Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 104, pp. 1–31.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, DOI: 10.58997/ejde.2025.104

GLOBAL SOLUTIONS AND BLOW-UP FOR WAVE EQUATIONS WITH VARIABLE COEFFICIENTS AND BOUNDARY SUPERCRITICAL SOURCE

TAE GAB HA

Abstract. In this study, we consider a wave equation with variable coefficients, boundary damping, and supercritical source terms. This study aims to study the local and global existence and classify the decay rate of energy depending on the growth near zero in the damping term. Furthermore, we demonstrate that the weak solution blows up in finite time with positive and nonpositive initial energies.

1. Introduction

In this study, we investigate the local and global existence, energy decay rates, and finite time blow-up of the solution to the wave equation

$$\begin{gathered} u_{tt} - \mu(t)Lu = f(x,t) \quad \text{in } \Omega \times (0,+\infty), \\ u = 0 \quad \text{on } \Gamma_0 \times (0,+\infty), \\ \mu(t)\frac{\partial u}{\partial \nu_L} + q(u_t) = h(u) \quad \text{on } \Gamma_1 \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \end{gathered} \tag{1.1}$$

where

$$Lu = \operatorname{div}(A(x)\nabla u) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\big(a_{ij}(x)\frac{\partial u}{\partial x_j}\big).$$

Here, $A(x) = (a_{ij}(x))$ is a symmetric and positive matrix and $\frac{\partial u}{\partial \nu_L} = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\nu_i$, where $\nu = (\nu_1, \dots, \nu_n)$ is the outward unit normal to Γ. Ω is a bounded domain of $\mathbb{R}^n (n \geq 3)$ with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint with $\operatorname{meas}(\Gamma_0) \neq 0$.

Problem (1.1) has been widely studied over the past few decades. The condition $h(s)s \leq 0$ indicates that h represents the attractive force. When $h(s)s \geq 0$ as in the present case, h represents the source term. This scenario is more delicate than the attractive force because the solution of (1.1) can blow up. The damping-source interplay in the system (1.1) arises naturally in numerous contexts, such as classical mechanics, fluid dynamics, and quantum field theory (cf. [29, 41]). The interaction between the two competitive forces, that is, the damping and source terms, makes the problem attractive from a mathematical perspective.

For the present case when h is polynomial nonlinear source term such as $h(u) = |u|^\gamma u$, the stability of (1.1) has been studied by many authors (see [7, 8, 14, 15, 16, 17, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30, 31, 36, 37, 38, 39, 43] and a list on references therein), where $h(u)$ is subcritical or critical source. However, very few results addressed wave equations influenced by supercritical sources (cf. [1, 3, 11, 12, 42]). For example, [42] proved the local and global existence, uniqueness and Hadamard well-posedness for the wave equation when source terms can be supercritical or super-supercritical. However, the author did not consider the energy decay and blow-up of the solutions. Guo et al. [11] considered a system of nonlinear wave equations with supercritical sources

2020 *Mathematics Subject Classification.* 35L05, 35L20, 35A01, 35B40, 35B44.

Key words and phrases. Wave equation with variable coefficients; supercritical source; blow-up; existence of solutions; energy decay rates.

Submitted June 5, 2025. Published November 4, 2025.

and damping terms. They proved global existence and exponential and algebraic uniform decay rates of energy, moreover, blow-up result for weak solutions with nonnegative initial energy. But as far as I know, the only problem with considering supercritical source is the constant coefficients case, that is, $A = I$ and dimension $n = 3$.

In the case of variable coefficients, that is $A \neq I$, boundary stability of the wave equation was considered in [4, 9, 15, 18, 22]. The wave equations with variable coefficients arise in mathematical modeling of inhomogeneous media in solid mechanics, electromagnetic, fluid flows through porous media. For the variable coefficients problem, the main tool is Riemannian geometry method, which was introduced by [46] and has been widely used in the literature, see [6, 9, 32, 33, 34, 44, 45] and a list of references therein. However, there were very few results considered the source term. For example, [4] proved the uniform decay rate of the energy to the viscoelastic wave equation with variable coefficients and acoustic boundary conditions without damping term. Recently, [26] studied the general decay rate of the energy for the wave equation with variable coefficients and Balakrishnan-Taylor damping and source term without imposing any restrictive growth near zero on the damping term. However, above-mentioned literature did not consider Riemannian geometry, and only treated a subcritical source. [19] proved the uniform energy decay rates of the wave equation with variable coefficients applying the Riemannian geometry method and modified multiplier method. But, it was considered a subcritical source. More recently, the author proved the existence of a solution, the energy decay rate, and the blow-up of the solution for a wave equation with an interior supercritical source ([23]). The method and process of the proof in this study are similar to [23]. However, there are some differences between this study and that of [23]. Reference [23] dealt with an interior supercritical source without considering Riemannian geometry. Hence, the parameter analysis results for the source term were different (see Figures 1 and 2). Moreover, because this study considers the boundary conditions and Riemannian geometry, the analysis is more complicated, although the method and process of the proof are similar to [23]. To the best of our knowledge, there is no existing literature addressing the variable coefficient problem that includes both damping and source terms on Riemannian geometry while also considering the boundary supercritical source.

Our main motivation is constituted by three dimensional case, in which the source term can be supercritical on variable coefficient problem. The differences from previous literature are as follows:

(i) Supercritical source for $n \geq 3$.
(ii) Variable coefficient problem having source term on Riemannian geometry.
(iii) Blow-up result with positive initial energy as well as nonpositive initial energy.

To overcome difficulties and prove the above statements, first, we refine the energy space and a constant used in potential well method, because we do not guarantee $H_0^1(\Omega) \hookrightarrow L^{\gamma+2}(\Gamma_1)$ since the source term is supercritical. Also we have a hypothesis on damping term for proving existence of solutions and energy decay rates (see Remark 2.2). Second, we use the Faedo-Galerkin method because nonlinear semigroup arguments considered in the previous literatures cannot be used since this paper deals with an operator $-\mu(t)L$, which depends on t. Third, we refine the key point constants used to prove blow-up result. So, this paper has improved and generalized previous literatures.

The goal of this paper is to prove the existence result using the Faedo-Galerkin method and truncated approximation method, and classify the energy decay rate applying the method developed in [46, 35]. Moreover, we prove the blow-up of the weak solution with positive initial energy as well as nonpositive initial energy. This paper is organized as follows: In Section 2, we recall the notation, hypotheses and some necessary preliminaries and introduce our main result. In Section 3, we prove the local existence of weak solutions, and show the global existence of weak solution in each two conditions in Section 4. In Section 5, we prove the uniform decay rate under suitable conditions on the initial data and boundary damping by the differential geometric approach. In Section 6, we prove the blow-up of the weak solution with positive initial energy as well as nonpositive initial energy. by using contradiction method.

2. Preliminaries

We begin this section by introducing notation and stating our main results. We define the Hilbert space $H_0^1(\Omega) = \{u \in H^1(\Omega); u = 0 \text{ on } \Gamma_0\}$ with the norm $\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$ and $\mathcal{H} = \{u \in H_0^1(\Omega); u \in L^{\gamma+2}(\Gamma_1)\}$ with the norm $\|u\|_{\mathcal{H}} = \|u\|_{H_0^1(\Omega)} + \|u\|_{L^{\gamma+2}(\Gamma_1)}$. $\|\cdot\|_p$ and $\|\cdot\|_{p,\Gamma}$ are denoted by the $L^p(\Omega)$ norm and the $L^p(\Gamma)$ norm, respectively, and $\langle u, v\rangle = \int_\Omega u(x)v(x)dx$ and $\langle u, v\rangle_\Gamma = \int_\Gamma u(x)v(x)\, d\Gamma$. Moreover, we require some notations for Riemannian geometry, as mentioned in [46] and the references therein. For reader comprehension, we have repeated these here.

Let $A(x) = (a_{ij}(x))$ be a symmetric and positive definite matrix for all $x \in \mathbb{R}^n$ $(n \geq 3)$ and $a_{ij}(x)$ be smooth functions on $\mathbb{R}^n$ satisfying

$$c_1 \sum_{i=1}^n \xi_i^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j, \quad \forall x \in \mathbb{R}^n, \quad 0 \neq \xi = (\xi_1, \dots, \xi_n)^T \in \mathbb{R}^n, \tag{2.1}$$

where c_1 is a positive constant. Set

$$G(x) = \big(g_{ij}(x)\big) = A^{-1}(x).$$

For each $x \in \mathbb{R}^n$, we define the inner product $g(\cdot,\cdot) = \langle \cdot,\cdot\rangle_g$ and the norm $|\cdot|_g$ on the tangent space $\mathbb{R}^n_x = \mathbb{R}^n$ as

$$\begin{gathered} g(X,Y) = \langle X, Y\rangle_g = \sum_{i,j=1}^n g_{ij}(x)\alpha_i\beta_j, \quad |X|_g = \langle X, X\rangle_g^{\frac{1}{2}}, \\ \forall X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x. \end{gathered} \tag{2.2}$$

Subsequently, $(\mathbb{R}^n, g)$ is a Riemannian manifold with Riemann metric g. $\nabla_g u$ and D_g are denoted by the gradient of u and Levi-Civita connection in the Riemannian metric g, respectively. It follows that

$$\nabla_g u = \sum_{i=1}^n \Big(\sum_{j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_j}\Big)\frac{\partial}{\partial x_i} = A(x)\nabla u, \quad |\nabla_g u|_g^2 = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}. \tag{2.3}$$

Let H be the vector field on $(\mathbb{R}^n, g)$. Subsequently, the covariant differential D_gH of H determines a bilinear form on $\mathbb{R}^n_x \times \mathbb{R}^n_x$, for each $x \in \mathbb{R}^n$ as

$$D_gH(X,Y) = \langle D_{gX}H, Y\rangle_g, \quad \forall X, Y \in \mathbb{R}^n_x, \tag{2.4}$$

where $D_{gX}H$ is the covariant derivative of the vector field H with respect to X.

(H1) Hypothesis on Ω. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $n \geq 3$ with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here Γ_0 and Γ_1 are closed and disjoint, where $\operatorname{meas}(\Gamma_0) \neq 0$. A vector field H exists on the Riemannian manifold $(\mathbb{R}^n, g)$ such that

$$D_gH(X,X) \geq \sigma|X|_g^2, \quad \forall X \in \mathbb{R}^n_x, \quad x \in \overline{\Omega}, \tag{2.5}$$

where σ is a positive constant and

$$H \cdot \nu \leq 0 \quad \text{on } \Gamma_0 \quad \text{and} \quad H \cdot \nu \geq \delta > 0 \quad \text{on } \Gamma_1, \tag{2.6}$$

where ν represents the unit outward normal vector to Γ. Moreover we assume that

$$\mu(0)\frac{\partial u_0}{\partial \nu_L} + g(u_1) = h(u_0) \quad \text{on } \ \Gamma_1. \tag{2.7}$$

(H2) Hypothesis on μ and f. Let $\mu \in W^{1,\infty}(0,\infty) \cap W^{1,1}(0,\infty)$ be a function satisfying the conditions

$$\mu(t) \geq \mu_0 > 0 \quad \text{and} \quad \mu'(t) \leq 0 \quad \text{a.e. in } [0,\infty), \tag{2.8}$$

where μ_0 is a positive constant. We assume that

$$f \in H^1(0,\infty; L^2(\Omega)). \tag{2.9}$$

(H3) Hypothesis on q. Let $q : \mathbb{R} \to \mathbb{R}$ be a nondecreasing C^1 function such that $q(0) = 0$ and suppose that there exist positive constants c_3, c_4, and a strictly increasing and odd function β of C^1 class on $[-1, 1]$, such that

$$|\beta(s)| \le |q(s)| \le |\beta^{-1}(s)| \quad \text{if} \quad |s| \le 1, \tag{2.10}$$

$$c_3|s|^{\rho+1} \le |q(s)| \le c_4|s|^{\rho+1} \quad \text{if} \quad |s| > 1, \tag{2.11}$$

where β^{-1} denotes the inverse function of β, and $\rho \ge \frac{2(n-2)\gamma-2}{n-(n-2)\gamma}$.

(H4) Hypothesis on γ. Let γ be a constant satisfying the condition

$$\frac{1}{n-2} < \gamma \le \frac{n-1}{n-2}. \tag{2.12}$$

Lemma 2.1 ([46]). *Let $u, v \in C^1(\overline{\Omega})$ and H, X be vector fields on $(\mathbb{R}^n, g)$. Then*

$$H(u) = \langle \nabla_g u, H\rangle_g, \quad \langle H(x), A(x)X(x)\rangle_g = H(x) \cdot X(x), \tag{2.13}$$

$$\operatorname{div}(uH) = u \operatorname{div}(H) + H(u), \quad \int_\Omega \operatorname{div}(H)\, dx = \int_\Gamma H \cdot \nu \, d\Gamma, \tag{2.14}$$

$$\int_\Omega u \ \operatorname{div}(\nabla_g v)\, dx = \int_\Gamma u \frac{\partial v}{\partial \nu_L}\, d\Gamma - \int_\Omega \langle \nabla_g u, \nabla_g v\rangle_g \, dx, \tag{2.15}$$

$$\langle \nabla_g u, \nabla_g(H(u))\rangle_g = D_g H(\nabla_g u, \nabla_g u) + \frac{1}{2}\operatorname{div}(|\nabla_g u|_g^2 H) - \frac{1}{2}|\nabla_g u|_g^2 \operatorname{div}(H). \tag{2.16}$$

Hypothesis (2.5) was introduced by Yao [46] for the exact controllability of the wave equation with variable coefficients. The existence of such a vector field depends on the sectional curvature of the Riemannian manifold $(\mathbb{R}^n, g)$. There are several methods and examples in [46] to find out a vector field H that is satisfied the hypothesis (2.5). Specially, if $A = I$, the constant coefficient case, the condition (2.5) is automatically satisfied by choosing $H = x - x_0$ for any $x_0 \in \mathbb{R}^n$.

Remark 2.2. In view of the critical Sobolev imbedding $H^{1/2}(\Gamma_1) \hookrightarrow L^{\frac{2(n-1)}{n-2}}(\Gamma_1)$, the mapping $k(u) = |u|^\gamma u$ is not locally Lipschitz from $H_0^1(\Omega)$ into $L^2(\Gamma_1)$ for the supercritical values $\frac{1}{n-2} < \gamma \le \frac{n-1}{n-2}$. However, by the hypothesis on ρ $\left(\rho \ge \frac{2(n-2)\gamma-2}{n-(n-2)\gamma}\right)$, $k(u)$ is locally Lipschitz from $H_0^1(\Omega)$ into $L^{\frac{\rho+2}{\rho+1}}(\Gamma_1)$.

For $n > 3$, if $\frac{2}{n-2} \le \gamma \le \frac{n-1}{n-2}$, then the inequality $\rho \ge \gamma$ always holds true under the assumption $\rho \ge \frac{2(n-2)\gamma-2}{n-(n-2)\gamma}$ (see Figure 1).

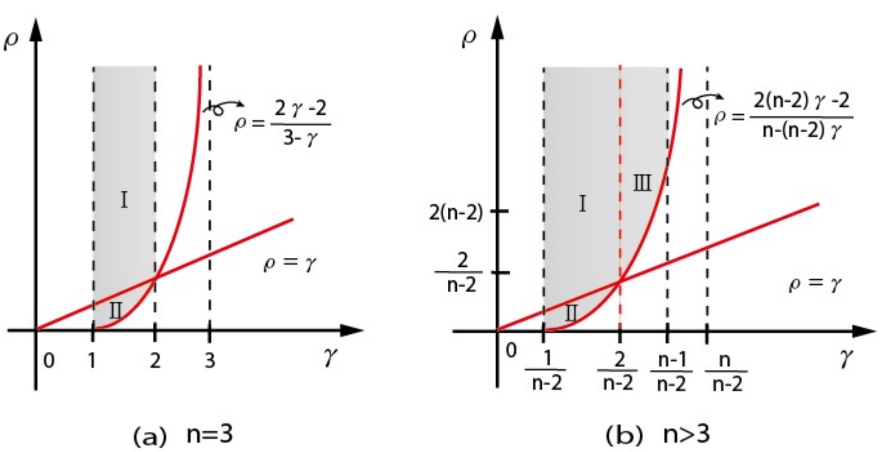

FIGURE 1. Admissible range of the damping parameter ρ and the exponent of the source γ.

