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LINEAR STABILITY OF THE COUETTE FLOW FOR NON-ISENTROPIC COMPRESSIBLE FLUIDS

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Abstract. In this article, we study the linear stability of a two-dimensional non-isentropic compressible fluid with vanishing shear viscosity in the context of Couette flow on an infinitely long flat torus $\mathbb{T}\times\mathbb{R}$. By employing explicit weighted energy estimates and the Fourier multipliers method, we first establish the inviscid damping of the incompressible component of the velocity. Subsequently, we derive an upper bound which is superlinear in time for the compressible part of the fluid. Furthermore, we demonstrate an enhanced dissipation phenomenon for the velocity field under certain quality conditions pertaining to the initial density, initial temperature, and incompressible component of the initial velocity field.

1. Introduction and main result

In this article, we study the long-time asymptotic behavior of the linearized two dimensional non-isentropic compressible Navier-Stokes equations in a domain $\mathbb{T}\times\mathbb{R}$. The governing equations (in non-dimensional variables) are

$$\begin{aligned} \varrho_t + \mathbf{u}\cdot\nabla\varrho + \varrho\operatorname{div}\mathbf{u} &= 0,\\ \varrho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) + \frac{1}{\gamma M^2}\nabla P &= \frac{1}{\mathrm{Re}}\left(\mu\Delta\mathbf{u} + (\nu+\mu)\nabla\operatorname{div}\mathbf{u}\right),\\ \varrho(\vartheta_t + \mathbf{u}\cdot\nabla\vartheta) + (\gamma-1)P\operatorname{div}\mathbf{u} &= \frac{\gamma\mu}{\sigma\mathrm{Re}}\Delta\vartheta + \frac{\gamma(\gamma-1)M^2}{\mathrm{Re}}\left(\frac{\mu}{2}|\nabla\mathbf{u}+\nabla\mathbf{u}^\top|^2 + \nu|\operatorname{div}\mathbf{u}|^2\right). \end{aligned} \tag{1.1}$$

Here $t\geq 0$ is time, $(x,y)\in\mathbb{T}\times\mathbb{R}$ is the spatial coordinate and $\mathbb{T}=\mathbb{R}/\mathbb{Z}$. The unknown $\mathbf{u}$ is the velocity vector, ϱ is the density, ϑ is the temperature, $P=\varrho\vartheta$ is the pressure. $\gamma>1$ is the ratio of specific heats, $M>0$ is the Mach number of the reference state, $\mathrm{Re}>0$ is the Reynolds number, and $\sigma>0$ is the Prandtl number. The two constant viscosity coefficients μ and ν are the shear viscosity and the volume viscosity respectively. The equations (1.1) then express respectively the conservation of mass, the balance of momentum, and the balance of energy under internal pressure, viscosity forces, and the conduction of thermal energy.

A comprehensive understanding of the stability of compressible or incompressible shear flows is a fundamental problem in fluid mechanics and has been the subject of both theoretical and practical interest in astrophysics and engineering, see [1, 2, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 24, 25, 26, 29, 30] for the compressible fluid and [3, 4, 5, 10, 11, 21, 23, 27, 28] for incompressible fluid. The aim of the present paper is to study the long-time asymptotic behavior of the linearized non-isentropic compressible Navier-Stokes equations around the Couette flow. That is we seek a stationary solution of (1.1) with a constant mean pressure which have the form

$$\varrho_{sh}=\varrho_{sh}(y),\quad \mathbf{u}_{sh}=\begin{pmatrix} y\\ 0\end{pmatrix},\quad \vartheta_{sh}=\vartheta_{sh}(y),\quad\text{with}\quad \varrho_{sh}(y)\vartheta_{sh}(y)=1. \tag{1.2}$$

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Obviously, when $\mu \neq 0$, because of the strong nonlinear term $|\nabla \mathbf{u} + \nabla \mathbf{u}^\top|^2$ appeared in the third equation of (1.1), it is straightforward to verify that $\vartheta_{sh}(y)$ must satisfy the restricted relation

$$\frac{\gamma\mu}{\sigma \mathrm{Re}} \partial_{yy}\vartheta_{sh}(y) = -\frac{\gamma\mu(\gamma-1)M^2}{\mathrm{Re}}. \tag{1.3}$$

To solve (1.3), we can choose $\vartheta_{sh}(y)$ as

$$\vartheta_{sh}(y) = \vartheta_r[r + (1-r)y - (1-\frac{1}{\vartheta_r})y^2] \tag{1.4}$$

where $r > 0$ is the temperature ratio and ϑ_r is the recovery temperature defined as

$$\vartheta_r := 1 + \frac{(\gamma-1)\sigma M^2}{2}.$$

Because of the complicated form of $\vartheta_{sh}(y)$, to study the long-time asymptotic behaviour of (1.1) around the stationary solution defined in (1.2) and (1.4) is a very difficult problem. To the best of our knowledge, there are only a few results in this direction, see [1, 2, 6, 7, 8, 13, 14, 15, 16, 17, 18, 19, 20, 25, 26].

Because of the mathematical challenges, to approach the problem, here, we only consider a simple case of (1.1) with the shear viscosity coefficient $\mu = 0$ and the volume viscosity $\nu \neq 0$. In this case, system (1.1) can be rewritten as

$$\begin{aligned} &\varrho_t + \mathbf{u}\cdot\nabla\varrho + \varrho \operatorname{div}\mathbf{u} = 0, \\ &\varrho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) + \frac{1}{\gamma M^2}\nabla(\varrho\vartheta) = \frac{\nu}{\mathrm{Re}}\nabla \operatorname{div}\mathbf{u}, \\ &\varrho(\vartheta_t + \mathbf{u}\cdot\nabla\vartheta) + (\gamma-1)\varrho\vartheta \operatorname{div}\mathbf{u} = \frac{\nu\gamma(\gamma-1)M^2}{\mathrm{Re}}|\operatorname{div}\mathbf{u}|^2. \end{aligned} \tag{1.5}$$

It is straightforward to verify that the Couette flow,

$$\varrho_{sh} = 1, \quad \mathbf{u}_{sh} = \begin{pmatrix} y \\ 0 \end{pmatrix}, \quad \vartheta_{sh} = 1, \tag{1.6}$$

is a stationary solution of (1.5). Our goal is to understand the stability and large-time behavior of perturbations near this Couette flow.

Before presenting our main result, let us give a short review of the extensive mathematical results about stability analysis on the compressible Navier-Stokes equations. Glatzel [14, 15] studied the linear inviscid and viscous stability properties of the compressible Couette flow via a normal mode analysis in simplified flow model with constant viscous coefficients and a constant density profile. Duck et al. [13] proved the linear stability of the plane Couette flow for the non-isentropic compressible Navier-Stokes equations. Chagelishvili et al. [8] considered the inviscid stability of the 2D Couette flow. By means of some formal approximation, they showed that the energy of acoustic perturbations grows linear in time due to the transfer of energy from the mean flow to perturbations. Taking advantage of a fourth-order finite-difference method and a spectral collocation method, Hu et al. [16] studied the viscous linear stability of supersonic Couette flow for a perfect gas governed by Sutherland viscosity law. Kagei [17] proved that the plane Couette flow in an infinite layer is asymptotically stable if the Reynolds and Mach numbers are sufficiently small. Li et al. [20] investigated the stability analysis of the plane Couette flow for the 3D compressible Navier-Stokes equations with Navier-slip boundary condition at the bottom boundary. They shown that the plane Couette flow is asymptotically stable for small perturbation provided that the slip length, Reynolds and Mach numbers satisfy some restricted relation. Recently, Antonelli et al. [1] studied the linear stability properties of the 2D isentropic compressible Euler equations linearized around a shear flow given by a monotone profile, close to the Couette flow, with constant density, in the domain $\mathbb{T}\times\mathbb{R}$. Later then, they in [2] also studied the linear stability properties of perturbations around the homogeneous Couette flow for a 2D isentropic inviscid or viscous compressible fluid. Moreover, in the inviscid case, they proved the inviscid damping for the solenoidal component of the velocity field and Lyapunov type instability for the density and the irrotational component of the velocity field. In the viscous case, they obtained the enhanced dissipation phenomenon. Zeng et al. [29] considered the linear stability of the three dimensional isentropic compressible

