h_m(\|u_k\|_{C(\overline{\Omega})}) \to \infty$ as $k \to \infty$, which contradicts (3.52). $\square$

Corollary 3.4. *Assume* (H1)–(H4) *hold and let* $\{u_k\} \subset H^1(\Omega)$ *be a sequence of weak solutions to* (1.1). *Then the following statements are equivalent:*

(i) $\|u_k\|_{L^{2^*}(\Omega)} \le C_1$ *and* $\|u_k\|_{L^{2_*}(\partial\Omega)} \le C_1$,
(ii) $\|u_k\|_{C(\overline{\Omega})} \le C_3$,
(iii) $\|u_k\|_{H^1(\Omega)} \le C_2$.

for some constants C_i *independent of* k, $i = 1, 2, 3$.

Proof. We prove that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

The proof of $(i) \Rightarrow (ii)$ follows directly from the Corollary 3.3. Now, using the elliptic regularity result, see the estimate (5.1) in the Theorem 5.1, and the Gagliardo-Nirenberg interpolation, the proof of $(ii) \Rightarrow (iii)$ is done. Finally, Sobolev's embedding and the continuity of the trace operator, proves that $(iii) \Rightarrow (i)$. □

4. Appendix: Regularity for the Neumann non homogeneous linear problem

In this appendix, we recall the regularity of weak solution to the linear problem with non homogeneous data both at the interior and on the boundary. Let us consider the linear nonhomogeneous Neumann problem

$$\begin{aligned} -\Delta u + u &= g(x), \quad x \in \Omega, \\ \frac{\partial u}{\partial \eta} &= g_B(x), \quad x \in \partial\Omega, \end{aligned} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^N$, $(N > 2)$, is an open, connected and bounded domain with C^2 boundary.

Theorem 4.1. *Let us consider the problem* (4.1)*, there exists a positive constant* $C > 0$ *independent of* u, h *and* g_B *such that the following holds:*

(i) If $\partial\Omega \in C^{0,1}$, $g \in L^q(\Omega)$ *and* $g_B \in L^{q_B}(\partial\Omega)$ *with* $q \ge 1$ *and* $q_B \ge 1$*, then there exists a unique* $u \in W^{1,m}(\Omega)$ *and*

$$\|u\|_{W^{1,m}(\Omega)} \le C \left(\|g\|_{L^q(\Omega)} + \|g_B\|_{L^{q_B}(\partial\Omega)} \right), \tag{4.2}$$

where $m = \min\{\frac{Nq}{N-q}, \frac{Nq_B}{N-1}\}$ *whenever* $1 \le q < N$*, or* $m = \min\{q, \frac{Nq_B}{N-1}\}$ *whenever* $q \ge N$*. Furthermore, if* $q > \frac{N}{2}$ *and* $q_B > N - 1$*, then*

$$\|u\|_{C^{\nu}(\overline{\Omega})} \le C \left(\|g\|_{L^q(\Omega)} + \|g_B\|_{L^{q_B}(\partial\Omega)} \right),$$

where $\nu = 1 - \frac{N}{m}$*,* $(m > N)$*.*

(ii) If $\partial\Omega \in C^{1,1}$, $g \in C^{\nu}(\Omega) \cap L^q(\Omega)$ *and* $g_B \in L^{q_B}(\partial\Omega)$ *with* $q > \frac{N}{2}$ *and* $q_B > N - 1$*, then there exists a unique* $u \in C^{\nu}(\overline{\Omega}) \cap C^{2,\nu}(\Omega)$.

(iii) If $\partial\Omega \in C^{2,\nu}$, $g \in C^{\nu}(\overline{\Omega})$ *and* $g_B \in C^{1,\nu}(\partial\Omega)$ *with* $\nu \in (0,1)$*, then there exists a unique* $u \in C^{2,\nu}(\overline{\Omega})$ *and*

$$\|u\|_{C^{2,\nu}(\overline{\Omega})} \le C \left(\|g\|_{C^{\nu}(\overline{\Omega})} + \|g_B\|_{C^{1,\nu}(\partial\Omega)} \right),$$

where C *is a positive constant independent of* u, g *and* g_B.

(iv) If $\partial\Omega \in C^2$, $g \in L^p(\Omega)$ *and* $g_B \in W^{1-\frac{1}{p},p}(\partial\Omega)$*, then* $u \in W^{2,p}(\Omega)$ *and*

$$\|u\|_{W^{2,p}(\Omega)} \le C \Big(\|g\|_{L^p(\Omega)} + \|g_B\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \Big),$$

where C *is a positive constant independent of* u, g *and* g_B.

(iv) If $\partial\Omega \in C^{1,\nu}$ *with* $\nu \in (0,1]$, $g \in C^{\nu}(\Omega)$ *and* $g_B \in C^{\nu}(\partial\Omega) \cap L^{\infty}(\partial\Omega)$ *then if* u *is a bounded weak solution to* (4.1)*, then* $u \in C^{1,\beta}(\overline{\Omega}) \cap C^{2,\beta}(\Omega)$*, where* β *depends on* ν *and* N*.*

Proof. (i) It follows from [7, Ch.3 Sec. 6] or [10, Lem. 2.2] that there exists a unique $u \in W^{1,p}(\Omega)$ solving (4.1). Now if $p > N$, using the Sobolev embedding theorem, one has $u \in C^{\alpha}(\overline{\Omega})$. Then by applying [5, Thm. 6.13] for the corresponding nonhomogeneous Dirichlet problem, we have that $u \in C^{1,\alpha}(\overline{\Omega})$, see also [10].