Definition 2.3. A function $u(x,t)$ is called a weak solution of (1.1) on $\Omega \times (0,T)$ if $u \in C_w(0,T;\mathcal{H}) \cap C_w^1(0,T;L^2(\Omega))$, $u_t|_{\Gamma_1} \in L^{\rho+2}(0,T;\Gamma_1)$ and satisfies (1.1) in the distribution sense, i.e.,

$$\begin{aligned}&\int_0^T \int_\Omega -u_t\phi_t\,dx\,dt + \int_\Omega u_t\phi dx\Big|_0^T + \int_0^T \mu(t)\int_\Omega \langle \nabla_g u, \nabla_g \phi\rangle_g\,dx\,dt\\&+\int_0^T\int_{\Gamma_1} q(u_t)\phi\,d\Gamma\,dt - \int_0^T\int_{\Gamma_1} h(u)\phi\,d\Gamma\,dt\\&=\int_0^T\int_\Omega f(x,t)\phi\,dx\,dt,\end{aligned}$$

for any $\phi \in C_w(0,T;\mathcal{H}) \cap C_w^1(0,T;L^2(\Omega))$, $\phi|_{\Gamma_1} \in L^{\rho+2}(0,T;\Gamma_1)$ and $u(x,0) = u_0(x) \in \mathcal{H}$, $u_t(x,0) = u_1(x) \in L^2(\Omega)$.

One can easily check that $\mathcal{H} = H_0^1(\Omega)$ when $\frac{1}{n-2} < \gamma \le \frac{2}{n-2}$. Moreover, if $n = 3$, then we can replace $\mathcal{H}$ by $H_0^1(\Omega)$, since $H_0^1(\Omega) \hookrightarrow L^{\gamma+2}(\Gamma_1)$ (see Figure 2).

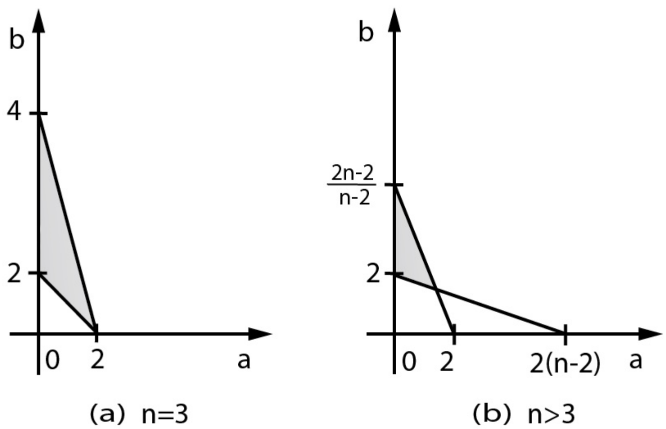

FIGURE 2. Admissible range of parameters a and b with respect to the trace imbedding $H_0^1(\Omega) \hookrightarrow L^{a\gamma+b}(\Gamma_1)$.

The energy associated with problem (1.1) when $h(u) = |u|^\gamma u$ is

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\mu(t)\|\,|\nabla_g u(t)|_g\|_2^2 - \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2}.$$

We now state our main results.

Theorem 2.4. *Suppose that* (H1)–(H4) *hold and let* $h(u) = |u|^\gamma u$. *Then for the given initial data* $(u_0,u_1) \in \mathcal{H} \times^2 (\Omega)$, *there exist* $T > 0$ *and a weak solution of problem* (1.1). *Moreover, the following energy identity holds for all* $0 \le t \le T$,

$$E(t) + \int_0^t\int_{\Gamma_1} q(u_s)u_s\,d\Gamma\,ds - \frac{1}{2}\int_0^t u'(s)\|\,|\nabla_g u(s)|_g\|_2^2\,ds - \int_0^t\int_\Omega f(x,s)u_s dx\,ds = E(0). \tag{2.17}$$

Furthermore, if one of the 2 assumptions hold: $\rho \ge \gamma$ *or*

$$E(0) < d_0 \quad \text{and} \quad \|\,|\nabla_g u_0|_g\|_2 < \lambda_0 \tag{2.18}$$

with $f = 0$, *where*

$$\lambda_0 = \Big(\frac{\mu_0}{K_0^{\gamma+2}}\Big)^{1/\gamma} \qquad d_0 = \frac{\gamma\mu_0}{2(\gamma+2)}\lambda_0^2, \quad K_0 = \sup_{u\in\mathcal{H},u\neq 0}\Big(\frac{\|u\|_{\gamma+2,\Gamma_1}}{\|\,|\nabla_g u|_g\|_2}\Big).$$

Then the solution $u(x,t)$ *of* (1.1) *is global.*

Theorem 2.5. *Suppose that the hypotheses in Theorem 2.4 and* (2.18) *with $\rho \le \gamma$ and $f = 0$ hold. Then we have following energy decay rates:*

Case 1: *β is linear. Then*
$$E(t) \le C_1 e^{-\omega t},$$
where ω is a positive constant.

Case 2: *β has polynomial growth near zero, that is, $\beta(s) = s^{\rho+1}$. Then*
$$E(t) \le \frac{C_2}{(1+t)^{2/\rho}}.$$

Case 3: *β does not necessarily have polynomial growth near zero. Then*
$$E(t) \le C_3 \Big(F^{-1}\big(\frac{1}{t}\big)\Big)^2,$$
where $F(s) = s\beta(s)$ and C_i $(i = 1, 2, 3)$ are positive constants that depends only on $E(0)$.

Theorem 2.6. *Suppose that hypotheses* (H1)–(H4) *hold with $\rho < \gamma$ and $f = 0$. Moreover, assume that*
$$(u_0, u_1) \in \{(u_0, u_1) \in \mathcal{H} \times L^2(\Omega); \| |\nabla_g u_0|_g \|_2 > \lambda_0,\ -1 < E(0) < d_0\}$$
and
$$\beta^{-1}(1) \le \Big(\frac{(\gamma+2)(\mu_0\gamma\lambda_0^2 - 2(\gamma+2)E_1)^2}{8(\gamma+1)\operatorname{meas}(\Gamma_1)(\mu_0\lambda_0^2 - 2E_1)}\Big)^{\frac{\gamma+1}{\gamma+2}}, \tag{2.19}$$
where
$$E_1 = \begin{cases} 0 & \text{if } E(0) < 0, \\ \text{positive constant with} \quad E(0) < E_1 < d_0,\ E_1 < E(0) + 1 & \text{if } E(0) \ge 0. \end{cases}$$
Then the weak solution of problem (1.1) *blows up in finite time.*

We have summarized our results as follows.

(1) Local existence is obtained for the regions I, II, and III in Figure 1.
(2) Global existence is obtained for the region I and III, or the region II with the condition (2.18) in Figure 1.
(3) Energy decay rate is obtained for region II in Figure 1 with the condition (2.18).
(4) For the region II in Figure 1 with the condition (2.19), we obtain the blow-up in finite time.

3. Proof of Theorem 2.4: local existence

The proof is based on three steps according to the following condition of the source term h:

(1) Existence of the global solution when h is globally Lipschitz from $H_0^1(\Omega)$ to $L^2(\Gamma_1)$.
(2) Existence of the local solution when h is locally Lipschitz from $H_0^1(\Omega)$ to $L^2(\Gamma_1)$.
(3) Existence of the local solution when h is locally Lipschitz from $H_0^1(\Omega)$ to $L^{\frac{\rho+2}{\rho+1}}(\Gamma_1)$.

Since the mapping $|u|^\gamma u$ is locally Lipschitz from $H_0^1(\Omega)$ to $L^{\frac{\rho+2}{\rho+1}}(\Gamma_1)$ (see Remark 2.2), the existence of the local solution can be guaranteed even if $h(u) = |u|^\gamma u$.

3.1. Globally Lipschitz source.

We first deal with the case where the source h is globally Lipschitz from $H_0^1(\Omega)$ to $L^2(\Gamma_1)$. In this case, we have the following result.

Proposition 3.1. *Assume that* (H1)-(-H3) *hold. In addition, assume that $(u_0, u_1) \in \mathcal{H} \times L^2(\Omega)$ and $h : H_0^1(\Omega) \to L^2(\Gamma_1)$ is globally Lipschitz continuous. Then problem* (1.1) *has a unique global solutions $u \in C_w(0, T; \mathcal{H}) \cap C_w^1(0, T; L^2(\Omega))$ for arbitrary $T > 0$.*

Our goal in this subsection is to show the local existence result for problem (1.1). We construct an approximate solution by using the Faedo-Galerkin method. Let $\{w_j\}_{j\in\mathbb{N}}$ be a basis in $H_0^1(\Omega)$ and define $V_m = span\{w_1, w_2, \dots, w_m\}$. Let u_0^m and u_1^m be sequences of V_m such that $u_0^m \to u_0$

strongly in $H_0^1(\Omega)$ and $u_1^m \to u_1$ strongly in $L^2(\Omega)$. We search for a function, for each $\eta \in (0,1)$ and $m \in \mathbb{N}$, such that

$$u^{\eta m}(t) = \sum_{j=1}^{m} \delta^{jm}(t) w_j$$

satisfying the approximate perturbed equation

$$\begin{aligned}
&\int_\Omega u_{tt}^{\eta m} w dx + \mu(t) \int_\Omega \langle \nabla_g u^{\eta m}, \nabla_g w \rangle_g dx + \eta \int_{\Gamma_1} u_t^{\eta m} w \, d\Gamma \\
&+ \int_{\Gamma_1} q(u_t^{\eta m}) w \, d\Gamma - \int_{\Gamma_1} h(u^{\eta m}) w \, d\Gamma \\
&= \int_\Omega f(t) w dx, \quad w \in V_m, \\
&\quad u_0^{\eta m} = \sum_{j=1}^{m} \langle u_0, w_j \rangle w_j, \quad u_1^{\eta m} = \sum_{j=1}^{m} \langle u_1, w_j \rangle w_j.
\end{aligned} \tag{3.1}$$

Since (3.1) is a normal system of ordinary differential equations, there exist solutions $u^{\eta m}$ to problem (3.1). A solution u to problem (1.1) on some interval $[0, t_m)$, $t_m \in (0, T]$ will be obtained as the limit of $u^{\eta m}$ as $m \to \infty$ and $\eta \to 0$. Next, we show that $t_m = T$ and the local solution is uniformly bounded independent of m, η and t. For this purpose, let us replace w by $u_t^{\eta m}$ in (3.1) we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big[\frac{1}{2}\|u_t^{\eta m}\|_2^2 + \frac{1}{2}\mu(t)\| \, |\nabla_g u^{\eta m}|_g\|_2^2 + \frac{1}{\gamma+2}\|u^{\eta m}\|_{\gamma+2,\Gamma_1}^{\gamma+2}\Big] + \eta \|u_t^{\eta m}\|_{2,\Gamma_1}^2 + \int_{\Gamma_1} q(u_t^{\eta m}) \, u_t^{\eta m} \, d\Gamma \\
&= \frac{1}{2}\mu'(t)\| \, |\nabla_g u^{\eta m}|_g\|_2^2 + \int_{\Gamma_1} h(u^{\eta m}) u_t^{\eta m} \, d\Gamma + \int_{\Gamma_1} |u^{\eta m}|^\gamma u^{\eta m} u_t^{\eta m} \, d\Gamma + \int_\Omega f(t) u_t^{\eta m} dx.
\end{aligned} \tag{3.2}$$

Now we estimate $\int_{\Gamma_1} q(u_t^{\eta m}) \, u_t^{\eta m} \, d\Gamma$, $\int_{\Gamma_1} |u^{\eta m}|^\gamma u^{\eta m} u_t^{\eta m} \, d\Gamma$ and $\int_{\Gamma_1} h(u^{\eta m}) u_t^{\eta m} \, d\Gamma$. From the hypotheses on q, we have

$$\begin{aligned}
\int_{\Gamma_1} q(u_t^{\eta m}) \, u_t^{\eta m} \, d\Gamma &= \int_{|u_t^{\eta m}| \le 1} q(u_t^{\eta m}) \, u_t^{\eta m} \, d\Gamma + \int_{|u_t^{\eta m}| > 1} q(u_t^{\eta m}) \, u_t^{\eta m} \, d\Gamma \\
&\ge \int_{|u_t^{\eta m}(t)| > 1} q(u_t^{\eta m}) \, u_t^{\eta m} \, d\Gamma \\
&\ge c_3 \int_{\Gamma_1} |u_t^{\eta m}|^{\rho+2} \, d\Gamma - c_3 \int_{|u_t^{\eta m}| \le 1} |u_t^{\eta m}|^{\rho+2} \, d\Gamma \\
&\ge c_3 \|u_t^{\eta m}\|_{\rho+2,\Gamma_1}^{\rho+2} - c_3 \operatorname{meas}(\Gamma_1).
\end{aligned} \tag{3.3}$$

From $\rho \ge \frac{2(n-2)\gamma - 2}{n-(n-2)\gamma}$, we have the imbedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2(n-1)}{n-2}}(\Gamma_1) \hookrightarrow L^{(\gamma+1)\frac{\rho+2}{\rho+1}}(\Gamma_1)$, so that by Young's inequality with $\frac{\rho+1}{\rho+2} + \frac{1}{\rho+2} = 1$ we deduce that

$$\begin{aligned}
\int_{\Gamma_1} |u^{\eta m}|^\gamma u^{\eta m} u_t^{\eta m} \, d\Gamma &\le C(\epsilon_0) \|u^{\eta m}\|_{(\gamma+1)\frac{\rho+2}{\rho+1},\Gamma_1}^{\gamma+1} + \epsilon_0 \|u_t^{\eta m}\|_{\rho+2,\Gamma_1}^{\rho+2} \\
&\le C(\epsilon_0) \| \, |\nabla_g u^{\eta m}|_g\|_2^{\gamma+1} + \epsilon_0 \|u_t^{\eta m}\|_{\rho+2,\Gamma_1}^{\rho+2} \\
&\le C(\epsilon)\big(1 + \| \, |\nabla_g u^{\eta m}|_g\|_2\big)^2 + \epsilon_0 \|u_t^{\eta m}\|_{\rho+2,\Gamma_1}^{\rho+2} \\
&\le C(\epsilon)\big(1 + \| \, |\nabla_g u^{\eta m}|_g\|_2^2\big) + \epsilon_0 \|u_t^{\eta m}\|_{\rho+2,\Gamma_1}^{\rho+2}.
\end{aligned} \tag{3.4}$$