Navier-Stokes equations on $\mathbb{T} \times \mathbb{R} \times \mathbb{T}$. They proved the enhanced dissipation phenomenon and the lift-up phenomenon around the Couette flow $(y, 0, 0)^\top$. The motivation of the present paper is to generalize the results obtained by Antonelli et al. [1, 2] to the non-isentropic compressible Navier-Stokes equations with vanished shear viscosity.

We denote

$$\rho = \varrho - \varrho_{sh}, \quad \mathbf{v} = \mathbf{u} - \mathbf{u}_{sh}, \quad \theta = \vartheta - \vartheta_{sh}.$$

The linearized system of (1.5) around the Couette flow (1.6) reads as follows

$$\begin{gathered} \partial_t \rho + y\partial_x \rho + \operatorname{div} \mathbf{v} = 0, \\ \partial_t \mathbf{v} + y\partial_x \mathbf{v} + \begin{pmatrix} v^y \\ 0 \end{pmatrix} + \frac{1}{\gamma M^2}(\nabla\rho + \nabla\theta) = \nu \nabla \operatorname{div} \mathbf{v}, \\ \partial_t \theta + y\partial_x \theta + (\gamma - 1) \operatorname{div} \mathbf{v} = 0. \end{gathered} \tag{1.7}$$

The above linear system is very different from the isentropic compressible Euler equation in [1, 30] and the isentropic compressible Navier-Stokes equation in [2]. On the one hand, as we only have dissipation on the compressible part of $\mathbf{v}$, the behavior of $(\rho, \nabla \operatorname{div} \mathbf{v}, \theta)$ is similar to the isentropic compressible Navier-Stokes equation discussed in [2], i.e., the density, the compressible part of the velocity field, and the temperature experience a Lyapunov type instability. In contrast, because of the lack of dissipation on the incompressible part of $\mathbf{v}$, the incompressible part of the velocity experiences an inviscid damping just like the classical incompressible Euler equation. On the other hand, we define

$$\omega = \nabla^\perp \cdot \mathbf{v}, \quad \text{with} \quad \nabla^\perp = (-\partial_y, \partial_x)^\top,$$

we can find that the equation of the incompressible part of the velocity connects with the compressible part of the velocity through the equation

$$\partial_t \omega + y\partial_x \omega - \operatorname{div} \mathbf{v} = 0. \tag{1.8}$$

Hence, assuming the initial data satisfies some quality relation (see (1.17) in the following) and exploiting the special linear structure of (1.7), we can transfer the dissipation of $\nabla \operatorname{div} \mathbf{v}$ to the incompressible part of the velocity field ω. As a result, we can prove the enhanced dissipation phenomenon of the the velocity field. The phenomenon of enhanced dissipation has been widely studied in the physics literature [2, 3, 4, 11, 23], and has recently received a lot of attention from the mathematical community. Here, we give more explanation about the enhanced dissipation phenomenon. For example, considering the equation

$$\partial_t F + y\partial_x F = \nu(\partial_{xx} + \partial_{yy})F, \quad F(x, y, 0) = F_0(x, y). \tag{1.9}$$

Applying the Fourier transform to the governing equation and using the coordinate transformation, we obtain the evolution equation in the frequency domain,

$$\partial_t \widehat{F} - k\partial_\eta \widehat{F} = -\nu(k^2 + \eta^2)\widehat{F}, \quad \widehat{F}(k, \eta, 0) = \widehat{F}_0(k, \eta), \tag{1.10}$$

where (k, η) denotes the dual variables in Fourier space corresponding to the spatial coordinates. The Fourier transform convention used in this derivation, along with its fundamental properties, will be specified in the subsequent section.

Making the natural change of variables

$$\xi := \eta + kt, \quad H(k, \xi, t) := \widehat{F}(k, \eta, t),$$

we find that

$$\partial_t H(k, \xi, t) = -\nu(k^2 + (\xi - kt)^2)\, H(k, \xi, t), \quad H(k, \xi, 0) = \widehat{F}_0(k, \xi).$$

Integrating in time yields

$$H(k, \xi, t) = \widehat{F}_0(k, \xi) e^{-\nu \int_0^t (k^2 + (\xi - k\tau)^2)\, d\tau}.$$

Therefore,

$$\begin{aligned} \widehat{F}(k, \eta, t) = H(k, \xi, t) &= \widehat{F}_0(k, \eta + kt) e^{-\nu \int_0^t k^2 + (\eta + k(t-\tau))^2\, d\tau} \\ &= \widehat{F}_0(k, \eta + kt) e^{-\nu(k^2+\eta^2)t} e^{-\frac{1}{3}\nu k^2 t^3 - \nu k\eta t^2}. \end{aligned} \tag{1.11}$$

This explicit representation reflects the enhanced dissipation. The dissipation time scale is $O(\nu^{-\frac{1}{3}})$, which is much faster than the standard dissipation time scale $O(\nu^{-1})$. Clearly the dissipation rate is inhomogeneous and depends on the frequencies k.

Before going into details of our theorem, we introduce some notation. We define

$$\alpha = \operatorname{div} \mathbf{v}, \quad \omega = \nabla^{\perp} \cdot \mathbf{v}, \quad \text{with} \quad \nabla^{\perp} = (-\partial_y, \partial_x)^{\top},$$

according to the Helmholtz projection operators, we have

$$\mathbf{v} = (v^x, v^y)^{\top} := \mathbb{P}[\mathbf{v}] + \mathbb{Q}[\mathbf{v}] \tag{1.12}$$

with

$$\mathbb{P}[\mathbf{v}] := \nabla^{\perp}\Delta^{-1}\omega, \quad \mathbb{Q}[\mathbf{v}] := \nabla\Delta^{-1}\alpha. \tag{1.13}$$

From the above definition, one infers that

$$v^y = \partial_y(\Delta^{-1})\alpha + \partial_x(\Delta^{-1})\omega, \tag{1.14}$$

from which, we can rewrite (1.7) in terms of $(\rho, \alpha, \omega, \theta)$ that

$$\begin{gathered}
\partial_t \rho + y\partial_x \rho + \alpha = 0,\\
\partial_t \alpha + y\partial_x \alpha + 2\partial_x(\partial_y(\Delta^{-1})\alpha + \partial_x(\Delta^{-1})\omega) + \frac{1}{\gamma M^2}(\Delta\rho + \Delta\theta) = \nu\Delta\alpha,\\
\partial_t \omega + y\partial_x \omega - \alpha = 0,\\
\partial_t \theta + y\partial_x \theta + (\gamma - 1)\alpha = 0.
\end{gathered} \tag{1.15}$$

Obviously, the above system is a closed system regarding of $(\rho, \alpha, \omega, \theta)$. Let

$$\begin{aligned}
\widehat{f}(k,\eta) &= \frac{1}{2\pi} \iint_{\mathbb{T}\times\mathbb{R}} e^{-i(kx+\eta y)} f(x,y)\, dx\, dy,\\
f(x,y) &= \frac{1}{2\pi} \sum_k \int_{\mathbb{R}} e^{i(kx+\eta y)} \widehat{f}(k,\eta)\, d\eta,
\end{aligned}$$

then we define $f \in H^s(\mathbb{T}\times\mathbb{R})$ if

$$\|f\|_{H^s}^2 = \sum_k \int \langle k,\eta\rangle^{2s} |\hat{f}|^2(k,\eta)\, d\eta < +\infty, \quad \text{with} \quad \langle k,\eta\rangle = \sqrt{1+k^2+\eta^2}.$$

Now, we can state the main result of the present paper.