(ii) From part (i) we have that $u \in C^{\alpha}(\overline{\Omega})$. Since $\partial\Omega \in C^{1,1}$, Ω satisfies the exterior sphere condition at every point on the boundary and using the fact that $g \in C^{\alpha}(\overline{\Omega})$, reasoning as above it follows from [5, Thm. 6.13] that $u \in C^{\alpha}(\overline{\Omega}) \cap C^{2,\alpha}(\overline{\Omega})$.

(iii) See [1, Page 55] or [7, Chap.3 Sec. 3].

(iv) See [1, Page 55] or [7, Chap.3 Sec. 9].

(v) By [8, Thm. 2], one has $u \in C^{1,\beta}(\overline{\Omega})$. Then using the bootstrap for the differential equation in Ω, we obtain the desired regularity in Ω. $\square$

5. Appendix: Regularity of weak solutions

In this section, we establish auxiliary results on further regularity of weak solutions to (1.1), by assuming that conditions on the growth of the nonlinearities are subcritical or even critical. Using a Moser type procedure, it is known that $u \in L^q(\Omega) \cap L^q(\partial\Omega)$ for all $q < \infty$ (see [9, Theorem 3.1]). Moreover, using elliptic regularity theory, we state the following result that guarantees, in particular, Hölder regularity of any weak solution to (1.1).

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^N$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $f_B : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions, such that*

$$|f(x,s)| \le |a(x)|\big(1 + |s|^{2^*_{N/r}-1}\big),$$
$$|f_B(x,s)| \le |a_B(x)|\left(1 + |s|^{2_{*,N/r_B}-1}\right),$$

where

$$a(x) \in L^r(\Omega), \quad \text{with } \frac{N}{2} < r \le \infty,$$
$$a_B(x) \in L^{r_B}(\partial\Omega), \quad \text{with } N-1 < r_B \le \infty.$$

Let $u \in H^1(\Omega)$ be a weak solution to (1.1), then $u \in L^q(\Omega) \cap L^q(\partial\Omega)$ for all $1 \le q < \infty$. Moreover, $u \in W^{1,m}(\Omega) \cap C^\nu(\overline{\Omega})$, and the following estimates hold

$$\|u\|_{W^{1,m}(\Omega)} \le C\left(\|f(\cdot,u)\|_{L^r(\Omega)} + \|f_B(\cdot,u)\|_{L^{r_B}(\partial\Omega)}\right), \tag{5.1}$$
$$\|u\|_{C^\nu(\overline{\Omega})} \le C\left(\|f(\cdot,u)\|_{L^r(\Omega)} + \|f_B(\cdot,u)\|_{L^{r_B}(\partial\Omega)}\right), \tag{5.2}$$

where $m = \min\left\{r^*, \frac{Nr_B}{N-1}\right\}$, if $1 \le r < N$, or $m = \min\left\{r, \frac{Nr_B}{N-1}\right\}$, if $r \ge N$ and $\nu = 1 - \frac{N}{m}$. Also

$$\|u\|_{L^\infty(\partial\Omega)} \le \|u\|_{C(\overline{\Omega})} = \|u\|_{L^\infty(\Omega)}.$$

Proof. Let $u \in H^1(\Omega)$ be a weak solution to (1.1). Then $u \in L^q(\Omega) \cap L^q(\partial\Omega)$ for all $q < \infty$ (see [9, Theorem 3.1]).

Next, we use elliptic regularity theory. By Hölder's inequality, we have

$$f(\cdot,u) \in L^q(\Omega), \quad \text{for every } 1 < q < r,$$
$$f_B(x,u) \in L^{q_B}(\partial\Omega), \quad \text{for every } 1 < q_B < r_B,$$

By elliptic regularity (see Theorem (4.1)), $u \in W^{1,m}(\Omega)$ for $m = \min\{q^*, \frac{Nq_B}{N-1}\}$ whenever $1 \le q < N$, or $m = \min\{q, \frac{Nq_B}{N-1}\}$, whenever $q \ge N$.

Thanks to $r > N/2$ and $r_B > N-1$, we can always choose

$$q \in (N/2, r), \quad q_B \in (N-1, r_B).$$

Then $m > N$, so $u \in C^\nu(\overline{\Omega})$ for $\nu = 1 - \frac{N}{m}$.

Moreover, since $u \in C^\nu(\overline{\Omega})$, $a \in L^r(\Omega)$ and $\tilde{f} \in C(\overline{\Omega})$, using the Hölder inequality, then, the product $|a(\cdot)|\tilde{f}(|u(\cdot)|) \in L^r(\Omega)$. Hence, $f(\cdot,u(\cdot)) \in L^r(\Omega)$.

Similarly, if $u \in C^\nu(\overline{\Omega})$, $a_B \in L^{r_B}(\partial\Omega)$ and $\tilde{f_B} \in C(\partial\Omega)$, by Hölder inequality, then, the product $|a_B(\cdot)|\tilde{f_B}(|u(\cdot)|) \in L^{r_B}(\partial\Omega)$. Hence, we can conclude that $f_B(\cdot,u(\cdot)) \in L^{r_B}(\partial\Omega)$. Then (5.1) and (5.2) hold, completing the proof. $\square$

Acknowledgements. Rosa Pardo was supported by the MICINN, Spain, grant PID2022-137074NB-I00, and by the UCM, Spain, Grupo 920894. Edgar Antonio received support from a CONAHCYT scholarship, and from the Consortium of Mexican Universities-Ibero-American University Post-graduate Association (CUMex-AUIP) for the flight ticket and living expenses at the Universidad Complutense de Madrid from January 7 to August 2, 2024.

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