Under the assumption that h is globally Lipschitz from $H_0^1(\Omega)$ into $L^2(\Gamma_1)$ we have

$$\|h(u^{\eta m})\|_{2,\Gamma_1} \le \|h(u^{\eta m}) - h(0)\|_{2,\Gamma_1} + \|h(0)\|_{2,\Gamma_1} \le L_h \|\nabla u^{\eta m}\|_2 + \|h(0)\|_{2,\Gamma_1} \le C_4(\|\nabla u^{\eta m}\|_2 + 1),$$

where L_h is the Lipschitz constant and C_4 is for some positive constant, so that by Hölder's and Young's inequalities and from the fact (2.1) and the imbedding $L^{\rho+2}(\Gamma_1) \hookrightarrow L^2(\Gamma_1)$, we deduce that

$$
\begin{aligned}
\Big|\int_{\Gamma_1} h(u^{\eta m})u_t^{\eta m}\,d\Gamma\Big| &\le C(\epsilon)\|h(u^{\eta m})\|_{2,\Gamma_1}^2 + \epsilon\|u_t^{\eta m}\|_{2,\Gamma_1}^2 \\
&\le C(\epsilon)\Big(\frac{1}{c_1}\|\,|\nabla_g u^{\eta m}|_g\|_2^2 + 1\Big) + \epsilon\|u_t^{\eta m}\|_{2,\Gamma_1}^2, \\
&\le C(\epsilon)\Big(\frac{1}{c_1}\|\,|\nabla_g u^{\eta m}|_g\|_2^2 + 1\Big) + \epsilon C_{\rho+2,2}^2\|u_t^{\eta m}\|_{\rho+2,\Gamma_1}^2 \\
&\le C(\epsilon)\Big(\frac{1}{c_1}\|\,|\nabla_g u^{\eta m}|_g\|_2^2 + 1\Big) + \epsilon 2^{\rho+1}C_{\rho+2,2}^2\big(1 + \|u_t^{\eta m}\|_{\rho+2,\Gamma_1}^{\rho+2}\big),
\end{aligned}
\tag{3.5}
$$

where $C_{\rho+2,2}$ is an imbedding constant. Replacing (3.3), (3.4) and (3.5) in (3.2) we obtain

$$
\begin{aligned}
&\frac{d}{dt}\Big[\frac{1}{2}\|u_t^{\eta m}\|_2^2 + \frac{1}{2}\mu(t)\|\,|\nabla_g u^{\eta m}|_g\|_2^2 + \frac{1}{\gamma+2}\|u^{\eta m}\|_{\gamma+2,\Gamma_1}^{\gamma+2}\Big] \\
&+ \eta\|u_t^{\eta m}\|_{2,\Gamma_1}^2 + \Big(c_3 - \epsilon_0 - \epsilon 2^{\rho+1}C_{\rho+2,2}^2\Big)\|u_t^{\eta m}\|_{\rho+2,\Gamma_1}^{\rho+2} \\
&\le c_3\,\mathrm{meas}(\Gamma_1) + C(\epsilon_0) + C(\epsilon) + \epsilon 2^{\rho+1}C_{\rho+2,2}^2 \\
&\quad + \Big(\frac{1}{2}\mu'(t) + C(\epsilon_0) + \frac{C(\epsilon)}{c_1}\Big)\|\,|\nabla_g u^{\eta m}|_g\|_2^2 + \frac{1}{2}\|f(t)\|_2^2 + \frac{1}{2}\|u_t^{\eta m}\|_2^2.
\end{aligned}
\tag{3.6}
$$

By integrating (3.6) over $(0,t)$ with $t \in (0,t_m)$ we have

$$
\begin{aligned}
&\frac{1}{2}\|u_t^{\eta m}\|_2^2 + \frac{1}{2}\mu(t)\|\,|\nabla_g u^{\eta m}|_g\|_2^2 + \frac{1}{\gamma+2}\|u^{\eta m}\|_{\gamma+2,\Gamma_1}^{\gamma+2} + \eta\int_0^t \|u_s^{\eta m}\|_{2,\Gamma_1}^2\,ds \\
&+ \Big(c_3 - -\epsilon_0 - \epsilon 2^{\rho+1}C_{\rho+2,2}^2\Big)\int_0^t \|u_s^{\eta m}\|_{\rho+2,\Gamma_1}^{\rho+2}\,ds \\
&\le \Big(c_3\,\mathrm{meas}(\Gamma_1) + C(\epsilon_0) + C(\epsilon) + \epsilon 2^{\rho+1}C_{\rho+2,2}^2\Big)T \\
&\quad + \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\mu(0)\|\,|\nabla_g u_0|_g\|_2^2 + \frac{1}{\gamma+2}\|u_0\|_{\gamma+2,\Gamma_1}^{\gamma+2} \\
&\quad + \Big(\frac{1}{2}\|\mu'\|_{L^\infty(0,T)} + C(\epsilon_0) + \frac{C(\epsilon)}{c_1}\Big)\int_0^t \|\,|\nabla_g u^{\eta m}(s)|_g\|_2^2\,ds \\
&\quad + \frac{1}{2}\int_0^t \|f(s)\|_2^2\,ds + \frac{1}{2}\int_0^t \|u_s^{\eta m}(s)\|_2^2\,ds.
\end{aligned}
\tag{3.7}
$$

Therefore, choosing $\epsilon_0 = c_3/4$ and $\epsilon = \frac{c_3}{2^{\rho+3}C_{\rho+2,2}^2}$ and then by Gronwall's lemma we obtain

$$
\|u_t^{\eta m}\|_2^2 + \|\,|\nabla_g u^{\eta m}|_g\|_2^2 + \|u^{\eta m}\|_{\gamma+2,\Gamma_1}^{\gamma+2} + \int_0^t \|u_s^{\eta m}(s)\|_{2,\Gamma_1}^2\,ds + \int_0^t \|u_s^{\eta m}(s)\|_{\rho+2,\Gamma_1}^{\rho+2}\,ds \le C_5, \tag{3.8}
$$

where C_5 is a positive constant which is independent of m, η, and t. Estimate (3.8) implies that

$$
u^{\eta m} \quad \text{is uniformly bounded in} \quad L^\infty(0,T;\mathcal{H}), \tag{3.9}
$$

$$
u_t^{\eta m} \quad \text{is uniformly bounded in} \quad L^\infty(0,T;L^2(\Omega)). \tag{3.10}
$$

We note that from (3.8), taking the hypotheses on q into account we also obtain

$$
\int_0^t\int_{\Gamma_1} |q(u_s^{\eta m}(s))|^2\,d\Gamma\,ds \le C_6, \tag{3.11}
$$

where C_6 is a positive constant independent of m, η and t.

From (3.8)-(3.11), there exists a subsequence of $\{u^{\eta m}\}$, which we still denote by $\{u^{\eta m}\}$, such that

$$
\begin{gathered}
u^{\eta m} \to u^\eta \quad \text{weak star in } L^\infty(0,T;\mathcal{H}), \\
u_t^{\eta m} \to u_t^\eta \quad \text{weakly star in} \quad L^\infty(0,T;L^2(\Omega)), \\
u^{\eta m} \to u^\eta \quad \text{weakly in} \quad L^2(0,T;\mathcal{H}),
\end{gathered}
$$

$$
\begin{gathered}
u_t^{\eta m} \to u_t^\eta \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)),\\
u_t^{\eta m} \to u_t^\eta \quad \text{weakly in} \quad L^2(0,T;L^2(\Gamma_1)),\\
u_{tt}^{\eta m} \to u_{tt}^\eta \quad \text{weakly in} \quad L^2(0,T;H^{-1}(\Omega)),\\
q(u_t^{\eta m}) \to \psi \quad \text{weakly in} \quad L^2(0,T;L^2(\Gamma_1)).
\end{gathered}
$$

Since $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ is compact, we have, thanks to Aubin-Lions Theorem that

$$
u^{\eta m} \to u^\eta \quad \text{strongly in} \quad L^2(0,T;L^2(\Gamma_1)),
$$

and consequently, by using the Lions lemma, we deduce

$$
h(u^{\eta m}) \to h(u^\eta) \quad \text{weakly in} \quad L^2(0,T;L^2(\Gamma_1)).
$$

The above convergences permit us to pass to the limit in the (3.1). Since $\{w_j\}$ is a basis of $H_0^1(\Omega)$ and V_m is dense in $H_0^1(\Omega)$, after passing to the limit we obtain

$$
\begin{aligned}
&\int_0^T \int_\Omega u_{tt}^\eta, v dx\theta(t)\,dt + \int_0^T \mu(t)\int_\Omega \langle \nabla_g u^\eta, \nabla_g v\rangle_g dx\theta(t)\,dt + \eta\int_0^T\int_{\Gamma_1} u_t^\eta v\,d\Gamma\theta(t)\,dt\\
&+ \int_0^T\int_{\Gamma_1} \psi v\,d\Gamma\theta(t)\,dt - \int_0^T\int_{\Gamma_1} h(u^\eta)v dx\theta(t)\,dt\\
&= \int_0^T\int_\Omega f(x,t)v dx\theta(t)\,dt,
\end{aligned}
\tag{3.12}
$$

for all $\theta \in D(0,T)$ and $v \in H_0^1(\Omega)$.

Since estimates (3.8) and (3.11) are also independent of η, we can pass to the limit as $\eta \to 0$ in u^η obtaining a function u by the same argument used to obtain u^η from $u^{\eta m}$, such that

$$
u^\eta \to u \quad \text{weakly in } L^2(0,T;\mathcal{H}), \tag{3.13}
$$
$$
u_t^\eta \to u_t \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \tag{3.14}
$$
$$
u_t^\eta \to u_t \quad \text{weakly in } L^2(0,T;L^2(\Gamma_1)), \tag{3.15}
$$
$$
u_{tt}^\eta \to u_{tt} \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)), \tag{3.16}
$$
$$
q(u_t^\eta) \to \psi \quad \text{weakly in } L^2(0,T;L^2(\Gamma_1)), \tag{3.17}
$$
$$
h(u_\eta) \to h(u) \quad \text{weakly in } L^2(0,T;L^2(\Gamma_1)). \tag{3.18}
$$

By the above convergences in (3.12), we have

$$
\begin{aligned}
&\int_0^T \int_\Omega u_{tt}, v dx\theta(t)\,dt + \int_0^T \mu(t)\int_\Omega \langle \nabla_g u, \nabla_g v\rangle_g dx\theta(t)\,dt\\
&+ \int_0^T\int_{\Gamma_1} \psi v\,d\Gamma\theta(t)\,dt - \int_0^T\int_{\Gamma_1} h(u)v dx\theta(t)\,dt\\
&= \int_0^T\int_\Omega f(x,t)v dx\theta(t)\,dt.
\end{aligned}
\tag{3.19}
$$

From (3.19) and taking $v \in D(\Omega)$, we conclude that

$$
u_{tt} - \mu(t)Lu = f \quad \text{in } D'(\Omega \times (0,T)) \tag{3.20}
$$

and since (3.17) and (3.18), it holds that

$$
\mu(t)\frac{\partial u}{\partial \nu} + \psi = h(u) \quad \text{in } L^2(0,T;L^2(\Gamma_1)).
$$

Our goal is to show that $\psi = q(u_t)$. Indeed, considering $w = u^{\eta m}$ in (3.1) and then integrating over $(0,T)$, we have

$$
\int_0^T \langle u_{tt}^{\eta m}, u^{\eta m}\rangle\,dt + \int_0^T \mu(t)\|\,|\nabla_g u^{\eta m}|_g\|_2^2\,dt + \eta\int_0^T \langle u_t^{\eta m}, u^{\eta m}\rangle_{\Gamma_1}\,dt
$$

$$
\begin{aligned}
&+\int_0^T \langle q(u_t^{\eta m}), u^{\eta m}\rangle_{\Gamma_1}\,dt-\int_0^T \langle h(u^{\eta m}), u^{\eta m}\rangle_{\Gamma_1}\,dt\\
&=\int_0^T \langle f, u^{\eta m}\rangle\,dt.
\end{aligned}
$$

Then from convergences (3.13)-(3.18) we obtain

$$
\begin{aligned}
&\lim_{m\to\infty,\eta\to 0}\int_0^T \mu(t)\|\,|\nabla_g u^{\eta m}|_g\|_2^2\,dt\\
&=-\int_0^T \langle u_{tt}, u\rangle\,dt-\int_0^T \langle \psi, u\rangle_{\Gamma_1}\,dt+\int_0^T \langle h(u), u\rangle_{\Gamma_1}\,dt+\int_0^T \langle f, u\rangle\,dt.
\end{aligned}
\tag{3.21}
$$

By combining (3.20) and (3.21), we have

$$
\lim_{m\to\infty,\eta\to 0}\int_0^T \mu(t)\|\,|\nabla_g u^{\eta m}|_g\|_2^2\,dt=\int_0^T \mu(t)\|\,|\nabla_g u|_g\|_2^2\,dt,
$$

which implies that

$$
|\nabla_g u^{\eta m}|_g\to|\nabla_g u|_g\quad\text{strongly in } L^2(0,T;L^2(\Omega)).
\tag{3.22}
$$

Next, considering $w=u_t^{\eta m}$ in (3.1) and then integrating over $(0,T)$, we have

$$
\begin{aligned}
&\int_0^T \langle u_{tt}^{\eta m}, u_t^{\eta m}\rangle\,dt+\int_0^T \mu(t)\int_\Omega \langle\nabla_g u^{\eta m},\nabla_g u_t^{\eta m}\rangle_g\,dx\,dt+\eta\int_0^T \|u_t^{\eta m}\|_{2,\Gamma_1}^2\,dt\\
&+\int_0^T \langle q(u_t^{\eta m}), u_t^{\eta m}\rangle_{\Gamma_1}\,dt-\int_0^T \langle h(u^{\eta m}), u_t^{\eta m}\rangle_{\Gamma_1}\,dt\\
&=\int_0^T \langle f, u_t^{\eta m}\rangle\,dt.
\end{aligned}
$$

From (3.14)-(3.18) and (3.22), we arrive at

$$
\lim_{m\to\infty,\eta\to 0}\int_0^T \langle q(u_t^{\eta m}), u_t^{\eta m}\rangle_{\Gamma_1}\,dt=\int_0^T \langle\psi, u_t\rangle_{\Gamma_1}\,dt.
\tag{3.23}
$$

On the other hand, since q is a nondecreasing monotone function, we obtain

$$
\int_0^T \langle q(u_t^{\eta m})-q(\varphi), u_t^{\eta m}-\varphi\rangle_{\Gamma_1}\,dt\ge 0
$$

for all $\varphi\in L^2(\Gamma_1)$. Thus, it implies that

$$
\int_0^T \langle q(u_t^{\eta m}),\varphi\rangle_{\Gamma_1}\,dt+\int_0^T \langle q(\varphi), u_t^{\eta m}-\varphi\rangle_{\Gamma_1}\,dt\le\int_0^T \langle q(u_t^{\eta m}), u_t^{\eta m}\rangle_{\Gamma_1}\,dt.
$$

By considering (3.15), (3.17) and (3.23), we obtain

$$
\int_0^T \langle\psi-q(\varphi), u_t-\varphi\rangle_{\Gamma_1}\,dt\ge 0,
\tag{3.24}
$$

which implies that $\psi=q(u_t)$.

We now show the uniqueness of the solution. Let u^1 and u^2 be two solutions of problem (1.1). Then $z=u^1-u^2$ satisfies

$$
\int_\Omega z_{tt}w dx+\mu(t)\int_\Omega\langle\nabla_g z,\nabla_g w\rangle_g dx+\int_{\Gamma_1}(q(u_t^1)-q(u_t^2))w\,d\Gamma=\int_{\Gamma_1}(h(u^1)-h(u^2)w\,d\Gamma\,dt,
$$

for all $w\in\mathcal{H}$. By replacing $w=z_t$ in above identity and observing that q is monotonically nondecreasing and $h:H_0^1(\Omega)\to L^2(\Gamma_1)$ is globally Lipschitz, it holds that

$$
\frac{d}{dt}\Big[\frac{1}{2}\|z_t\|_2^2+\frac{1}{2}\mu(t)\|\,|\nabla_g z|_g\|_2^2\Big]\le C_7\|\,|\nabla_g z|_g\|_2^2,
$$

Where C_7 is for some positive constant. By integrating from 0 to t and using Gronwall's Lemma, we conclude that $\|z_t\|_2=\|\,|\nabla_g z|_g\|_2=0$.