Theorem 1.1. *Let $\gamma > 1$, $0 < \nu < 1$ and $0 < M \le \nu^{-1}$. Assume that $(\rho^{in}, \alpha^{in}, \omega^{in}, \theta^{in}) \in H^{\frac{3}{2}}(\mathbb{T}\times\mathbb{R})$ is the initial data of (1.15) with*

$$\int_{\mathbb{T}} \rho^{in}\, dx = \int_{\mathbb{T}} \alpha^{in}\, dx = \int_{\mathbb{T}} \omega^{in}\, dx = \int_{\mathbb{T}} \theta^{in}\, dx = 0. \tag{1.16}$$

Then, there exists a positive constant C independent of γ, ν, M such that

$$\begin{aligned}
&\big\|\mathbb{P}[\mathbf{v}]^x(t)\big\|_{L^2}\\
&\le C\langle t\rangle^{-1/2}\gamma^{-1}\exp(CM(M+1))\Big(\frac{1}{M}\big(\|\rho^{in}\|_{H^{3/2}} + \|\theta^{in}\|_{H^{3/2}}\big) + \|\alpha^{in}\|_{H^{3/2}} + \gamma\|\omega^{in}\|_{H^{3/2}}\Big),\\
&\big\|\mathbb{P}[\mathbf{v}]^y(t)\big\|_{L^2}\\
&\le C\langle t\rangle^{-3/2}\gamma^{-1}\exp(CM(M+1))\Big(\frac{1}{M}\big(\|\rho^{in}\|_{H^{3/2}} + \|\theta^{in}\|_{H^{3/2}}\big) + \|\alpha^{in}\|_{H^{3/2}} + \gamma\|\omega^{in}\|_{H^{3/2}}\Big),
\end{aligned}$$

$$\begin{aligned}
&\big\|\mathbb{Q}[\mathbf{v}](t)\big\|L^2 + \frac{\gamma}{M}\|\rho(t)\| - L^2 + \frac{\gamma}{M}\|\theta(t)\|_{L^2}\\
&\le C\langle t\rangle^{1/2}\Big\{\Big\|\frac{(\gamma-1)\rho^{in} - \theta^{in}}{M}\Big\|_{L^2} + (\gamma+1)\exp(CM(M+1))\\
&\quad \times \Big(\frac{1}{M}\|\rho^{in}\|_{H^1} + \frac{1}{M}\|\theta^{in}\|_{H^1} + \|\alpha^{in}\|_{H^1} + \gamma\|\omega^{in}\|_{H^1}\Big)\Big\}.
\end{aligned}$$

Moreover, if ρ^{in}, θ^{in}, and ω^{in} additionally satisfy the relation

$$\rho^{in} + \gamma\omega^{in} + \theta^{in} = 0, \tag{1.17}$$

we can obtain the enhanced dissipation of the velocity field,

$$\begin{aligned}
&\big\|\mathbb{P}[\mathbf{v}]^x(t)\big\|_{L^2} \le C\langle t\rangle^{-1/2} e^{-\frac{1}{16}\nu^{1/3}t} \exp(CM(M+1))\Big(\|\alpha^{in}\|_{H^{3/2}} + \frac{1}{M}\|\omega^{in}\|_{H^{3/2}}\Big),\\
&\big\|\mathbb{P}[\mathbf{v}]^y(t)\big\|_{L^2} \le C\langle t\rangle^{-3/2} e^{-\frac{1}{16}\nu^{1/3}t} \exp(CM(M+1))\Big(\|\alpha^{in}\|_{H^{3/2}} + \frac{1}{M}\|\omega^{in}\|_{H^{3/2}}\Big),\\
&\quad\big\|\mathbb{Q}[\mathbf{v}](t)\big\|_{L^2} + \frac{1}{M}\|\rho(t)+\theta(t)\|_{L^2}\\
&\quad\le C\langle t\rangle^{1/2} e^{-\frac{1}{32}\nu^{1/3}t}(1+\gamma)\exp(CM(M+1))\Big(\|\alpha^{in}\|_{H^1} + \frac{1}{M}\|\omega^{in}\|_{H^1}\Big).
\end{aligned}$$

At first glance, the enhanced dissipation phenomenon of the velocity field is some surprising because of there is only dissipation for the compressible part of the velocity. This mainly benefits from the relation (1.17) which gives rise to $\omega = -\frac{1}{\gamma}(\rho+\theta)$. The special relation connects compressible and incompressible phenomena. Namely, an increase of the vorticity need to be compensated by a decrease for the density and the temperature.

In [1], Antonelli et al. studied the linear stability properties of the 2D isentropic compressible Euler equations linearized around a shear flow given by a monotone profile, close to the Couette flow, with constant density, in the domain $\mathbb{T}\times\mathbb{R}$. For the non-isentropic compressible fluid, how to obtain a similar result is an interesting problem. This is left in the future work.

Remark 1.2. For $0<\nu<1$, Theorem 1.1 holds for the whole subsonic regime. Formally let $M\to 0$, the behavior of fluid subsonic regime may be very similar to the incompressible case. Physically, when $M\to 0^+$, our system indeed formally converges to an incompressible regime, as the dominant pressure term $\frac{1}{\gamma M^2}\nabla P$ enforces $\operatorname{div}\mathbf{v}\approx 0$. However, the convergence is subtle because: (i) The temperature equation remains active in our non-isentropic model. (ii) The volume viscosity ν maintains dissipation even as $M\to 0$. We recognize that this transition between compressible and incompressible regimes presents important theoretical questions. A rigorous examination of this asymptotic limit will be the focus of our ongoing research efforts.

2. Proof of the main theorem

2.1. **Preliminary and a priori estimates.** First of all, we are concerned with the dynamics of the x-averages of the perturbations. To reveal the distinction between the zero mode case $k=0$ and the nonzero modes $k\neq 0$. We define

$$f_0(y) := \frac{1}{2\pi}\int_{\mathbb{T}} f(x,y)\,dx, \quad f_{\neq}(x,y) := f(x,y) - f_0(y),$$

which represents the projection onto 0 frequency and the projection onto non-zero frequencies.

Because the structure of the Couetee flow and that the equations are linear, it is clear that the zero mode in x has an independent dynamics with respect to other modes. Consequently, in our analysis we can decouple the evolution of the $k=0$ mode from the rest of the perturbation. Integration in x equations in (1.15), one infer that

$$\begin{gathered}
\partial_t\rho_0 = -\alpha_0,\\
\partial_t\alpha_0 = -\frac{1}{\gamma M^2}\partial_{yy}\rho_0 - \frac{1}{\gamma M^2}\partial_{yy}\theta_0 + \nu\partial_{yy}\alpha_0,\\
\partial_t\omega_0 = \alpha_0,\\
\partial_t\theta_0 = -(\gamma-1)\alpha_0.
\end{gathered} \tag{2.1}$$

From (2.1), we can further get α_0 satisfies the damped wave equations

$$\partial_{tt}\alpha_0 - \nu\partial_t\partial_{yy}\alpha_0 - \frac{1}{M^2}\partial_{yy}\alpha_0 = 0, \quad \text{in } \mathbb{R}, \tag{2.2}$$

and $\rho_0+\theta_0$ satisfies the wave equation

$$\partial_{tt}(\rho_0+\theta_0) - \frac{1}{M^2}\partial_{yy}(\rho_0+\theta_0) = 0, \quad \text{in } \mathbb{R}. \tag{2.3}$$

Hence, given $\rho_0^{in} = \alpha_0^{in} = \theta_0^{in} = \omega_0^{in} = 0$, we can get that for all $t \geq 0$,

$$\rho_0(t) = \alpha_0(t) = \theta_0(t) = \omega_0(t) = 0.$$

Consequently, in our analysis we can decouple the evolution of the $k = 0$ mode from the rest of the perturbation. Let us consider the coordinate transform

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} x - yt \\ y \end{pmatrix}.$$

Under the new coordinate transform, the differential operators change as follows

$$\partial_x = \partial_X, \quad \partial_y = \partial_Y - t\partial_X, \quad \Delta = \Delta_L := \partial_{XX} + (\partial_Y - t\partial_X)^2.$$

We define

$$\begin{gathered} R(t, X, Y) = \rho(t, X + tY, Y), \quad A(t, X, Y) = \alpha(t, X + tY, Y), \\ \Omega(t, X, Y) = \omega(t, X + tY, Y), \quad \Theta(t, X, Y) = \theta(t, X + tY, Y). \end{gathered}$$

Then, the linear system (1.15) reduces to the following system in the new coordinates,

$$\begin{gathered} \partial_t R = -A, \\ \partial_t A = \nu\Delta_L A - 2\partial_X(\partial_Y - t\partial_X)(\Delta_L^{-1})A - 2\partial_{XX}(\Delta_L^{-1})\Omega - \frac{1}{\gamma M^2}(\Delta_L R + \Delta_L \Theta), \\ \partial_t \Omega = A, \\ \partial_t \Theta = -(\gamma - 1)A. \end{gathered} \tag{2.4}$$

We want to analyze the system (2.4) on the frequency space, in analogy with respect to the incompressible Couette flow. So we define the symbol associated with $-\Delta_L$ as

$$p(t, k, \eta) = k^2 + (\eta - kt)^2,$$

and denote the symbol associated to the operator $2\partial_X(\partial_Y - t\partial_X)$ as

$$(\partial_t p)(t, k, \eta) = -2k(\eta - kt).$$

In the moving frame, for the Laplacian operator, there hold the following inequalities.

Lemma 2.1. *Let $p = -\widehat{\Delta}_L = k^2 + (\eta - kt)^2$, then for any function $f \in H^{s+2\beta}(\mathbb{T} \times \mathbb{R})$, it holds that*

$$\|p^{-\beta} f\|_{H^s} \leq C\frac{1}{\langle t\rangle^{2\beta}}\|f\| - H^{s+2\beta}, \quad \|p^{\beta} f\|_{H^s} \leq C\langle t\rangle^{2\beta}\|f\|_{H^{s+2\beta}}, \tag{2.5}$$

for any $\beta > 0$.

Proof. The bound (2.5) follows just by Plancherel Theorem and the basic inequalities for japanese brackets $\langle k, \eta\rangle \leq C\langle \eta - \xi\rangle\langle k, \xi\rangle$. □

In proving Theorem 1.1, we use some main ideas from [1] and [2] but we are faced with a number of technical difficulties because of a more complicated system. We first get by taking the Fourier transform of (2.4) that

$$\begin{gathered} \partial_t \widehat{R} = -\widehat{A}, \\ \partial_t \widehat{A} = -\nu p\widehat{A} + \frac{\partial_t p}{p}\widehat{A} - \frac{2k^2}{p}\widehat{\Omega} + \frac{p}{\gamma M^2}(\widehat{R} + \widehat{\Theta}), \\ \partial_t \widehat{\Omega} = \widehat{A}, \\ \partial_t \widehat{\Theta} = -(\gamma - 1)\widehat{A}. \end{gathered} \tag{2.6}$$

To exploit the special structure of the system (2.6), we introduce an unknown good function Φ as

$$\Phi = \frac{R + \Theta}{\gamma} \tag{2.7}$$

from which we can rewrite (2.6) as

$$\begin{aligned}
&\partial_t\widehat{\Phi} = -\widehat{A},\\
&\partial_t\widehat{A} = -\nu p\widehat{A} + \frac{\partial_t p}{p}\widehat{A} - \frac{2k^2}{p}\widehat{\Omega} + \frac{p}{M^2}\widehat{\Phi}.
\end{aligned} \tag{2.8}$$

To break through the barrier involved in the term Ω in (2.8), we deduce from

$$\partial_t(R+\gamma\Omega+\Theta)=0$$

that it holds

$$R+\gamma\Omega+\Theta = R^{in}+\gamma\Omega^{in}+\Theta^{in}$$

which combining with (2.7) leads to

$$\Omega = \Phi^{in}+\Omega^{in}-\Phi. \tag{2.9}$$

Hence, substituting (2.9) into (2.8), we obtain a closed system only involved in $\widehat{\Phi}$, and $\widehat{A}$ other than the initial data

$$\begin{aligned}
&\partial_t\widehat{\Phi} = -\widehat{A},\\
&\partial_t\widehat{A} = -\nu p\widehat{A} + \frac{\partial_t p}{p}\widehat{A} + (\frac{p}{M^2}+\frac{2k^2}{p})\widehat{\Phi} - \frac{2k^2}{p}(\widehat{\Phi}^{in}+\widehat{\Omega}^{in}).
\end{aligned} \tag{2.10}$$

In the following, to obtain the enhanced dissipation, we introduce the "ghost multiplier" which has been used in [3, 4].

Let multiplier m solve the linear ODE for $k\neq 0$,

$$\begin{aligned}
&\frac{\partial_t m}{m} = -\frac{\nu^{1/3}}{[\nu^{1/3}|t-\frac{\eta}{k}|]^2+1},\\
&m(0,k,\eta)=1.
\end{aligned}$$

Notice that there is a constant c (independent of k, η, t, and ν) such that $c < m(t,k,\eta)\le 1$. In particular, its presence does not change a norm

$$\|m(t,\nabla)\langle\nabla\rangle^\sigma f\|_{L^2}\approx\|\langle\nabla\rangle^\sigma f\|_{L^2}, \quad \text{with } \widehat{\langle\nabla\rangle^\sigma} f(k,\eta) := \big(1+(k^2+\eta^2)\big)^{\sigma/2}\widehat{f}(k,\eta).$$

The crucial property that m satisfies is

$$1\lesssim \nu^{-1/6}\Big(\sqrt{-\frac{\partial_t m}{m}(t,k,\eta)}+\nu^{1/2}|k,\eta-kt|\Big) \quad \text{for } k\neq 0, \tag{2.11}$$

which implies that

$$\|f_{\neq}\|_{L^2}^2 \lesssim \nu^{-1/3}\Big(\Big\|\sqrt{-\frac{\partial_t m}{m}}f_{\neq}\Big\|_{L^2}^2 + \nu\|\nabla_L f_{\neq}\|_{L^2}^2\Big). \tag{2.12}$$

The following lemma plays a crucial role in our subsequent analysis.