3.2. **Locally Lipschitz source.** In this subsection, we loosen the globally Lipschitz condition on the source by allowing h to be locally Lipschitz continuous. More precisely, we have the following result.

Proposition 3.2. *Assume that* (H!)-(H4) *hold. In addition, assume that* $(u_0, u_1) \in \mathcal{H} \times L^2(\Omega)$ *and* $h : H_0^1(\Omega) \to L^2(\Gamma_1)$ *is locally Lipschitz continuous satisfying* $c_5|s|^{\gamma+1} \le |h(s)| \le c_6|s|^{\gamma+1}$, *where* c_5, c_6 *are for some positive constants. Then problem* (1.1) *has a unique local solution* $u \in C_w(0,T;\mathcal{H}) \cap C_w^1(0,T;L^2(\Omega))$ *for some* $T > 0$.

Proof. We define

$$h_K(u) = \begin{cases} h(u) & \text{if } \|\,|\nabla_g u|_g\|_2 \le K, \\ h\big(\frac{Ku}{\|\,|\nabla_g u|_g\|_2}\big) & \text{if } \|\,|\nabla_g u|_g\|_2 > K, \end{cases}$$

where K is a positive constant. With this truncated h_K, we consider the problem

$$\begin{gathered} u_{tt} - \mu(t)Lu = f(x,t) \quad \text{in } \Omega \times (0,+\infty), \\ u = 0 \quad \text{on } \Gamma_0 \times (0,+\infty), \\ \mu(t)\frac{\partial u}{\partial \nu_L} + q(u_t) = h_K(u) \quad \text{on } \Gamma_1 \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \end{gathered} \tag{3.25}$$

Since $h_K : H_0^1(\Omega) \to L^2(\Gamma_1)$ is globally Lipschitz continuous for each K (see [10]), then by Proposition 3.1, the truncated problem (3.25) has a unique global solution $u_K \in C_w(0,T;\mathcal{H}) \cap C_w^1(0,T;L^2(\Omega))$ for any $T > 0$. Moreover by [30] there exists a sequence of functions u_K^l, which converges to u_K in the class $C_w(0,T;H^2(\Omega)) \cap C_w^1(0,T;H_0^1(\Omega))$. For simplifying the notation in the rest of the proof, we shall express u_K^l as u.

By the regularity of u, we can multiply (3.25) by u_t and integrate on $\Omega \times (0,t)$, where $0 < t < T$. Then we obtain by using the fact $\mu'(s) < 0$ for all $s > 0$,

$$\begin{aligned} &\frac{1}{2}(\|u_t\|_2^2 + \mu(t)\|\,|\nabla_g u|_g\|_2^2) + \frac{1}{\gamma+2}\|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} + \int_0^t \int_{\Gamma_1} q(u_s(x,s))u_s(x,s)\,d\Gamma\,ds \\ &\le \frac{1}{2}(\|u_1\|_2^2 + \mu(0)\|\,|\nabla_g u_0|_g\|_2^2) + \frac{1}{\gamma+2}\|u_0\|_{\gamma+2,\Gamma_1}^{\gamma+2} + \int_0^t \int_\Omega f(x,s)u_s(x,s)dx\,ds \\ &\quad + \Big(1 + \frac{1}{c_5}\Big)\int_0^t \int_{\Gamma_1} |h_K(u(x,s))|\,|u_s(x,s)|\,d\Gamma\,ds. \end{aligned} \tag{3.26}$$

We note that $h_K : H_0^1(\Omega) \to L^{\frac{\rho+2}{\rho+1}}(\Gamma_1)$ is globally Lipschitz with Lipschitz constant $L_h(K)$ (see [10, 13]). Hence we estimate the last term on the right-hand side of (3.26) as

$$\begin{aligned} &\Big(1 + \frac{1}{c_5}\Big)\int_0^t \int_{\Gamma_1} |h_K(u(x,s))|\,|u_s(x,s)|\,d\Gamma\,ds \\ &\le \Big(1 + \frac{1}{c_5}\Big)\int_0^t \|h_K(u(s))\|_{\frac{\rho+2}{\rho+1},\Gamma_1}\|u_s\|_{\rho+2,\Gamma_1}\,ds \\ &\le \epsilon_1\int_0^t \|u_s(s)\|_{\rho+2,\Gamma_1}^{\rho+2}\,ds + C(\epsilon_1)\int_0^t \|h_K(u(s))\|_{\frac{\rho+2}{\rho+1},\Gamma_1}^{\frac{\rho+2}{\rho+1}}\,ds \\ &\le \epsilon_1\int_0^t \|u_s(s)\|_{\rho+2,\Gamma_1}^{\rho+2}\,ds + C(\epsilon_1)\Big(\frac{2}{\sqrt{c_1}}L_h(K)\Big)^{\frac{\rho+2}{\rho+1}}\int_0^t \|\,|\nabla_g u|_g\|_2^2\,ds \\ &\quad + tC(\epsilon_1)\Big(\Big(\frac{2}{\sqrt{c_1}}L_h(K)\Big)^{\frac{\rho+2}{\rho+1}} + 2^{-(\rho+1)}|h(0)|^{\frac{\rho+2}{\rho+1}}\operatorname{meas}(\Gamma_1)\Big). \end{aligned} \tag{3.27}$$

From the hypothesis on q, we have

$$\int_0^t \int_{\Gamma_1} q(u_s(x,s))u_s(x,s)\,d\Gamma\,ds \ge c_3\int_0^t \|u_s(s)\|_{\rho+2,\Gamma_1}^{\rho+2}\,ds - tc_3\operatorname{meas}(\Gamma_1). \tag{3.28}$$

By replacing (3.27) and (3.28) in (3.26) and choosing $\epsilon_1 \le c_3$, we obtain

$$\begin{aligned} &\|u_t\|_2^2 + \| \, |\nabla_g u|_g\|_2^2 + \|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} + \int_0^t \|u_s(s)\|_{\rho+2,\Gamma_1}^{\rho+2}\,ds \\ &\le C_9 + C_1(L_h(K))T + C_2(L_h(K)) \int_0^t \|u_s(s)\|_2^2 + \| \, |\nabla_g u(s)|_g\|_2^2\,ds \end{aligned} \tag{3.29}$$

for all $t \in [0,T]$, where

$$C_9 = \frac{1}{2C_8}(\|u_1\|_2^2 + \mu(0)\| \, |\nabla_g u_0|_g\|_2^2 + \|u_0\|_{\gamma+2,\Gamma_1}^{\gamma+2} + \|f\|_{L^2(0,T;L^2(\Omega))}),$$

$$C_1(L_h(K)) = \frac{1}{C_8} C(\epsilon_1)\Big(\Big(\frac{2}{\sqrt{c_1}} L_h(K)\Big)^{\frac{\rho+2}{\rho+1}} + 2^{-(\rho+1)}|h(0)|^{\frac{\rho+2}{\rho+1}} \operatorname{meas}(\Gamma_1)\Big) + \frac{1}{C_8} c_3 \operatorname{meas}(\Gamma_1),$$

$$C_2(L_h(K)) = \frac{1}{C_8}\Big(C(\epsilon_1)\Big(\frac{2}{\sqrt{c_1}} L_h(K)\Big)^{\frac{\rho+2}{\rho+1}} + \frac{1}{2}\Big),$$

for $C_8 = \min\{\frac{1}{\gamma+2}, \frac{\mu_0}{2}, c_3 - \epsilon_1\}$. Thus by Gronwall's inequality, (3.29) becomes

$$\|u_t\|_2^2 + \| \, |\nabla_g u|_g\|_2^2 + \|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} + \int_0^t \|u_s(s)\|_{\rho+2,\Gamma_1}^{\rho+2}\,ds \le (C_9 + C_1(L_h(K))T) e^{C_2(L_h(K))t},$$

for all $t \in [0,T]$. If we choose

$$T = \min\Big\{\frac{1}{C_1(L_h(K))}, \frac{1}{C_2(L_h(K))} \ln 2\Big\}, \tag{3.30}$$

then

$$\|u_t\|_2^2 + \| \, |\nabla_g u|_g\|_2^2 + \|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} + \int_0^t \|u_s(s)\|_{\rho+2,\Gamma_1}^{\rho+2}\,ds \le 2(C_9+1) \le K^2 \quad \text{for all} \quad t \in [0,T], \tag{3.31}$$

provided we choose $K^2 \ge 2(C_9 + 1)$. Consequently, (3.31) gives us that $\| \, |\nabla_g u|_g\|_2 \le K$ for all $t \in [0,T]$. Therefore, by the definition of h_K, we have that $h_K(u) = h(u)$ on $[0,T]$. By the uniqueness of solutions, the solution of the truncated problem (3.25) accords with the solution of the original, non-truncated problem (1.1) for $t \in [0,T]$, which means that the proof is complete. □

3.3. **Completion of the proof for the local existence.** To establish the existence of solutions, we need to extend the result in Proposition 3.2 where the source h is locally Lipschitz from $H_0^1(\Omega)$ into $L^{\frac{\rho+2}{\rho+1}}(\Gamma_1)$. For the construction of the Lipschitz approximation for the source, we employ another truncated function introduced in [40]. Let $\delta_n \in C_0^\infty(\mathbb{R})$ be a cut off function such that

$$0 \le \delta_n \le 1, \quad \delta_n(s) = \begin{cases} 1, & \text{if } |s| \le n, \\ 0, & \text{if } |s| \ge 2n, \end{cases}$$

and $|\delta_n'(s)| \le \frac{C}{n}$ for some constant C independent from n. We also define

$$h_n(u) = h(u)\delta_n(u). \tag{3.32}$$

Then the truncated function h_n satisfies the following lemma. The proof is a routine series of estimates as in [1, 13], so we omit it.

Lemma 3.3. *The following statements hold.*

(1) $h_n : H_0^1(\Omega) \to L^2(\Gamma_1)$ *is globally Lipschitz continuous.*
(2) $h_n : H_0^{1-\epsilon}(\Omega) \to L^{\frac{\rho+2}{\rho+1}}(\Gamma_1)$ *is locally Lipschitz continuous with Lipschitz constant independent of* n.

With the truncated source h_n defined in (3.32), by Proposition 3.2 and Lemma 3.3, we have a unique local solution $u^n \in C_w(0,T;\mathcal{H}) \cap C_w^1(0,T;L^2(\Omega))$ satisfying the following approximation of (1.1)

$$
\begin{gathered}
u_{tt}-\mu(t)Lu=f(x,t) \quad \text{in } \Omega\times(0,+\infty),\\
u=0 \quad \text{on } \Gamma_0\times(0,+\infty),\\
\mu(t)\frac{\partial u}{\partial \nu_L}+q(u_t)=h_n(u) \quad \text{on } \Gamma_1\times(0,+\infty),\\
u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x).
\end{gathered} \tag{3.33}
$$

From Lemma 3.3, the life span T of each solution u^n, given in (3.30), is independent of n since the local Lipschitz constant of the mapping $h_n : H_0^1(\Omega)\to L^{\frac{\rho+2}{\rho+1}}(\Gamma_1)$ is independent of n. Also we known that T depends on K, where $K^2\ge 2(C_9+1)$, however, since $\|u_1^n\|_2^2+\|\,|\nabla_g u_0^n|_g\|_2^2+\|u_0^n\|_{\gamma+2,\Gamma_1}^{\gamma+2}\to\|u_1\|_2^2+\|\,|\nabla_g u_0|_g\|_2^2+\|u_0\|_{\gamma+2,\Gamma_1}^{\gamma+2}$, we can choose K sufficiently large so that K is independent of n. By (3.31),

$$
\|u_t^n\|_2^2+\|\,|\nabla_g u^n|_g\|_2^2+\|u^n\|_{\gamma+2,\Gamma_1}^{\gamma+2}\le K^2 \tag{3.34}
$$

for all $t\in[0,T]$. Therefore, there exists a function u and a subsequence of $\{u^n\}$, which we still denote by $\{u^n\}$, such that

$$
u^n\to u \quad \text{weak star in } L^\infty(0,T;\mathcal{H}), \tag{3.35}
$$

$$
u_t^n\to u_t \quad \text{weak star in } L^\infty(0,T;L^2(\Omega)). \tag{3.36}
$$

By (3.34), (3.35) and (3.36), we infer that

$$
\|u_t\|_2^2+\|\,|\nabla_g u|_g\|_2^2+\|u\|_{\gamma+2,\Gamma_1}^{\gamma+2}\le K^2 \tag{3.37}
$$

for all $t\in[0,T]$. Moreover, by Aubin-Lions Theorem, we have

$$
u^n\to u \quad \text{strongly in } L^\infty(0,T;H^{1-\epsilon}(\Omega)), \tag{3.38}
$$

for $0<\epsilon<1$. Since u^n is a solution of (3.33), it holds that

$$
\begin{aligned}
&\int_0^T\int_\Omega u_{tt}^n,\phi\,dx\,dt+\int_0^T\mu(t)\int_\Omega\langle\nabla_g u^n,\nabla_g\phi\rangle_g\,dx\,dt+\int_0^T\int_{\Gamma_1}q(u_t^n)\phi\,d\Gamma\,dt\\
&=\int_0^T\int_\Omega f(x,t)\phi\,dx\,dt+\int_0^T\int_{\Gamma_1}h_n(u^n)\phi\,d\Gamma\,dt,
\end{aligned} \tag{3.39}
$$

for any $\phi\in C_w(0,T;\mathcal{H})\cap C_w^1(0,T;L^2(\Omega))$, $\phi\in L^{\rho+2}(0,T;\Gamma_1)$.

Now we show that

$$
\lim_{n\to\infty}\int_0^T\int_{\Gamma_1}h_n(u^n)\phi\,d\Gamma\,dt=\int_0^T\int_{\Gamma_1}h(u)\phi\,d\Gamma\,dt. \tag{3.40}
$$

Indeed, we have

$$
\begin{aligned}
&\Big|\int_0^T\int_{\Gamma_1}(h_n(u^n)-h(u))\phi\,d\Gamma\,dt\Big|\\
&\le\int_0^T\int_{\Gamma_1}|h_n(u^n)-h_n(u)|\,|\phi|\,d\Gamma\,dt+\int_0^T\int_{\Gamma_1}|h_n(u)-h(u)|\,|\phi|\,d\Gamma\,dt.
\end{aligned} \tag{3.41}
$$

By (2) in Lemma 3.3 and (3.38), we obtain

$$
\begin{aligned}
&\int_0^T\int_{\Gamma_1}|h_n(u^n)-h_n(u)|\,|\phi|\,d\Gamma\,dt\\
&\le\Big(\int_0^T\int_{\Gamma_1}|h_n(u^n)-h_n(u)|^{\frac{\rho+2}{\rho+1}}\,d\Gamma\,dt\Big)^{\frac{\rho+1}{\rho+2}}\Big(\int_0^T\int_{\Gamma_1}|\phi|^{\rho+2}\,d\Gamma\,dt\Big)^{\frac{1}{\rho+2}}\\
&\le C(K)\|\phi\|_{L^{\rho+2}(0,T;\Gamma_1)}\Big(\int_0^T\|u^n-u\|_{H^{1-\epsilon}(\Omega)}^{\frac{\rho+2}{\rho+1}}\,dt\Big)^{\frac{\rho+1}{\rho+2}}\to 0.
\end{aligned} \tag{3.42}
$$

Since $\delta_n(u(x)) \to 1$ a.e. in Ω, we have $h_n(u) \to h(u)$ a.e. Then we also have $|h_n(u)-h(u)|^{\frac{\rho+2}{\rho+1}} \leq 2^{\frac{\rho+2}{\rho+1}}|h(u)|^{\frac{\rho+2}{\rho+1}}$ and $h(u) \in L^{\frac{\rho+2}{\rho+1}}(\Gamma_1)$, for $u \in H_0^1(\Omega)$. Thus by the Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned}
&\int_0^T \int_{\Gamma_1} |h_n(u) - h(u)|\ |\phi|\, d\Gamma\, dt \\
&\leq \Big(\int_0^T \int_{\Gamma_1} |h_n(u) - h(u)|^{\frac{\rho+2}{\rho+1}}\, d\Gamma\, dt\Big)^{\frac{\rho+1}{\rho+2}} \Big(\int_0^T \int_{\Gamma_1} |\phi|^{\rho+2}\, d\Gamma\, dt\Big)^{\frac{1}{\rho+2}} \\
&\leq \|\phi\|_{L^{\rho+2}(0,T;\Gamma_1)} \Big(\int_0^T \int_{\Gamma_1} |h(u)|^{\frac{\rho+2}{\rho+1}} |\delta_n(u) - 1|^{\frac{\rho+2}{\rho+1}}\, d\Gamma\, dt\Big)^{\frac{\rho+1}{\rho+2}} \to 0.
\end{aligned} \tag{3.43}$$

Convergences (3.42) and (3.43), (3.41) gives us (3.40).