Lemma 2.2. *For any $(\rho^{in},\alpha^{in},\omega^{in},\theta^{in})\in H^s(\mathbb{T}\times\mathbb{R})$ with $s\ge 0$. Assume that $\gamma>1$, $0<\nu<1$, and $0<M\le\nu^{-1}$. Then there exists a positive constant C independent of γ,ν,M such that*

$$\begin{aligned}
&\frac{1}{M}\|(p^{-1/4}\widehat{\Phi})(t)\|_{H^s} + \|(p^{-3/4}\widehat{A})(t)\|_{H^s}\\
&\le C\exp(CM(M+1))\Big(\frac{1}{M}\|\widehat{\Phi}^{in}\|_{H^s} + \|\widehat{A}^{in}\|_{H^s} + \|\widehat{\Phi}^{in}+\widehat{\Omega}^{in}\|_{H^s}\Big).
\end{aligned} \tag{2.13}$$

Proof. For any $s\ge 0$, we define two weighted functions involved in $\widehat{\Phi}$ and $\widehat{A}$ as

$$Z_1(t) := \frac{1}{M}\langle k,\eta\rangle^s(m^{-1}p^{-1/4}\widehat{\Phi})(t), \tag{2.14}$$

$$Z_2(t) := \langle k,\eta\rangle^s(m^{-1}p^{-3/4}\widehat{A})(t). \tag{2.15}$$

From the equations in (2.10) and definitions of Z_1 and Z_2, a simple computations gives

$$\begin{gathered}\partial_t Z_1 = -\frac{\partial_t m}{m} Z_1 - \frac{1}{4}\frac{\partial_t p}{p} Z_1 - \frac{1}{M} p^{1/2} Z_2, \\ \partial_t Z_2 = -\Big(\frac{\partial_t m}{m} + \nu p\Big) Z_2 + \frac{1}{4}\frac{\partial_t p}{p} Z_2 \Big(\frac{1}{M} p^{1/2} + 2M \frac{k^2}{p^{3/2}}\Big) Z_1 - \langle k, \eta\rangle^s \frac{2m^{-1}k^2}{p^{\frac{7}{4}}} (\widehat{\Phi}^{in} + \widehat{\Omega}^{in}).\end{gathered} \tag{2.16}$$

Now, by multiplying the first equation by $\bar{Z}_1$ and the second equation by $\bar{Z}_2$ in (2.16) respectively, we obtain

$$\frac{1}{2}\frac{d}{dt}|Z_1|^2 = -\frac{\partial_t m}{m}|Z_1|^2 - \frac{1}{4}\frac{\partial_t p}{p}|Z_1|^2 - \frac{1}{M} p^{1/2}\mathrm{Re}(\bar{Z}_1 Z_2), \tag{2.17}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|Z_2|^2 &= -\Big(\frac{\partial_t m}{m} + \nu p\Big)|Z_2|^2 + \frac{1}{4}\frac{\partial_t p}{p}|Z_2|^2 + \frac{1}{M} p^{1/2}\mathrm{Re}(Z_1 \bar{Z}_2) \\ &\quad + 2M\frac{k^2}{p^{3/2}}\mathrm{Re}(Z_1\bar{Z}_2) - \langle k, \eta\rangle^s \frac{2m^{-1}k^2}{p^{\frac{7}{4}}}\mathrm{Re}((\widehat{\Phi}^{in} + \widehat{\Omega}^{in})\bar{Z}_2).\end{aligned} \tag{2.18}$$

From $p(t,k,\eta) = k^2 + (\eta - kt)^2 > 0$, one has for any $t > \eta/k$ it holds $\partial_t p/p > 0$, the third term on the right-hand side of the first equation in (2.16) acts as a damping term for Z_1. Instead, $\partial_t p/p < 0$ for $t < \eta/k$, hence it induces a growth on Z_1. However, the situation is opposite for the second equation involved in Z_2. That is to say, for $t > \eta/k$, the term $(\partial_t p/p)Z_2$ induces a growth, for $t < \eta/k$, the term $(\partial_t p/p)Z_2$ acts as a damping term. Thus, there is a competition between Z_1 and Z_2. To balance this relation, we have to consider the time derivative of the mixed terms involved in Z_1, Z_2:

$$\frac{d}{dt}\Big(\frac{\partial_t p}{p^{3/2}} Z_1\Big) = -\frac{\partial_t m}{m}\frac{\partial_t p}{p^{3/2}} Z_1 + \Big(\frac{2k^2}{p^{3/2}} - \frac{7}{4}\frac{(\partial_t p)^2}{p^{\frac{5}{2}}}\Big) Z_1 - \frac{1}{M}\frac{\partial_t p}{p} Z_2 \tag{2.19}$$

from this and the second equation in (2.16), we obtain

$$\begin{aligned}\frac{M}{4}\frac{d}{dt}\Big(\frac{\partial_t p}{p^{3/2}}\mathrm{Re}(\bar{Z}_1 Z_2)\Big) &= -\frac{1}{4}\frac{\partial_t p}{p}(|Z_2|^2 - |Z_1|^2) + \frac{M}{4}\Big(\frac{2k^2}{p^{\frac{3}{2}}} - \frac{3}{2}\frac{(\partial_t p)^2}{p^{5/2}}\Big)\mathrm{Re}(\bar{Z}_1 Z_2) \\ &\quad - \frac{M}{2}\frac{\partial_t m}{m}\frac{\partial_t p}{p^{3/2}}\mathrm{Re}(\bar{Z}_1 Z_2) - \nu\frac{M}{4}\frac{\partial_t p}{p^{1/2}}\mathrm{Re}(\bar{Z}_1 Z_2) + M^2\frac{k^2\partial_t p}{2p^3}|Z_1|^2 \\ &\quad - \langle k, \eta\rangle^s m^{-1}\frac{Mk^2\partial_t p}{2p^{\frac{13}{4}}}\mathrm{Re}\left((\widehat{\Phi}^{in} + \widehat{\Omega}^{in})\bar{Z}_2\right).\end{aligned} \tag{2.20}$$

It is obvious that the first term on the right=hand side of (2.20) could cancel two bad terms $-\frac{1}{4}\frac{\partial_t p}{p}|Z_1|^2$ appeared in (2.17) and $\frac{1}{4}\frac{\partial_t p}{p}|Z_1|^2$ appeared in (2.18).

Because of the lack of a diffusive term in the equation of $\widehat{\Phi}$, we have to exploit the special structural characteristics (wave structure) of (2.16) to find hidden dissipation for Z_1. So, we also need to consider the time derivative of the mixed terms involved in Z_1, Z_2 with different weight as

$$\frac{d}{dt}\left(p^{-1/2} Z_1\right) = -\frac{\partial_t m}{m} p^{-1/2} Z_1 - \frac{3}{4}\frac{\partial_t p}{p^{3/2}} Z_1 - \frac{1}{M} Z_2 \tag{2.21}$$

which combined with the second equation in (2.16) give rise to

$$\begin{aligned}-\frac{d}{dt}\left(p^{-1/2}\mathrm{Re}(\bar{Z}_1 Z_2)\right) &= -\frac{1}{M}\Big(1 + 2M^2\frac{k^2}{p^2}\Big)|Z_1|^2 + \frac{1}{2}\frac{\partial_t p}{p^{3/2}}\mathrm{Re}(\bar{Z}_1 Z_2) \\ &\quad + 2\frac{\partial_t m}{mp^{1/2}}\mathrm{Re}(\bar{Z}_1 Z_2) + \frac{1}{M}|Z_2|^2 + \nu p^{1/2}\mathrm{Re}(\bar{Z}_1 Z_2) \\ &\quad + \langle k, \eta\rangle^s \frac{2m^{-1}k^2}{p^{9/4}}\mathrm{Re}((\widehat{\Phi}^{in} + \widehat{\Omega}^{in})\bar{Z}_1).\end{aligned} \tag{2.22}$$

Finally, to define a coercive energy functional, we need to consider the time derivative of the term $p^{-3/2}\partial_t pZ_1$ or $p^{-3/2}\partial_t pZ_2$. Here, we choose the former

$$\begin{aligned}
&\frac{M^2}{2}\frac{d}{dt}\Big|\frac{\partial_t p}{p^{3/2}}Z_1\Big|^2\\
&= M^2\Big(\frac{2k^2\partial_t p}{p^3}-\frac{7}{4}\frac{(\partial_t p)^3}{p^4}\Big)|Z_1|^2 - M^2\frac{\partial_t m}{m}\frac{(\partial_t p)^2}{p^3}|Z_1|^2 - M\frac{(\partial_t p)^2}{p^{\frac52}}\mathrm{Re}(\bar{Z}_1 Z_2).
\end{aligned}\tag{2.23}$$

Now, we define the energy functional

$$\begin{aligned}
\mathcal{E}(t) &= \frac12\Big(1+M^2\frac{(\partial_t p)^2}{p^3}\Big)|Z_1|^2(t)+\frac12|Z_2|^2(t)+\Big(\frac{M}{4}\frac{\partial_t p}{p^{3/2}}\mathrm{Re}(\bar{Z}_1 Z_2)\Big)(t)\\
&\quad -\Big(\frac{M\nu^{1/3}}{4}p^{-1/2}\mathrm{Re}(\bar{Z}_1 Z_2)\Big)(t).
\end{aligned}\tag{2.24}$$

Multiplying by $\frac{M\nu^{1/3}}{4}$ on both hand side of (2.22) then summing (2.17), (2.18), (2.20), and (2.23) gives

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}(t)+\Big(\frac{\partial_t m}{m}+\frac{\nu^{1/3}}{4}\big(1+2M^2\frac{k^2}{p^2}\big)+M^2\frac{\partial_t m}{m}\frac{(\partial_t p)^2}{p^3}\Big)|Z_1|^2+\Big(\frac{\partial_t m}{m}+\nu p\Big)|Z_2|^2\\
&=\frac{\nu^{1/3}}{4}|Z_2|^2+\frac{M\nu^{\frac43}}{4}p^{1/2}\mathrm{Re}(\bar{Z}_1 Z_2)-\frac{\nu M}{4}\frac{\partial_t p}{p^{1/2}}\mathrm{Re}(\bar{Z}_1 Z_2)+M^2\Big(\frac{5k^2\partial_t p}{2p^3}-\frac74\frac{(\partial_t p)^3}{p^4}\Big)|Z_1|^2\\
&\quad +M\Big(\frac{\nu^{1/3}}{8}\frac{\partial_t p}{p^{\frac32}}+\frac52\frac{k^2}{p^{3/2}}-\frac{11}{8}\frac{(\partial_t p)^2}{p^{\frac52}}+\frac{\nu^{1/3}}{2}\frac{\partial_t m}{mp^{1/2}}\Big)\mathrm{Re}(\bar{Z}_1 Z_2)\\
&\quad +\langle k,\eta\rangle^s\frac{M\nu^{1/3}m^{-1}k^2}{2p^{9/4}}\mathrm{Re}((\widehat{\Phi}^{in}+\widehat{\Omega}^{in})\bar{Z}_1)-\langle k,\eta\rangle^s\frac{Mm^{-1}k^2\partial_t p}{2p^{\frac{13}{4}}}\mathrm{Re}((\widehat{\Phi}^{in}+\widehat{\Omega}^{in})\bar{Z}_2)\\
&:=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4+\mathcal{I}_5+\mathcal{I}_6+\mathcal{I}_7.
\end{aligned}\tag{2.25}$$

Now we are in a position to bound the terms on the right-hand side of (2.25). First, from (2.11), it holds

$$\frac{\partial_t m}{m}+\nu p\ge \nu^{1/3}.\tag{2.26}$$

Hence, the first term $\mathcal{I}_1$ can be absorbed directly in the left. We next consider $\mathcal{I}_2$. With the aid of the Cauchy-Schwarz inequality, one has

$$|\mathcal{I}_2|\le\frac{M\nu^{1/3}}{8}(\nu|Z_1|^2+\nu p|Z_2|^2)\le\frac{M\nu}{8}(\nu^{1/3}|Z_1|^2)+\frac{M\nu^{1/3}}{8}(\nu p|Z_2|^2).\tag{2.27}$$

As a result, to absorb $\mathcal{I}_2$ by the left, we need the assumption $M\nu\le 1$.

Since $|\partial_t p|\le 2|k|p^{1/2}$, we can bound the terms $\mathcal{I}_3$ as follows

$$\begin{aligned}
|\mathcal{I}_3| &\le \frac{\nu}{4}\Big(\frac{2M|k|p^{1/2}}{p^{1/2}}\mathrm{Re}(\bar{Z}_1 Z_2)\Big)\\
&\le \frac{\nu}{4}\Big(4\frac{M^2k^2}{p}|Z_1|^2+\frac14(p|Z_2|^2)\Big)\\
&\le \frac{\nu M^2k^2}{p}|Z_1|^2+\frac{1}{16}\nu p|Z_2|^2.
\end{aligned}\tag{2.28}$$

In the same manner, from $|\partial_t p|\le 2|k|p^{1/2}$ and the fact that $|k|p^{-3/2}\le 1$, we have

$$|\mathcal{I}_4|\le 19M^2\frac{|k|^3}{p^{5/2}}|Z_1|^2\le CM^2\frac{k^2}{p}|Z_1|^2.\tag{2.29}$$

In the following, we bound the terms in $\mathcal{I}_5$. Thanks to $|\partial_t p|\le 2|k|p^{1/2}$ again, we have

$$\begin{aligned}
|\mathcal{I}_5| &\le \frac{M\nu^{1/3}}{4}\frac{|k|}{p^2}\mathrm{Re}(\bar{Z}_1 Z_2)+\frac{M}{4}\frac{|k|^2}{p^{3/2}}\mathrm{Re}(\bar{Z}_1 Z_2)+\frac{M\nu^{1/3}}{2}\frac{\partial_t m}{mp^{1/2}}\mathrm{Re}(\bar{Z}_1 Z_2)\\
&:=\mathcal{I}_{5,1}+\mathcal{I}_{5,2}+\mathcal{I}_{5,3}.
\end{aligned}\tag{2.30}$$

From $\nu \le 1$ and $p^{-\frac{3}{2}} \le 1$, we can bound $\mathcal{I}_{5,1}$ as

$$|\mathcal{I}_{5,1}| \le CM\frac{k^2}{p}(|Z_1|^2 + |Z_2|^2). \tag{2.31}$$

The term $\mathcal{I}_{5,2}$ can be controlled similarly if noticing the fact that $|k|p^{-1} \le 1$.