On the other hand, by using similar arguments from (3.21) to (3.24), we obtain

$$q(u_t^n) \to q(u_t) \quad \text{weakly in } L^2(0,T;L^2(\Gamma_1)). \tag{3.44}$$

Convergences (3.36), (3.37), (3.40) and (3.44) permit us to pass to the limit in (3.39) and conclude the following result.

Proposition 3.4. *Assume that* (H!)-(H4) *hold. In addition, assume that* $(u_0, u_1) \in \mathcal{H} \times L^2(\Omega)$ *and* $h : H_0^1(\Omega) \to L^{\frac{\rho+2}{\rho+1}}(\Gamma_1)$ *is locally Lipschitz continuous. Then problem* (1.1) *has a local solution* $u \in C_w(0,T;\mathcal{H}) \cap C_w^1(0,T;L^2(\Omega))$ *for some* $T > 0$.

Let $h(u) = |u|^\gamma u$, then $h : H_0^1(\Omega) \to L^{\frac{\rho+2}{\rho+1}}(\Gamma_1)$ is locally Lipschitz continuous (see Remark 2.2). Thus by Proposition 3.4, the proof of the local existence statement in Theorem 2.4 is complete.

3.4. **Energy identity.** It is well known that to prove the uniqueness of weak solutions, we will justify the energy identity (2.17). The energy identity can be derived formally by multiplying (1.1) by u_t. But, such a calculation is not justified, since u_t is not sufficiently regular to be the test function in as required in Definition **??**. To overcome this problem, we employ the operator $T^\epsilon = (I - \epsilon L)^{-1}$ to smooth function in space, which is mentioned in Appendix A of [13]. We recall important properties of T^ϵ which play an essential role when establishing the energy identity.

Lemma 3.5 ([13])**.** *Let* $u^\epsilon = T^\epsilon u$. *Then following statements hold.*

(1) *If* $u \in L^2(\Omega)$, *then* $\|u^\epsilon\|_2 \leq \|u\|_2$ *and* $u^\epsilon \to u$ *in* $L^2(\Omega)$ *as* $\epsilon \to 0$.
(2) *If* $u \in H_0^1(\Omega)$, *then* $\|\nabla u^\epsilon\|_2 \leq \|\nabla u\|_2$ *and* $u^\epsilon \to u$ *in* $H_0^1(\Omega)$ *as* $\epsilon \to 0$.
(3) *If* $u \in L^p(\Gamma_1)$ *with* $1 < p < \infty$, *then* $\|u^\epsilon\|_{p,\Gamma_1} \leq \|u\|_{p,\Gamma_1}$ *and* $u^\epsilon \to u$ *in* $L^p(\Gamma_1)$ *as* $\epsilon \to 0$.

We will now justify the energy identity (2.17). We aapply the operator T^ϵ on every term of (1.1) and multiply by u_t^ϵ. Then we obtain by integrating in space and time

$$\begin{aligned}
&\int_0^t \int_\Omega u_{ss}^\epsilon u_s^\epsilon dx\, ds + \int_0^t \mu(s) \int_\Omega \langle \nabla_g u^\epsilon, \nabla_g u_s^\epsilon \rangle_g dx\, ds + \int_0^t \int_{\Gamma_1} T^\epsilon(q(u_s)) u_s^\epsilon\, d\Gamma\, ds \\
&= \int_0^t \int_\Omega f(x,s) u_s^\epsilon dx\, ds + \int_0^t \int_{\Gamma_1} T^\epsilon(h(u)) u_s^\epsilon\, d\Gamma\, ds.
\end{aligned} \tag{3.45}$$

Since $u \in H_0^1(\Omega)$ and $u_t \in L^2(\Omega)$, we have by Lemma 3.5, $u^\epsilon \to u$ in $H_0^1(\Omega)$ and $u_t^\epsilon \to u_t$ in $L^2(\Omega)$. Therefore using this convergences, we have

$$\begin{aligned}
&\lim_{\epsilon \to 0} \Big(\int_0^t \int_\Omega u_{ss}^\epsilon u_s^\epsilon dx\, ds + \int_0^t \mu(s) \int_\Omega \langle \nabla_g u^\epsilon, \nabla_g u_s^\epsilon \rangle_g dx\, ds\Big) \\
&= \frac{1}{2}\big(\|u_t\|_2^2 + \|\, |\nabla_g u|_g\|_2^2 - \|u_1\|_2^2 - \|\, |\nabla_g u_0|_g\|_2^2\big).
\end{aligned} \tag{3.46}$$

Since $u_t, q(u_t) \in L^2(\Gamma_1)$, we easily check that

$$\lim_{\epsilon \to 0} \int_0^t \int_{\Gamma_1} T^\epsilon(q(u_s)) u_s^\epsilon\, d\Gamma\, ds = \int_0^t \int_{\Gamma_1} (q(u_s)) u_s\, d\Gamma\, ds. \tag{3.47}$$

Recall that $u_t \in L^{\rho+2}(\Gamma_1)$ and $h(u) \in L^{\frac{\rho+2}{\rho+1}}(\Gamma_1)$. By Lemma 3.5, we have $u_t^\epsilon \to u_t$ in $L^{\rho+2}(\Gamma_1)$ and $T^\epsilon(h(u)) \to h(u)$ in $L^{\frac{\rho+2}{\rho+1}}(\Gamma_1)$. Thus by Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{\epsilon\to 0}\int_0^t\int_{\Gamma_1} T^\epsilon(h(u))u_s^\epsilon\, d\Gamma\, ds = \int_0^t\int_{\Gamma_1}(h(u))u_s\, d\Gamma\, ds. \tag{3.48}$$

Convergences (3.46)-(3.48) permit us to pass to the limit in (3.45), consequently, the energy identity (2.17) holds.

3.5. **Strong time continuity.** Our objective of this subsection is to prove the strong time continuity of the weak solution u, as stated in Propositions 3.1, 3.2, and 3.4.

Proposition 3.6. *Assume that* (H!)-(H4) *hold. Let u be a local weak solution to problem* (1.1)*. Then the local weak solution u enjoys strong time continuity, i.e.,*

$$u \in C(0,T;\mathcal{H}) \cap C^1(0,T;L^2(\Omega)).$$

Proof. To establish strong continuity, i.e., $u \in C(0,T;\mathcal{H}) \cap C^1(0,T;L^2(\Omega))$, it is sufficient to show that the corresponding norm functions, $t \mapsto \|u(t)\|_{\mathcal{H}}$ and $t \mapsto \|u_t(t)\|_2$, are continuous on $[0,T]$.

The energy identity (2.17) holds for any $s,t \in [0,T]$,

$$E(t)-E(s) = \int_s^t \Big(\frac{1}{2}\mu'(\tau)\|\,|\nabla_g u(\tau)|_g\|_2^2 d\tau - \int_{\Gamma_1} q(u_\tau)u_\tau\, d\Gamma d\tau + \int_\Omega f(x,\tau)u_\tau dx\Big)d\tau.$$

From the a priori estimates derived during the proof of local existence, we have established that $u \in L^\infty(0,T;\mathcal{H})$, $u_t \in L^\infty(0,T;L^2(\Omega))$, and $u_t \in L^{\rho+2}(0,T;\Gamma_1)$. These regularity results, combined with the conditions on μ, f and q, ensure that the integrand on the right-hand side belongs to $L^1(0,T)$. A function defined by the integral of an L^1 function is absolutely continuous, and therefore continuous. Thus, we conclude that $E(t)$ is a continuous function on $[0,T]$.

Let us define the functional

$$\Psi(t) := \|u_t(t)\|_2^2 + \mu(t)\|\,|\nabla_g u(t)|_g\|_2^2.$$

Using the definition of the energy $E(t)$, we can write $\Psi(t)$ as

$$\Psi(t) = 2E(t) + \frac{2}{\gamma+2}\|u(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2}.$$

Since $E(t)$ is continuous, the continuity of $\Psi(t)$ depends on the continuity of the norm function $t \mapsto \|u(t)\|_{\gamma+2,\Gamma_1}$. To prove this, we use a standard compactness argument. We know that $u \in L^\infty(0,T;H_0^1(\Omega))$ and $u_t \in L^\infty(0,T;L^2(\Omega))$. By the Aubin-Lions lemma,

$$u \in C(0,T;L^2(\Omega)).$$

Similarly, by the compact Sobolev trace imbedding $H_0^1(\Omega) \hookrightarrow L^q(\Gamma_1)$ for $q < \frac{2(n-1)}{n-2}$, we have

$$u \in C(0,T;L^q(\Gamma_1))$$

for any such q. Therefore since $u \in L^\infty(0,T;L^{\gamma+2}(\Gamma_1))$, a standard interpolation argument shows that the mapping $t \mapsto u(t)$ is also strongly continuous in the intermediate space $L^{\gamma+2}(\Gamma_1)$. Thus, the norm function $t \mapsto \|u(t)\|_{\gamma+2,\Gamma_1}$ is continuous. Since $\Psi(t)$ is a sum and product of continuous functions, we conclude that $\Psi(t)$ is continuous on $[0,T]$.

By Proposition 3.4, we have $u_t \in C_w(0,T;L^2(\Omega))$ and $u \in C_w(0,T;\mathcal{H})$. On the other hand, the property of weak convergence implies that norms are lower semi-continuous. For any $s \in [0,T]$:

$$\|u_t(s)\|_2^2 \le \liminf_{t\to s}\|u_t(t)\|_2^2 \quad\text{and}\quad \|\,|\nabla_g u(s)|_g\|_2^2 \le \liminf_{t\to s}\|\,|\nabla_g u(t)|_g\|_2^2.$$

This implies that

$$\Psi(s) = \|u_t(s)\|_2^2 + \mu(s)\|\,|\nabla_g u(s)|_g\|_2^2 \le \liminf_{t\to s}\Psi(t).$$

However, we proved that $\Psi(t)$ is continuous, which means

$$\lim_{t\to s}\Psi(t) = \Psi(s).$$

This requires $\limsup_{t\to s}\Psi(t)=\Psi(s)$ as well. The only way for the inequality from lower semicontinuity and the equality from continuity to both hold is if the limits of the individual norm terms also hold, i.e.,

$$\lim_{t\to s}\|u_t(t)\|_2^2=\|u_t(s)\|_2^2 \quad\text{and}\quad \lim_{t\to s}\|\,|\nabla_g u(t)|_g\|_2^2=\|\,|\nabla_g u(s)|_g\|_2^2.$$

The combination of weak continuity and norm continuity implies strong continuity. Therefore,

$$u\in C^1(0,T;L^2(\Omega)) \quad\text{and}\quad u\in C(0,T;H_0^1(\Omega)).$$

Since u is strongly continuous in both $H_0^1(\Omega)$ and $L^{\gamma+2}(\Gamma_1)$, it is strongly continuous in the sum space $\mathcal{H}$. This completes the proof that $u\in C(0,T;\mathcal{H})\cap C^1(0,T;L^2(\Omega))$. □

4. Proof of Theorem 2.4: global existence

In this section we prove that a local weak solution u on $[0,T]$ can be extended to $[0,\infty)$. From the standard continuation argument of ODE theory, it suffices to show that $\|u_t\|_2^2+\|\,|\nabla_g u|_g\|_2^2+\|u\|_{\gamma+2,\Gamma_1}^{\gamma+2}$ is bounded independent of t. We now consider the following two cases:

4.1. $\rho\ge\gamma$**.** Using the energy identity (2.17), we obtain

$$\begin{aligned}&\frac{d}{dt}\Big[\frac12\|u_t\|_2^2+\frac12\mu(t)\|\,|\nabla_g u|_g\|_2^2+\frac{1}{\gamma+2}\|u\|_{\gamma+2,\Gamma_1}^{\gamma+2}\Big]\\&=-\int_{\Gamma_1}q(u_t)u_t\,d\Gamma+\frac12\mu'(t)\|\,|\nabla_g u|_g\|_2^2+2\int_{\Gamma_1}|u|^\gamma uu_t\,d\Gamma+\int_\Omega f(t)u_t dx.\end{aligned}\tag{4.1}$$

By the same argument as (3.3), we have

$$\int_{\Gamma_1}q(u_t)u_t\,d\Gamma\ge c_3\|u_t\|_{\rho+2,\Gamma_1}^{\rho+2}-c_3\,\mathrm{meas}(\Gamma_1).\tag{4.2}$$

Using the Hölder and Young inequalities with $\frac{\gamma+1}{\gamma+2}+\frac{1}{\gamma+2}=1$ and the imbedding $L^{\rho+2}(\Gamma_1)\hookrightarrow L^{\gamma+2}(\Gamma_1)$, we deduce that

$$\begin{aligned}2\int_{\Gamma_1}|u|^\gamma uu_t\,d\Gamma&\le C(\epsilon_2)\|u\|_{\gamma+2,\Gamma_1}^{\gamma+2}+\epsilon_2C_{\rho+2,\gamma+2}^{\gamma+2}\|u_t\|_{\rho+2,\Gamma_1}^{\gamma+2}\\&\le C(\epsilon_2)\|u\|_{\gamma+2,\Gamma_1}^{\gamma+2}+\epsilon_2 2^{\rho+1}C_{\rho+2,\gamma+2}^{\gamma+2}\big(1+\|u_t\|_{\rho+2,\Gamma_1}^{\rho+2}\big),\end{aligned}\tag{4.3}$$

where $C_{\rho+2,\gamma+2}$ is an imbedding constant. By replacing (4.2) and (4.3) in (4.1) and using the Young inequality and (2.8), we obtain

$$\begin{aligned}&\frac{d}{dt}\Big[\frac12\|u_t\|_2^2+\frac12\mu(t)\|\,|\nabla_g u|_g\|_2^2+\frac{1}{\gamma+2}\|u\|_{\gamma+2,\Gamma_1}^{\gamma+2}\Big]\\&\le\frac12\|u_t\|_2^2+C(\epsilon_2)\|u\|_{\gamma+2,\Gamma_1}^{\gamma+2}+(c_3\,\mathrm{meas}(\Gamma_1)+\epsilon_2 2^{\rho+1}C_{\rho+2,\gamma+2}^{\gamma+2}+\frac12\|f\|_2^2)\\&\quad+(\epsilon_2 2^{\rho+1}C_{\rho+2,\gamma+2}^{\gamma+2}-c_3)\|u_t\|_{\rho+2,\Gamma_1}^{\rho+2}.\end{aligned}\tag{4.4}$$

Let

$$\widetilde{E}(t)=\frac12\|u_t\|_2^2+\frac12\mu(t)\|\,|\nabla_g u|_g\|_2^2+\frac{1}{\gamma+2}\|u\|_{\gamma+2,\Gamma_1}^{\gamma+2}.$$

Choosing $\epsilon_2=\frac{c_3}{2^{\rho+1}C_{\rho+2,\gamma+2}^{\gamma+2}}$, we rewrite (4.4) as

$$\widetilde{E}'(t)\le C_{10}+C_{11}\widetilde{E}(t),$$

where C_{10} and C_{11} are positive constants. Now applying Gronwall's inequality, we have that $\widetilde{E}(t)\le(C_{12}\widetilde{E}(0)+C_{13})e^{C_{12}t}$, where C_{12} and C_{13} are positive constants. Consequently, since $\widetilde{E}(0)$ is bounded we conclude that $\|u_t\|_2^2+\|\,|\nabla_g u|_g\|_2^2+\|u\|_{\gamma+2,\Gamma_1}^{\gamma+2}$ is bounded.