For the last term $\mathcal{I}_{5,3}$, we can use $M\nu^{1/3} \le 1$ and $p^{-1/2} \le 1$ to obtain

$$|\mathcal{I}_{5,3}| \le C\frac{\partial_t m}{m}(|Z_1|^2 + |Z_2|^2). \tag{2.32}$$

Substituting the above estimates involved in $\mathcal{I}_{5,1}, \mathcal{I}_{5,2}, \mathcal{I}_{5,3}$ into (2.30), we obtain

$$|\mathcal{I}_5| \le CM\frac{k^2}{p}(|Z_1|^2 + |Z_2|^2) + C\frac{\partial_t m}{m}(|Z_1|^2 + |Z_2|^2). \tag{2.33}$$

From $\nu \le 1$, $p^{-1} \le 1$, $|\partial_t p| \le 2|k|p^{1/2}$ and the multiplier m^{-1} is a bound Fourier multiplier, we obtain

$$\frac{Mm^{-1}k^2\partial_t p}{2p^{13/4}} + \frac{M\nu^{1/3}m^{-1}k^2}{2p^{9/4}} \le CM\frac{|k|^2}{p};$$

this and the Young inequality give rise to

$$|\mathcal{I}_6| + |\mathcal{I}_7| \le CM\frac{k^2}{p}\Big(\langle k,\eta\rangle^{2s}|\widehat{\Phi}^{in} + \widehat{\Omega}^{in}|^2 + (|Z_1|^2 + |Z_2|^2)\Big). \tag{2.34}$$

Noticing that

$$M^2\frac{\partial_t m}{m}\frac{(\partial_t p)^2}{p^3} > 0,$$

and then inserting (2.27), (2.28), (2.29), (2.30), (2.33), (2.34) into (2.25), we obtain

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}(t) + \frac{\nu^{1/3}}{16}\Big((1 + 4M^2\frac{k^2}{p^2})|Z_1|^2 + |Z_2|^2\Big)\\
&\le CM\frac{k^2}{p}\langle k,\eta\rangle^{2s}|\widehat{\Phi}^{in} + \widehat{\Omega}^{in}|^2 + C\Big(M(M+1)\frac{k^2}{p} + \frac{\partial_t m}{m}\Big)\mathcal{E}(t).
\end{aligned} \tag{2.35}$$

As

$$4M^2\frac{k^2}{p^2} \ge M^2\frac{(\partial_t p)^2}{p^3},$$

we obtain

$$\frac{d}{dt}\mathcal{E}(t) + \frac{\nu^{1/3}}{16}\mathcal{E}(t) \le CM\frac{k^2}{p}\langle k,\eta\rangle^{2s}|\widehat{\Phi}^{in} + \widehat{\Omega}^{in}|^2 + C\Big(M(M+1)\frac{k^2}{p} + 2\frac{\partial_t m}{m}\Big)\mathcal{E}(t). \tag{2.36}$$

It is easy to check that

$$\int_0^t \frac{k^2}{p(\tau)}\,d\tau = \int_0^t \frac{d\tau}{(\frac{\eta}{k} - \tau)^2 + 1} = \arctan(\frac{\eta}{k} - t) - \arctan(\frac{\eta}{k}).$$

As a result, applying Gronwall's inequality to (2.36) we have

$$\mathcal{E}(t) \le C\left(\mathcal{E}(0) + \langle k,\eta\rangle^{2s}|\widehat{\Phi}^{in} + \widehat{\Omega}^{in}|^2\right)\exp(CM(M+1). \tag{2.37}$$

From $|\partial_t p| < p$, it's not hard to check that

$$\mathcal{E}(t) \approx \frac{1}{4}\Big((1 + M^2\frac{(\partial_t p)^2}{p^3})|Z_1|^2 + |Z_2|^2\Big)(t)$$

which combines with the fact that m is a bounded Fourier multiplier and the definitions of Z_1, Z_2, we can obtain

$$\sum_k \int \mathcal{E}(t)\,d\eta \approx \frac{1}{M^2}\|p^{-1/4}\widehat{\Phi}(t)\|_{H^s}^2 + \|p^{-3/4}\widehat{A}(t)\|_{H^s}^2 \tag{2.38}$$

which implies that

$$\begin{aligned} &\frac{1}{M}\|(p^{-1/4}\widehat{\Phi})(t)\|_{H^s}+\|(p^{-3/4}\widehat{A})(t)\|_{H^s} \\ &\le C\exp(CM(M+1))\Big(\frac{1}{M}\|\widehat{\Phi}^{in}\|_{H^s}+\|\widehat{A}^{in}\|_{H^s}+\|\widehat{\Phi}^{in}+\widehat{\Omega}^{in}\|_{H^s}\Big). \end{aligned}\tag{2.39}$$

This completes the proof.. □

2.2. **Proof of Theorem 1.1 for general $\rho^{in},\theta^{in},\omega^{in}$.** Thanks to the previous Lemma, we are ready to conclude the proof of Theorem 1.1. First, from (2.9) and Lemma 2.2, we have

$$\begin{aligned} \|\Omega(t)\|_{H^s} &= \|\Phi(t)-\Phi^{in}-\Omega^{in}\|_{H^s} \\ &= M\|p^{1/4}(M^{-1}p^{-1/4}\widehat{\Phi})(t)\|_{H^s}+\|\Phi^{in}+\Omega^{in}\|_{H^s} \\ &\le CM\langle t\rangle^{1/2}\|M^{-1}p^{-1/4}\widehat{\Phi}(t)\|_{H^{s+\frac12}}+\|\Phi^{in}\|_{H^s}+\|\Omega^{in}\|_{H^s} \\ &\le C\gamma^{-1}\exp(CM(M+1))\langle t\rangle^{1/2}C_{in,s+\frac12} \end{aligned}\tag{2.40}$$

with

$$C_{in,s+\frac12}:=\frac{1}{M}\|\rho^{in}+\theta^{in}\|_{H^{s+\frac12}}+\|\alpha^{in}\|_{H^{s+\frac12}}+\gamma\|\omega^{in}\|_{H^{s+\frac12}}.$$

Recall the definition of $\mathbb{P}[\mathbf{v}]$ in (1.13), it holds

$$\|\mathbb{P}[\mathbf{v}]^x(t)\|_{L^2}=\|\partial_y\Delta^{-1}\omega(t)\|_{L^2}=\|(\partial_Y-t\partial_X)(\Delta_L^{-1}\Omega)(t)\|_{L^2}\le C\|((-\Delta_L)^{-1/2}\Omega)(t)\|_{L^2}.$$

Therefore, from $p^{1/2}\langle kt\rangle\ge C\langle k,\eta\rangle\langle kt\rangle\ge C\langle\eta\rangle$ and estimate (2.40) we obtain

$$\|\mathbb{P}[\mathbf{v}]^x(t)\|_{L^2}\le C\frac{1}{\langle t\rangle}\|\Omega(t)\|_{H^1}\le C\langle t\rangle^{-1/2}\gamma^{-1}\exp(CM(M+1))C_{in,\frac32}.\tag{2.41}$$

In the same manner, we can deal with the second component of $\mathbb{P}[\mathbf{v}]^y$,

$$\begin{aligned} \|\mathbb{P}[\mathbf{v}]^y(t)\|_{L^2} &= \|\partial_x\Delta^{-1}\omega\|_{L^2}=\|\partial_X(\Delta_L^{-1}\Omega)(t)\|_{L^2} \\ &= \|\frac{k}{p}\Omega(t)\|_{L^2}\le C\langle t\rangle^{-2}\|\Omega(t)\|_{H^1} \\ &\le C\langle t\rangle^{-3/2}\gamma^{-1}\exp(CM(M+1))C_{in,3/2}. \end{aligned}\tag{2.42}$$

Finally, we estimate the compressible part of the velocity. On the one hand, from the Helmholtz decomposition and the change of coordinates, we obtain

$$\begin{aligned} \|\mathbb{Q}[\mathbf{v}](t)\|_{L^2}+\frac{1}{M}\|\rho(t)+\theta(t)\|_{L^2} &= \|(-\Delta)^{-1/2}\alpha(t)\|_{L^2}+\frac{1}{M}\|\rho(t)+\theta(t)\|_{L^2} \\ &= \|(-\Delta_L)^{-1/2}A(t)\|_{L^2}+\frac{1}{M}\|R(t)+\Theta(t)\|_{L^2} \\ &= \|(-\Delta_L)^{-1/2}A(t)\|_{L^2}+\frac{\gamma}{M}\|\Phi(t)\|_{L^2}. \end{aligned}\tag{2.43}$$

As a result, from (2.43), Lemma 2.2 and that $p\le\langle t\rangle^2\langle k,\eta\rangle^2$, we deduce that

$$\begin{aligned} \|\mathbb{Q}[\mathbf{v}](t)\|_{L^2}+\frac{1}{M}\|\rho(t)+\theta(t)\|_{L^2} &= \|p^{1/4}(p^{-3/4}\widehat{A})(t)\|_{L^2}+\frac{\gamma}{M}\|p^{1/4}(p^{-1/4}\widehat{\Phi})(t)\|_{L^2} \\ &\le C\langle t\rangle^{1/2}\Big(\|(p^{-3/4}\widehat{A})(t)\|_{H^1}+\frac{\gamma}{M}\|(p^{-1/4}\widehat{\Phi})(t)\|_{H^1}\Big) \\ &\le C\langle t\rangle^{1/2}(1+\gamma)\gamma^{-1}\exp(CM(M+1))C_{in,1}. \end{aligned}\tag{2.44}$$

On the other hand, by (1.15), we have $(\partial_t+y\partial_x)((\gamma-1)\rho-\theta)=0$ which implies that

$$(\gamma-1)\rho-\theta=(\gamma-1)\rho^{in}-\theta^{in}.$$

Moreover, we have

$$\|\frac{(\gamma-1)\rho(t)-\theta(t)}{M}\|_{L^2}^2=\|\frac{(\gamma-1)\rho^{in}-\theta^{in}}{M}\|_{L^2}^2.$$

A simple computation gives

$$\frac{\gamma}{M}\rho = \frac{(\gamma-1)\rho-\theta}{M} + \frac{\rho+\theta}{M},$$
$$\frac{\gamma}{M}\theta = -\frac{(\gamma-1)\rho-\theta}{M} + (\gamma-1)\frac{\rho+\theta}{M}.$$

Hence

$$\begin{aligned}\frac{\gamma}{M}\|\rho(t)\|_{L^2} &\le C\Big\|\frac{(\gamma-1)\rho-\theta}{M}\Big\|_{L^2} + C\Big\|\frac{\rho+\theta}{M}\Big\|_{L^2}\\ &\le C\langle t\rangle^{1/2}\Big\{\Big\|\frac{(\gamma-1)\rho^{in}-\theta^{in}}{M}\Big\|_{L^2} + (1+\gamma)\gamma^{-1}\exp(CM(M+1))C_{in,1}\Big\},\end{aligned}$$

and

$$\begin{aligned}\frac{\gamma}{M}\|\theta(t)\|_{L^2} &\le C\Big\|\frac{(\gamma-1)\rho-\theta}{M}\Big\|_{L^2} + C(\gamma-1)\Big\|\frac{\rho+\theta}{M}\Big\|_{L^2}\\ &\le C(\gamma-1)\Big\|\frac{(\gamma-1)\rho^{in}-\theta^{in}}{M}\Big\|_{L^2} + C\langle t\rangle^{1/2}(\gamma-\gamma^{-1})\exp(CM(M+1))C_{in,1}.\end{aligned}$$

This proves the first case for general $\rho^{in},\theta^{in},\omega^{in}$.

2.3. **Proof of Theorem 1.1 with special $\rho^{in},\theta^{in},\omega^{in}$ satisfying** (1.17)**.** If

$$\rho^{in} + \gamma\omega^{in} + \theta^{in} = 0,$$

from $\partial_t(R+\gamma\Omega+\Theta)=0$, we can infer that

$$\Omega = -\frac{R+\Theta}{\gamma} = -\Phi.$$

Thus, we obtain a closed system only involved in $\widehat{\Phi}$ and $\widehat{A}$

$$\begin{aligned}&\partial_t\widehat{\Phi} = -\widehat{A},\\ &\partial_t\widehat{A} = -\nu p\widehat{A} + \frac{\partial_t p}{p}\widehat{A} + \Big(\frac{p}{M^2} + \frac{2k^2}{p}\Big)\widehat{\Phi}.\end{aligned} \tag{2.45}$$

For the above system, we can make a similar argument as in the proof of Lemma 2.2 to obtain another version of (2.36) which do not involve in $\widehat{\Phi}^{in},\widehat{\Omega}^{in}$ that

$$\frac{d}{dt}\mathcal{E}(t) + \frac{\nu^{1/3}}{16}\mathcal{E}(t) \le C\Big(M(M+1)\frac{k^2}{p} + \frac{\partial_t m}{m}\Big)\mathcal{E}(t), \tag{2.46}$$

Consequently, applying Gronwall's inequality to (2.46), we have

$$\mathcal{E}(t) \le C\exp(CM(M+1))e^{-\frac{\nu^{1/3}}{16}t}\mathcal{E}(0)$$

which combines with (2.38) to give

$$\begin{aligned}&\frac{1}{M}\|(p^{-1/4}\widehat{\Phi})(t)\|_{H^s} + \|(p^{-3/4}\widehat{A})(t)\|_{H^s}\\ &\le C\exp(CM(M+1))e^{-\frac{1}{32}\nu^{1/3}t}\Big(\frac{1}{\gamma M}\|\Phi^{in}\|_{H^s} + \|\alpha^{in}\|_{H^s}\Big).\end{aligned} \tag{2.47}$$

With this inequality in hand, we can follow the same argument as the derivation of (2.41), (2.42), and (2.44) to obtain

$$\|\mathbb{P}[\mathbf{v}]^x(t)\|_{L^2} \le C\exp(CM(M+1))\langle t\rangle^{-1/2}e^{-\frac{1}{16}\nu^{1/3}t}\Big(\frac{1}{\gamma M}\|\rho^{in}+\theta^{in}\|_{H^{3/2}} + \|\alpha^{in}\|_{H^{3/2}}\Big),$$

$$\|\mathbb{P}[\mathbf{v}]^y(t)\|_{L^2} \le C\exp(CM(M+1))\langle t\rangle^{-3/2}e^{-\frac{1}{16}\nu^{1/3}t}\Big(\frac{1}{\gamma M}\|\rho^{in}+\theta^{in}\|_{H^{3/2}} + \|\alpha^{in}\|_{H^{3/2}}\Big),$$

$$\begin{aligned}&\|\mathbb{Q}[\mathbf{v}](t)\|_{L^2} + \frac{1}{M}\|\rho(t)+\theta(t)\|_{L^2}\\ &\le C(1+\gamma)\exp(CM(M+1))\langle t\rangle^{1/2}e^{-\frac{1}{32}\nu^{1/3}t}\Big(\frac{1}{\gamma M}\|\rho^{in}+\theta^{in}\|_{H^1} + \|\alpha^{in}\|_{H^1}\Big).\end{aligned}$$

The proof of Theorem 1.1 is complete.

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