4.2. **Potential well.** First of all, we will find a stable region. We set

$$0 < K_0 := \sup_{u\in\mathcal{H}, u\neq 0}\Big(\frac{\|u\|_{\gamma+2,\Gamma_1}}{\|\,|\nabla_g u|_g\|_2}\Big) < \infty$$

and the functional

$$J(u) = \frac{\mu_0}{2}\|\,|\nabla_g u|_g\|_2^2 - \frac{1}{\gamma+2}\|u\|_{\gamma+2,\Gamma_1}^{\gamma+2}, \quad u\in\mathcal{H}. \tag{4.5}$$

We also define the function, for $\lambda > 0$,

$$j(\lambda) = \frac{\mu_0}{2}\lambda^2 - \frac{1}{\gamma+2}K_0^{\gamma+2}\lambda^{\gamma+2}, \tag{4.6}$$

then

$$\lambda_0 = \Big(\frac{\mu_0}{K_0^{\gamma+2}}\Big)^{1/\gamma}$$

is the absolute maximum point of j and

$$j(\lambda_0) = \frac{\gamma\mu_0}{2(\gamma+2)}\lambda_0^2 = d_0.$$

The energy associated with problem (1.1) is

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\mu(t)\|\,|\nabla_g u(t)|_g\|_2^2 - \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2}, \tag{4.7}$$

for $u\in\mathcal{H}$. By (2.8) and (4.5)-(4.7), we deduce

$$E(t) \geq J(u(t)) \geq \frac{\mu_0}{2}\|\,|\nabla_g u(t)|_g\|_2^2 - \frac{K_0^{\gamma+2}}{\gamma+2}\|\,|\nabla_g u(t)|_g\|_2^{\gamma+2} = j(\|\,|\nabla_g u(t)|_g\|_2). \tag{4.8}$$

Lemma 4.1. *Let u be a weak solution for problem* (1.1)*. Suppose that $E(0) < d_0$ and $\|\,|\nabla_g u_0|_g\|_2 < \lambda_0$. Then*

$$\|\,|\nabla_g u(t)|_g\|_2 < \lambda_0 \quad \textit{for all} \quad t\geq 0.$$

Proof. It is easy to verify that j is increasing for $0<\lambda<\lambda_0$, decreasing for $\lambda>\lambda_0$, $j(\lambda)\to -\infty$ as $\lambda\to+\infty$. Then since $d_0 > E(0) \geq j(\|\,|\nabla_g u_0|_g\|_2) \geq j(0) = 0$, there exist $\lambda_0' < \lambda_0 < \tilde{\lambda_0}$, which satisfy

$$j(\lambda_0') = j(\tilde{\lambda_0}) = E(0). \tag{4.9}$$

Considering that $E(t)$ is non-increasing, we have

$$E(t) \leq E(0) \quad \text{for all} \quad t\geq 0. \tag{4.10}$$

From (4.8) and (4.9), we deduce that

$$j(\|\,|\nabla_g u_0|_g\|_2) \leq E(0) = j(\lambda_0'). \tag{4.11}$$

Since $\|\,|\nabla_g u_0|_g\|_2 < \lambda_0$, $\lambda_0' < \lambda_0$ and j is increasing in $[0,\lambda_0)$, from (4.11) it holds that

$$\|\,|\nabla_g u_0|_g\|_2 \leq \lambda_0'. \tag{4.12}$$

Next, we prove that

$$\|\,|\nabla_g u(t)|_g\|_2 \leq \lambda_0' \quad \text{for all } t\geq 0. \tag{4.13}$$

We argue by contradiction. Suppose that (4.13) does not hold. Then there exists time t^* which satisfies

$$\|\,|\nabla_g u(t^*)|_g\|_2 > \lambda_0'. \tag{4.14}$$

If $\|\,|\nabla_g u(t^*)|_g\|_2 < \lambda_0$, from (4.8), (4.9) and (4.14) we can write

$$E(t^*) \geq j(\|\,|\nabla_g u(t^*)|_g\|_2) > j(\lambda_0') = E(0),$$

which contradicts (4.10).

If $\|\,|\nabla_g u(t^*)|_g\|_2 \geq \lambda_0$, then, in view of (4.12), there exists $\bar{\lambda_0}$ which satisfies

$$\|\,|\nabla_g u_0|_g\|_2 \leq \lambda_0' < \bar{\lambda_0} < \lambda_0 \leq \|\,|\nabla_g u(t^*)|_g\|_2. \tag{4.15}$$

Consequently, from the continuity of the function $t \mapsto \| |\nabla_g u(t)|_g\|_2$ there exists $\bar{t} \in (0, t^*)$ satisfying
$$\| |\nabla_g u(\bar{t})|_g\|_2 = \bar{\lambda_0}. \tag{4.16}$$
Then from (4.8), (4.9), (4.15) and (4.16), we obtain
$$E(\bar{t}) \geq j(\| |\nabla_g u(\bar{t})|_g\|_2) = j(\bar{\lambda_0}) > j(\lambda_0') = E(0),$$
which also contradicts (4.10). This completes the proof. □

From (4.8) and Lemma 4.1, we arrive at
$$E(t) \geq J(u(t)) > \| |\nabla_g u(t)|_g\|_2^2 \Big(\frac{\mu_0}{2} - \frac{K_0^{\gamma+2}}{\gamma+2}\lambda_0^\gamma\Big) = \mu_0 \| |\nabla_g u(t)|_g\|_2^2 \Big(\frac{1}{2} - \frac{1}{\gamma+2}\Big) \tag{4.17}$$
and, consequently,
$$J(t) \geq 0 \ (J(t) = 0 \text{ iff } u = 0) \quad \text{and} \quad \| |\nabla_g u(t)|_g\|_2^2 \leq \frac{2(\gamma+2)}{\mu_0\gamma} E(t). \tag{4.18}$$
By (4.17), we obtain
$$J(u(t)) > \frac{\mu_0\gamma}{2(\gamma+2)} \| |\nabla_g u(t)|_g\|_2^2. \tag{4.19}$$
Hence
$$\frac{1}{2}\|u_t(t)\|_2^2 + \frac{\mu_0\gamma}{2(\gamma+2)} \| |\nabla_g u(t)|_g\|_2^2 < \frac{1}{2}\|u_t(t)\|_2^2 + J(u(t)) \leq E(t) \leq E(0).$$
Therefore, there exists a positive constant C_{14} independent of t such that
$$\|u_t(t)\|_2^2 + \| |\nabla_g u(t)|_g\|_2^2 \leq C_{14} E(0). \tag{4.20}$$
Moreover, if we define the functional
$$I(u(t)) = \mu_0 \| |\nabla_g u(t)|_g\|_2^2 - \|u(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2},$$
then from the relationship $I(u(t)) = (\gamma+2)J(u(t)) - \frac{\mu_0\gamma}{2}\| |\nabla_g u(t)|_g\|_2^2$ and the strict inequality (4.19), we obtain
$$I(u(t)) > 0 \quad \text{for all } t \geq 0. \tag{4.21}$$
Consequently, from (4.20) and (4.21) we have
$$\|u_t(t)\|_2^2 + \| |\nabla_g u(t)|_g\|_2^2 + \|u(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2} \leq (1+\mu_0) C_{14} E(0).$$
This completes the proof of the global existence of solutions of (1.1).

5. Proof of Theorem 2.5: energy decay

In this section we prove the uniform decay rates for the solution of the problem
$$\begin{gathered} u_{tt} - \mu(t) L u = 0 \quad \text{in } \Omega \times (0, +\infty), \\ u = 0 \quad \text{on } \Gamma_0 \times (0, +\infty), \\ \mu(t)\frac{\partial u}{\partial \nu_L} + q(u_t) = |u|^\gamma u \quad \text{on } \Gamma_1 \times (0, +\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \end{gathered} \tag{5.1}$$
We consider the additional hypothesis on H:
$$\sigma \leq \operatorname{div}(H) \leq \frac{\sigma(\gamma+4)}{\gamma+2}. \tag{5.2}$$
Unless otherwise stated, the constant C is a generic positive constant, different in various occurrences. We define the energy associated with problem (5.1),
$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\mu(t) \| |\nabla_g u|_g\|_2^2 - \frac{1}{\gamma+2}\|u\|_{\gamma+2,\Gamma_1}^{\gamma+2}.$$
Then
$$E'(t) = \frac{1}{2}\mu'(t) \| |\nabla_g u|_g\|_2^2 - \int_{\Gamma_1} q(u_t) u_t \, d\Gamma \leq 0,$$

it follows that $E(t)$ is a non-increasing function.

First, we recall technical lemmas which will play an essential role when establishing the asymptotic behavior.

Lemma 5.1 ([35]). *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ a strictly increasing function of class C^1 such that*

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \to +\infty \quad \text{as } t \to +\infty.$$

Assume that there exists $\sigma \geq 0$ and $\omega > 0$ such that

$$\int_S^{+\infty} E^{1+\sigma}(t)\phi'(t)\,dt \leq \frac{1}{\omega} E^\sigma(0)E(S)$$

for all $S \geq 0$. Then E has the decay properties

- *if $\sigma = 0$, then $E(t) \leq E(0)e^{1-\omega\phi(t)}$ for all $t \geq 0$,*
- *if $\sigma > 0$, then $E(t) \leq E(0)\Big(\frac{1+\sigma}{1+\omega\sigma\phi(t)}\Big)^{\frac{1}{\sigma}}$ for al $t \geq 0$.*

Lemma 5.2 ([35]). *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ a strictly increasing function of class C^1 such that*

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \to +\infty \quad \text{as } t \to +\infty.$$

Assume that there exist $\sigma > 0$, $\sigma' \geq 0$ and $C > 0$ such that

$$\int_S^{+\infty} E^{1+\sigma}(t)\phi'(t)\,dt \leq CE^{1+\sigma}(S) + \frac{C}{(1+\phi(S))^{\sigma'}} E^\sigma(0)E(S), \quad 0 \leq S < +\infty.$$

Then, there exists $C > 0$ such that

$$E(t) \leq E(0)\frac{C}{(1+\phi(t))^{(1+\sigma')/\sigma}}, \quad \forall t > 0.$$

Let us now multiply equation (5.1) by $E^p(t)\phi'(t)\mathcal{M}u$, where

$$\mathcal{M}u = 2H(u) + (\operatorname{div}(H) - \sigma)u,$$

$p \geq 0$ and $\phi : \mathbb{R} \to \mathbb{R}$ is a concave nondecreasing function of class C^2, such that $\phi(t) \to +\infty$ as $t \to +\infty$, and then integrate the obtained result over $\Omega \times [S, T]$. Then we have

$$\begin{aligned}
0 &= \int_S^T E^p(t)\phi'(t)\int_\Omega \mathcal{M}u\Big(u_{tt} - \mu(t)Lu\Big)\,dx\,dt \\
&= \int_S^T E^p(t)\phi'(t)\int_\Omega u_{tt}\mathcal{M}u\,dx\,dt - \int_S^T E^p(t)\phi'(t)\int_\Omega (\operatorname{div}(H)-\sigma)u\mu(t)Lu\,dx\,dt \\
&\quad - 2\int_S^T E^p(t)\phi'(t)\int_\Omega H(u)\mu(t)Lu\,dx\,dt.
\end{aligned} \tag{5.3}$$

We note that

$$\begin{aligned}
&\int_S^T E^p(t)\phi'(t)\int_\Omega u_{tt}\mathcal{M}u\,dx\,dt \\
&= \Big[E^p(t)\phi'(t)\int_\Omega u_t\mathcal{M}u dx\Big]_S^T - \int_S^T (pE^{p-1}(t)E'(t)\phi'(t) + E^p(t)\phi''(t))\int_\Omega u_t\mathcal{M}u\,dx\,dt \\
&\quad - 2\int_S^T E^p(t)\phi'(t)\int_\Omega u_tH(u_t)\,dx\,dt - \int_S^T E^p(t)\phi'(t)\int_\Omega (\operatorname{div}(H)-\sigma)|u_t|^2\,dx\,dt
\end{aligned}$$

and

$$\begin{aligned}
&-\int_S^T E^p(t)\phi'(t)\int_\Omega (\operatorname{div}(H)-\sigma)u\mu(t)Lu\,dx\,dt \\
&= -\int_S^T E^p(t)\phi'(t)\mu(t)\int_{\Gamma_1} (\operatorname{div}(H)-\sigma)u\frac{\partial u}{\partial\nu_L}\,d\Gamma\,dt
\end{aligned}$$

$$+\int_S^T E^p(t)\phi'(t)\mu(t)\int_\Omega (\operatorname{div}(H)-\sigma)|\nabla_g u|_g^2\,dx\,dt\,.$$

Using Lemma 2.1 and that $H(u)\frac{\partial u}{\partial \nu_L}=|\nabla_g u|_g^2$ on Γ_0, we have

$$\begin{aligned}
&-2\int_S^T E^p(t)\phi'(t)\int_\Omega H(u)\mu(t)Lu\,dx\,dt\\
&=-\int_S^T E^p(t)\phi'(t)\mu(t)\int_{\Gamma_1}\Big(2\frac{\partial u}{\partial\nu_L}H(u)-|\nabla_g u|_g^2(H\cdot\nu)\Big)\,d\Gamma\,dt\\
&\quad-\int_S^T E^p(t)\phi'(t)\mu(t)\int_{\Gamma_0}|\nabla_g u|_g^2(H\cdot\nu)\,d\Gamma\,dt+2\int_S^T E^p(t)\phi'(t)\mu(t)\int_\Omega D_gH(\nabla_g u,\nabla_g u)\,dx\,dt\\
&\quad-\int_S^T E^p(t)\phi'(t)\mu(t)\int_\Omega |\nabla_g u|_g^2\operatorname{div}(H)\,dx\,dt.
\end{aligned}$$

By placing above identities in (5.3), we obtain

$$\begin{aligned}
&\sigma\int_S^T E^p(t)\phi'(t)\int_\Omega |u_t|^2\,dx\,dt+2\int_S^T E^p(t)\phi'(t)\mu(t)\int_\Omega D_gH(\nabla_g u,\nabla_g u)\,dx\,dt\\
&\quad-\sigma\int_S^T E^p(t)\phi'(t)\mu(t)\int_\Omega|\nabla_g u|_g^2\,dx\,dt-(\operatorname{div}(H)-\sigma)\int_S^T E^p(t)\phi'(t)\int_{\Gamma_1}|u|^{\gamma+2}\,d\Gamma\,dt\\
&=-\Big[E^p(t)\phi'(t)\int_\Omega u_t\mathcal{M}u dx\Big]_S^T+\int_S^T(pE^{p-1}(t)E'(t)\phi'(t)+E^p(t)\phi''(t))\int_\Omega u_t\mathcal{M}u\,dx\,dt\\
&\quad+2\int_S^T E^p(t)\phi'(t)\int_{\Gamma_1}|u|^\gamma uH(u)\,d\Gamma\,dt\\
&\quad+\int_S^T E^p(t)\phi'(t)\int_{\Gamma_1}q(u_t)\mathcal{M}u+\big(|u_t|^2-\mu(t)|\nabla_g u|_g^2\big)(H\cdot\nu)\,d\Gamma\,dt\\
&:=I_1+I_2+I_3+I_4.
\end{aligned}\tag{5.4}$$

Now we estimate terms on the right-hand side of (5.4).

Estimate for $I_1:=-\Big[E^p(t)\phi'(t)\int_\Omega u_t\mathcal{M}udx\Big]_S^T$. Using the Young inequality and the inequality

$$\int_\Omega |u|^2dx\le c_\Omega^*\int_\Omega|\nabla_g u|_g^2dx,\quad c_\Omega^*>0,\quad \forall u\in H_0^1(\Omega),$$

we obtain

$$\Big|\int_\Omega u_t\mathcal{M}udx\Big|\le CE(t),\tag{5.5}$$

consequently,

$$I_1\le -C\Big[E^p(t)\phi'(t)E(t)\Big]_S^T\le CE^{p+1}(S).\tag{5.6}$$

Estimate o $I_2:=\int_S^T\Big(pE^{p-1}(t)E'(t)\phi'(t)+E^p(t)\phi''(t)\Big)\int_\Omega u_t\mathcal{M}u\,dx\,dt$. From (5.5), we have

$$\begin{aligned}
|I_2|&\le C\int_S^T|pE^{p-1}(t)E'(t)\phi'(t)+E^p(t)\phi''(t)|E(t)\,dt\\
&\le CE^p(S)\int_S^T-E'(t)\,dt+CE^{p+1}(S)\int_S^T-\phi''(t)\,dt\\
&\le CE^{p+1}(S).
\end{aligned}\tag{5.7}$$

Estimate for $I_3:=2\int_S^T E^p(t)\phi'(t)\int_{\Gamma_1}|u|^\gamma uH(u)\,d\Gamma\,dt$. By the Young inequality with $\frac{\rho+1}{\rho+2}+\frac{1}{\rho+2}=1$ and from the fact $k(u)=|u|^\gamma u$ is locally Lipschitz from $H_0^1(\Omega)$ into $L^{\frac{\rho+2}{\rho+1}}(\Gamma_1)$, (2.1) and (4.18), we obtain

$$
\begin{aligned}
\int_{\Gamma_1} |u|^\gamma u H(u)\, d\Gamma &\le C(\epsilon_3) \int_{\Gamma_1} |u|^{\frac{(\gamma+1)(\rho+2)}{\rho+1}}\, d\Gamma + \epsilon_3 \int_{\Gamma_1} |H(u)|^{\rho+2}\, d\Gamma \\
&\le C(\epsilon_3) L_\gamma^{\frac{\rho+2}{\rho+1}} \|\nabla u\|_2^{\frac{\rho+2}{\rho+1}} + \epsilon_3 \sup_{x\in\overline{\Omega}} |H|_g^{\rho+2} \int_{\Gamma_1} |\nabla_g u|_g^{\rho+2}\, d\Gamma \\
&\le C(\epsilon_3) L_\gamma^{\frac{\rho+2}{\rho+1}} c_1^{-1} \| \, |\nabla_g u|_g \|_2^2 + \epsilon_3 \sup_{x\in\overline{\Omega}} |H|_g^{\rho+2} \int_{\Gamma_1} |\nabla_g u|_g^{\rho+2}\, d\Gamma \\
&\le C(\epsilon_3) L_\gamma^{\frac{\rho+2}{\rho+1}} \frac{2(\gamma+2)}{\mu_0 \gamma c_1} E(t) + \epsilon_3 \sup_{x\in\overline{\Omega}} |H|_g^{\rho+2} \int_{\Gamma_1} |\nabla_g u|_g^{\rho+2}\, d\Gamma;
\end{aligned}
\tag{5.8}
$$

consequently,

$$
I_3 \le C(\epsilon_3) E^{p+1}(S) + \epsilon_3 \sup_{x\in\overline{\Omega}} |H|_g^{\rho+2} \int_S^T E^p(t)\phi'(t) \int_{\Gamma_1} |\nabla_g u|_g^{\rho+2}\, d\Gamma\, dt. \tag{5.9}
$$

Estimate for $I_4 := \int_S^T E^p(t)\phi'(t) \int_{\Gamma_1} q(u_t)\mathcal{M}u + \big(|u_t|^2 - \mu(t)|\nabla_g u|_g^2\big)(H\cdot\nu)\, d\Gamma\, dt$. From the Young inequality with $\frac{\rho+1}{\rho+2} + \frac{1}{\rho+2} = 1$, we have

$$
\begin{aligned}
2\int_{\Gamma_1} q(u_t) H(u)\, d\Gamma &\le C(\epsilon_4) \int_{\Gamma_1} |q(u_t)|^{\frac{\rho+2}{\rho+1}}\, d\Gamma + \epsilon_4 \int_{\Gamma_1} |H(u)|^{\rho+2}\, d\Gamma \\
&\le C(\epsilon_4) \int_{\Gamma_1} |q(u_t)|^{\frac{\rho+2}{\rho+1}}\, d\Gamma + \epsilon_4 \sup_{x\in\overline{\Omega}} |H|_g^{\rho+2} \int_{\Gamma_1} |\nabla_g u|_g^{\rho+2}\, d\Gamma.
\end{aligned}
$$

Arguments similar to those for (5.8) yield

$$
(\operatorname{div}(H) - \sigma) \int_{\Gamma_1} q(u_t) u\, d\Gamma \le C \int_{\Gamma_1} |q(u_t)|^{\frac{\alpha}{\alpha-1}}\, d\Gamma + C \int_{\Gamma_1} |u|^\alpha\, d\Gamma \le C \int_{\Gamma_1} |q(u_t)|^{\frac{\alpha}{\alpha-1}}\, d\Gamma + CE(t),
$$

where $\alpha = \frac{(\gamma+1)(\rho+2)}{\rho+1}$. Then we obtain

$$
\begin{aligned}
I_4 &\le CE^{p+1}(S) + C(\epsilon_4) \int_S^T E^p(t)\phi'(t) \int_{\Gamma_1} |q(u_t)|^{\frac{\rho+2}{\rho+1}}\, d\Gamma\, dt + C \int_S^T E^p(t)\phi'(t) \int_{\Gamma_1} |q(u_t)|^{\frac{\alpha}{\alpha-1}}\, d\Gamma\, dt \\
&\quad + C \int_S^T E^p(t)\phi'(t) \int_{\Gamma_1} |u_t|^2\, d\Gamma\, dt + \epsilon_4 \sup_{x\in\overline{\Omega}} |H|_g^{\rho+2} \int_S^T E^p(t)\phi'(t) \int_{\Gamma_1} |\nabla_g u|_g^{\rho+2}\, d\Gamma\, dt \\
&\quad - \delta\mu_0 \int_S^T E^p(t)\phi'(t) \int_{\Gamma_1} |\nabla_g u|_g^2\, d\Gamma\, dt.
\end{aligned}
\tag{5.10}
$$

By replacing (5.6), (5.7), (5.9) and (5.10) in (5.4) and choosing ϵ_3, ϵ_4 small enough, from (5.2) we obtain

$$
\begin{aligned}
\int_S^T E^{p+1}(t)\phi'(t)\, dt &\le CE^{p+1}(S) + C \underbrace{\int_S^T E^p(t)\phi'(t) \int_{\Gamma_1} |u_t|^2\, d\Gamma\, dt}_{:=I_5} \\
&\quad + C \underbrace{\int_S^T E^p(t)\phi'(t) \int_{\Gamma_1} |q(u_t)|^{\frac{\rho+2}{\rho+1}}\, d\Gamma\, dt}_{:=I_6} \\
&\quad + C \underbrace{\int_S^T E^p(t)\phi'(t) \int_{\Gamma_1} |q(u_t)|^{\frac{\alpha}{\alpha-1}}\, d\Gamma\, dt}_{:=I_7}.
\end{aligned}
\tag{5.11}
$$

Now we estimate the last three terms on the right-hand side of (5.11).

Case 1: β is linear. Since β is linear, we can rewrite the hypothesis on q as

$$c_7|s| \le |q(s)| \le c_8|s| \quad \text{if } |s| \le 1,$$
$$c_3|s| \le c_3|s|^{\rho+1} \le |q(s)| \le c_4|s|^{\rho+1} \quad \text{if } |s| > 1,$$

for some positive constants c_7, c_8. Then we obtain

$$\int_{|u_t|\le 1} |u_t|^2\, d\Gamma \le c_7^{-1} \int_{|u_t|\le 1} u_t q(u_t)\, d\Gamma \le -c_7^{-1} E'(t), \tag{5.12}$$

$$\int_{|u_t|>1} |u_t|^2\, d\Gamma \le c_3^{-1} \int_{|u_t|>1} u_t q(u_t)\, d\Gamma \le -c_3^{-1} E'(t), \tag{5.13}$$

$$\int_{|u_t|\le 1} |q(u_t)|^{\frac{\rho+2}{\rho+1}}\, d\Gamma \le \int_{|u_t|\le 1} |q(u_t)|^2\, d\Gamma \le c_8 \int_{|u_t|\le 1} u_t q(u_t)\, d\Gamma \le -c_8 E'(t), \tag{5.14}$$

$$\int_{|u_t|>1} |q(u_t)|^{\frac{\rho+2}{\rho+1}}\, d\Gamma = \int_{|u_t|>1} |q(u_t)|^{\frac{1}{\rho+1}} |q(u_t)|\, d\Gamma \le c_4^{\frac{1}{\rho+1}} \int_{|u_t|>1} u_t q(u_t)\, d\Gamma \le -c_4^{\frac{1}{\rho+1}} E'(t), \tag{5.15}$$

$$\int_{|u_t|\le 1} |q(u_t)|^{\frac{\alpha}{\alpha-1}}\, d\Gamma \le c_8 \int_{|u_t|\le 1} u_t q(u_t)\, d\Gamma \le -c_8 E'(t) \tag{5.16}$$

and, since $\rho \le \gamma$, it holds that $\frac{\rho+1}{\alpha-1} \le 1$. Consequently,

$$\begin{aligned}
\int_{|u_t|>1} |q(u_t)|^{\frac{\alpha}{\alpha-1}}\, d\Gamma &= \int_{|u_t|>1} |q(u_t)|^{\frac{1}{\alpha-1}} |q(u_t)|\, d\Gamma \\
&\le c_4^{\frac{1}{\alpha-1}} \int_{|u_t|>1} u_t q(u_t)\, d\Gamma \\
&\le -c_4^{\frac{1}{\alpha-1}} E'(t).
\end{aligned} \tag{5.17}$$

From (5.12)-(5.17), we obtain

$$I_5 + I_6 + I_7 \le C E^{p+1}(S). \tag{5.18}$$

Combining (5.11) and (5.18), it follows that

$$\int_S^T E^{p+1}(t) \phi'(t)\, dt \le C E^p(0) E(S),$$

which implies by Lemma 5.1 with $p = 0$

$$E(t) \le E(0) e^{1 - \frac{\phi(t)}{C}}.$$

Let us set $\phi(t) := mt$, where m is for some positive constant, then $\phi(t)$ satisfies all the required properties and we obtain that the energy decays exponentially to zero.

Case 2: β has polynomial growth near zero. Assume that $\beta(s) = s^{\rho+1}$. Let $p = \frac{\rho}{2}$, then we rewrite (5.11) as

$$\begin{aligned}
&\int_S^T E^{\frac{\rho}{2}+1}(t) \phi'(t)\, dt \\
&\le C E(S) + C \int_S^T E^{\frac{\rho}{2}}(t) \phi'(t) \int_{\Gamma_1} |u_t|^2\, d\Gamma\, dt + C \int_S^T E^{\frac{\rho}{2}}(t) \phi'(t) \int_{\Gamma_1} |q(u_t)|^{\frac{\rho+2}{\rho+1}}\, d\Gamma\, dt \\
&\quad + C \int_S^T E^{\frac{\rho}{2}}(t) \phi'(t) \int_{\Gamma_1} |q(u_t)|^{\frac{\alpha}{\alpha-1}}\, d\Gamma\, dt.
\end{aligned} \tag{5.19}$$

By the hypotheses on q and the Hölder inequality with $\frac{2}{\rho+2} + \frac{\rho}{\rho+2} = 1$, we have

$$\int_{|u_t|\le 1} |u_t|^2\, d\Gamma \le \int_{|u_t|\le 1} (u_t q(u_t))^{\frac{2}{\rho+2}}\, d\Gamma \le C\Big(\int_{|u_t|\le 1} u_t q(u_t) d\Gamma\Big)^{\frac{2}{\rho+2}} \le C\big(-E'(t)\big)^{\frac{2}{\rho+2}},$$

$$\int_{|u_t|>1} |u_t|^2\, d\Gamma \le c_3^{2(\rho+2)} \int_{|u_t|>1} (u_t q(u_t))^{\frac{2}{\rho+2}}\, d\Gamma \le C\big(-E'(t)\big)^{\frac{2}{\rho+2}}.$$

Then

$$\begin{aligned}
&\int_S^T E^{\frac{\rho}{2}}(t)\phi'(t)\int_{\Gamma_1}|u_t|^2\,d\Gamma\,dt\\
&=\int_S^T E^{\frac{\rho}{2}}(t)\phi'(t)\int_{|u_t|\le 1}|u_t|^2\,d\Gamma\,dt+\int_S^T E^{\frac{\rho}{2}}(t)\phi'(t)\int_{|u_t|>1}|u_t|^2\,d\Gamma\,dt\\
&\le C\int_S^T E^{\frac{\rho}{2}}(t)\phi'(t)\big(-E'(t)\big)^{\frac{2}{\rho+2}}\,dt\\
&\le \epsilon_5\int_S^T E^{\frac{\rho}{2}+1}(t)\phi'(t)\,dt+C(\epsilon_5)E(S).
\end{aligned}\tag{5.20}$$

As for (5.14) and (5.15) we have

$$\int_{|u_t|\le 1}|q(u_t)|^{\frac{\rho+2}{\rho+1}}\,d\Gamma\le\int_{|u_t|\le 1}|q(u_t)|^2\,d\Gamma\le\int_{|u_t|\le 1}(u_tq(u_t))^{\frac{2}{\rho+2}}\,d\Gamma\le C\big(-E'(t)\big)^{\frac{2}{\rho+2}},$$
$$\int_{|u_t|>1}|q(u_t)|^{\frac{\rho+2}{\rho+1}}\,d\Gamma\le -c_4^{\frac{1}{\rho+1}}E'(t).$$

Then

$$\int_S^T E^{\frac{\rho}{2}}(t)\phi'(t)\int_{\Gamma_1}|q(u_t)|^{\frac{\rho+2}{\rho+1}}\,d\Gamma\,dt\le\epsilon_6\int_S^T E^{\frac{\rho}{2}+1}(t)\phi'(t)\,dt+C(\epsilon_6)E(S).\tag{5.21}$$

As for (5.16) and (5.17) we have

$$\int_{|u_t|\le 1}|q(u_t)|^{\frac{\alpha}{\alpha-1}}\,d\Gamma\le\int_{|u_t|\le 1}|q(u_t)|^2\,d\Gamma\le C\big(-E'(t)\big)^{\frac{2}{\rho+2}},$$
$$\int_{|u_t|>1}|q(u_t)|^{\frac{\alpha}{\alpha-1}}\,d\Gamma\le -c_4^{\frac{1}{\alpha-1}}E'(t).$$

Then

$$\int_S^T E^{\frac{\rho}{2}}(t)\phi'(t)\int_{\Gamma_1}|q(u_t)|^{\frac{\alpha}{\alpha-1}}\,d\Gamma\,dt\le\epsilon_7\int_S^T E^{\frac{\rho}{2}+1}(t)\phi'(t)\,dt+C(\epsilon_7)E(S).\tag{5.22}$$

By replacing (5.20)-(5.22) in (5.19) and choosing $\epsilon_5,\epsilon_6,\epsilon_7$ sufficiently small, we obtain

$$\int_S^T E^{\frac{\rho}{2}+1}(t)\phi'(t)\,dt\le CE(S),$$

which implies by Lemma 5.1 and choosing $\phi(t)=mt$,

$$E(t)\le\frac{CE(0)}{(1+t)^{2/\rho}}.$$

Case 3: β does not necessarily have polynomial growth near zero. We use the method of partitions of boundary modified the arguments in [35]. For every $t\ge 1$, we consider the following partitions of boundary depending $\phi'(t)$:

$$\Gamma_1^1=\{x\in\Gamma_1:|u_t(t)|\le\phi'(t)\},\quad \Gamma_1^2=\{x\in\Gamma_1:\phi'(t)<|u_t(t)|\le 1\},$$
$$\Gamma_1^3=\{x\in\Gamma_1:|u_t(t)|>1\}$$

if $\phi'(t)\le 1$; or

$$\Gamma_1^4=\{x\in\Gamma_1:|u_t(t)|\le 1<\phi'(t)\},\quad \Gamma_1^5=\{x\in\Gamma_1:1<|u_t(t)|\le\phi'(t)\},$$
$$\Gamma_1^6=\{x\in\Gamma_1:|u_t(t)|>\phi'(t)>1\}$$

if $\phi'(t)>1$. Then $\Gamma_1=\Gamma_1^1\cup\Gamma_1^2\cup\Gamma_1^3$ (or $\Gamma_1=\Gamma_1^4\cup\Gamma_1^5\cup\Gamma_1^6$). Let us estimate I_5, I_6 and I_7 on these partitions.

(i) Part on Γ_1^i, $i=3,5,6$. By same arguments as (5.13), (5.15) and (5.17) we obtain

$$\int_S^T E^p(t)\phi'(t)\int_{\Gamma_1^i}|u_t|^2\,d\Gamma\,dt\le CE^{p+1}(S),\tag{5.23}$$

$$\int_S^T E^p(t)\phi'(t)\int_{\Gamma_1^i}|q(u_t)|^{\frac{\rho+2}{\rho+1}}\,d\Gamma\,dt \le CE^{p+1}(S), \tag{5.24}$$

$$\int_S^T E^p(t)\phi'(t)\int_{\Gamma_1^i}|q(u_t)|^{\frac{\alpha}{\alpha-1}}\,d\Gamma\,dt \le CE^{p+1}(S). \tag{5.25}$$

(ii) Part on Γ_1^2. Using that β is increasing, ϕ' is non-increasing and (2.10), we obtain

$$\begin{aligned}\int_S^T E^p(t)\phi'(t)\int_{\Gamma_1^2}|u_t|^2\,d\Gamma\,dt &\le \int_S^T \frac{E^p(t)}{\beta(\phi'(t))}\int_{\Gamma_1^2}u_t q(u_t)\,d\Gamma\,dt\\ &\le \frac{1}{\beta(\phi'(T))}\int_S^T E^p(t)\big(-E'(t)\big)\,dt\\ &\le \frac{1}{\beta(\phi'(T))}E^{p+1}(S),\end{aligned} \tag{5.26}$$

$$\begin{aligned}\int_S^T E^p(t)\phi'(t)\int_{\Gamma_1^2}|q(u_t)|^{\frac{\rho+2}{\rho+1}}\,d\Gamma\,dt &\le \int_S^T E^p(t)\int_{\Gamma_1^2}|u'(t)|\,|q(u_t)|^2\,d\Gamma\,dt\\ &\le \beta^{-1}(1)\int_S^T E^p(t)\int_{\Gamma_1^2}u_t q(u_t)\,d\Gamma\,dt\\ &\le \beta^{-1}(1)E^{p+1}(S)\end{aligned} \tag{5.27}$$

and

$$\int_S^T E^p(t)\phi'(t)\int_{\Gamma_1^2}|q(u_t)|^{\frac{\alpha}{\alpha-1}}\,d\Gamma\,dt \le \int_S^T E^p(t)\int_{\Gamma_1^2}|u'(t)|\,|q(u_t)|^2\,d\Gamma\,dt \le \beta^{-1}(1)E^{p+1}(S). \tag{5.28}$$

(iii) Part on Γ_1^i, $i=1,4$. Using that $E(t)$ is non-increasing and β^{-1} is increasing, we have

$$\begin{aligned}\int_S^T E^p(t)\phi'(t)\int_{\Gamma_1^i}|u_t|^2\,d\Gamma\,dt &\le \frac{E^p(S)}{(\beta^{-1}(\phi'(T)))^2}\int_S^T\int_{\Gamma_1^i}\phi'(t)(\beta^{-1}(\phi'(t)))^2\,d\Gamma\,dt\\ &\le \frac{\operatorname{meas}(\Gamma_1)}{(\beta^{-1}(\phi'(T)))^2}E^p(S)\int_S^T \phi'(t)(\beta^{-1}(\phi'(t)))^2\,dt.\end{aligned} \tag{5.29}$$

From (2.10), we obtain

$$\begin{aligned}\int_S^T E^p(t)\phi'(t)\int_{\Gamma_1^i}|q(u_t)|^{\frac{\rho+2}{\rho+1}}\,d\Gamma\,dt &\le \int_S^T E^p(t)\phi'(t)\int_{\Gamma_1^i}|q(u_t)|^2\,d\Gamma\,dt\\ &\le \int_S^T E^p(t)\phi'(t)\int_{\Gamma_1^i}(\beta^{-1}(|u'(t)|))^2\,d\Gamma\,dt\\ &\le \operatorname{meas}(\Gamma_1)E^p(S)\int_S^T \phi'(t)(\beta^{-1}(\phi'(t)))^2\,dt\end{aligned} \tag{5.30}$$

and

$$\int_S^T E^p(t)\phi'(t)\int_{\Gamma_1^i}|q(u_t)|^{\frac{\alpha}{\alpha-1}}\,d\Gamma\,dt \le \operatorname{meas}(\Gamma_1)E^p(S)\int_S^T \phi'(t)(\beta^{-1}(\phi'(t)))^2\,dt. \tag{5.31}$$

Therefore, from (5.23)-(5.31), we deduce that

$$I_5+I_6+I_7 \le CE^{p+1}(S)+CE^p(S)\int_S^T \phi'(t)(\beta^{-1}(\phi'(t)))^2\,dt. \tag{5.32}$$

To estimate the last term of the right-hand side of (5.32), we need the following additional assumption over ϕ (see [35, p.434])

$$\int_1^\infty \phi'(t)(\beta^{-1}(\phi'(t)))^2\,dt \quad \text{converges.}$$

Then by replacing (5.32) in (5.11) we obtain

$$\begin{aligned} \int_S^T E^{p+1}(t)\phi'(t)\,dt &\le CE^{p+1}(S) + CE^p(S)\int_S^{+\infty} \phi'(t)(\beta^{-1}(\phi'(t)))^2\,dt \\ &\le CE^{p+1}(S) + CE^p(S)\int_{\phi(S)}^{+\infty}\Big(\beta^{-1}(\frac{1}{(\phi^{-1})'(s)})\Big)^2\,ds. \end{aligned} \tag{5.33}$$

We define $\psi(t) = 1 + \int_1^t \frac{1}{\beta(\frac{1}{s})}\,ds$, $t \ge 1$. Then ψ is strictly increasing and convex (cf. [35], [39]). We now take $\phi(t) = \psi^{-1}(t)$, then we can rewrite (5.33) as

$$\int_S^T E^{p+1}(t)\phi'(t)\,dt \le CE^{p+1}(S) + \frac{C}{\phi(S)}E^p(S),$$

which implies, by applying Lemma 5.2 with $p = 1$,

$$E(t) \le \frac{C}{\phi^2(t)} \quad \forall t > 0.$$

Let s_0 be a number such that $\beta(\frac{1}{s_0}) \le 1$. Since β is nondecreasing, we have

$$\psi(s) \le 1 + (s-1)\frac{1}{\beta(\frac{1}{s})} \le \frac{1}{F(\frac{1}{s})} \quad \forall s \ge s_0,$$

where $F(s) = s\beta(s)$, consequently, having in mind that $\phi = \psi^{-1}$, the above inequality yields

$$s \le \phi\Big(\frac{1}{F(\frac{1}{s})}\Big) = \phi(t) \quad \text{with } t = \frac{1}{F(\frac{1}{s})}.$$

Then we conclude that

$$\frac{1}{\phi(t)} \le F^{-1}(\frac{1}{t}).$$

Therefore the proof is complete.

6. Proof of Theorem 2.6: blow-up

First we introduce a following lemma that is essential role for proving the blow-up.

Lemma 6.1. *Under the hypotheses given in Theorem 2.6 the weak solution to problem* (1.1) *satisfies*

$$\|\,|\nabla_g u(t)|_g\|_2 > \lambda_0 \quad \textit{for all } 0 < t < T_{\max}.$$

Proof. We recall the function, for $\lambda > 0$,

$$j(\lambda) = \frac{\mu_0}{2}\lambda^2 - \frac{1}{\gamma+2}K_0^{\gamma+2}\lambda^{\gamma+2},$$

where $K_0 = \sup_{u\in\mathcal{H},u\neq 0}\Big(\frac{\|u\|_{\gamma+2,\Gamma_1}}{\|\,|\nabla_g u|_g\|_2}\Big)$. Then

$$\lambda_0 = \Big(\frac{\mu_0}{K_0^{\gamma+2}}\Big)^{1/\gamma}$$

is the absolute maximum point of j and

$$j(\lambda_0) = \frac{\gamma\mu_0}{2(\gamma+2)}\lambda_0^2 = d_0.$$

The energy associated with problem (1.1) is

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\mu(t)\|\,|\nabla_g u(t)|_g\|_2^2 - \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2}.$$

From the definition of j, we have

$$E(t) \ge \frac{\mu_0}{2}\|\,|\nabla_g u(t)|_g\|_2^2 - \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2} \ge j(\|\,|\nabla_g u(t)|_g\|_2) \quad \text{for all} \quad t \ge 0. \tag{6.1}$$

Note that j is increasing for $0 < \lambda < \lambda_0$, decreasing for $\lambda > \lambda_0$, and $j(\lambda) \to -\infty$ as $\lambda \to +\infty$.

Now we consider the initial energy $E(0)$ divided into two cases: $E(0) \ge 0$ and $E(0) < 0$.

Case 1: $E(0) \geq 0$**.** There exist $\lambda_1' < \lambda_0 < \lambda_1$ such that

$$j(\lambda_1) = j(\lambda_1') = E(0). \tag{6.2}$$

By considering that $E(t)$ is non-increasing, we have

$$E(t) \leq E(0) \quad \text{for all} \quad t > 0. \tag{6.3}$$

From (6.1) and (6.2) we deduce that

$$j(\| |\nabla_g u_0|_g \|_2) \leq E(0) = j(\lambda_1). \tag{6.4}$$

Since $\| |\nabla_g u_0|_g \|_2 > \lambda_0$, $\lambda_0 < \lambda_1$, and $j(\lambda)$ is decreasing for $\lambda_0 < \lambda$, from (6.4) we obtain

$$\| |\nabla_g u_0|_g \|_2 \geq \lambda_1. \tag{6.5}$$

Now we prove that

$$\| |\nabla_g u(t)|_g \|_2 \geq \lambda_1 \quad \text{for all } 0 < t < T_{\max} \tag{6.6}$$

by using the contradiction method. Suppose that (6.6) does not hold. Then there exists t^* in $(0, T_{\max})$ that satisfies

$$\| |\nabla_g u(t^*)|_g \|_2 < \lambda_1. \tag{6.7}$$

If $\| |\nabla_g u(t^*)|_g \|_2 > \lambda_0$, from (6.1), (6.2), and (6.7), we can write

$$E(t^*) \geq j(\| |\nabla_g u(t^*)|_g \|_2) > j(\lambda_1) = E(0),$$

which contradicts (6.3).

If $\| |\nabla_g u(t^*)|_g \|_2 \leq \lambda_0$, we have, in view of (6.5), that there exists $\bar{\lambda}$ which satisfies

$$\| |\nabla_g u(t^*)|_g \|_2 \leq \lambda_0 < \bar{\lambda} < \lambda_1 \leq \| |\nabla_g u_0|_g \|_2. \tag{6.8}$$

Consequently, from the continuity of the function $t \mapsto \| |\nabla_g u(t)|_g \|_2$ there exists $\bar{t} \in (0, t^*)$ satisfying $\| |\nabla_g u(\bar{t})|_g \|_2 = \bar{\lambda}$. Then from the last identity and taking (6.1), (6.2), and (6.8) into account we deduce that

$$E(\bar{t}) \geq j(\| |\nabla_g u(\bar{t})|_g \|_2) = j(\bar{\lambda}) > j(\lambda_1) = E(0),$$

which also contradicts (6.3).

Case 2: $E(0) < 0$**.** There is a $\lambda_2 > \lambda_0$ such that $j(\lambda_2) = E(0)$; consequently, by (6.1) we have

$$j(\| |\nabla_g u_0|_g \|_2) \leq E(0) = j(\lambda_2).$$

From the fact $j(\lambda)$ is decreasing for $\lambda_0 < \lambda$, we obtain

$$\| |\nabla_g u_0|_g \|_2 \geq \lambda_2.$$

By the same argument as in Case 1, we obtain

$$\| |\nabla_g u(t)|_g \|_2 \geq \lambda_2 \quad \text{for all } 0 < t < T_{\max}.$$

Thus the proof is complete. $\square$

Now we prove the blow-up result. To prove that $T_{\max}$ is finite, we argue by contradiction. Assume that the weak solution $u(t)$ can be extended to the whole interval $[0, \infty)$. Let E_1 be a real number such that

$$E_1 = \begin{cases} 0 & \text{if } E(0) < 0, \\ \text{a positive constant satisfying } \ E(0) < E_1 < d_0 \text{ and } E_1 < E(0) + 1 & \text{if } E(0) \geq 0. \end{cases}$$

By setting $G(t) := E_1 - E(t)$, we have

$$G'(t) = -E'(t) \geq 0, \tag{6.9}$$

which implies that $G(t)$ is nondecreasing, consequently,

$$0 < G_0 := E_1 - E(0) < 1 \tag{6.10}$$

and from Lemma 6.1, (2.8), and the definition of d_0, we have

$$\begin{aligned} G_0 \le G(t) &\le E_1 - \frac{\mu_0}{2}\| \, |\nabla_g u(t)|_g\|_2^2 + \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2} \\ &< d_0 - \frac{\mu_0}{2}\lambda_0^2 + \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2} \\ &\le \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2}. \end{aligned} \tag{6.11}$$

We define

$$M(t) = G^{1-\overline{\chi}}(t) + \tau N(t), \quad N(t) = \int_\Omega u_t u dx, \tag{6.12}$$

where $\overline{\chi}$ and τ are small positive constants to be chosen later. Then we have

$$M'(t) = (1-\overline{\chi})G^{-\overline{\chi}}(t)G'(t) + \tau N'(t). \tag{6.13}$$

Now we analyze the last term on the right-hand side of (6.13).

Lemma 6.2. *It holds*

$$\begin{aligned} N'(t) &\ge C_{15}\big(\|u_t\|_2^2 + \|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} + G(t) - G'(t)G_0^{\overline{\chi}-\chi}G^{-\overline{\chi}}(t)\big) \\ &\quad + \mu_0\Big(\frac{\theta}{2}-1\Big)\| \, |\nabla_g u|_g\|_2^2 - \theta E_1 - \frac{\zeta}{\epsilon_8}, \end{aligned} \tag{6.14}$$

where C_{15} is a positive constant, $0 < \chi < \frac{\gamma-\rho}{(\rho+2)(\gamma+2)}$, $\theta = \gamma+2-\epsilon_8$ with $0<\epsilon_8<\min\{1,\gamma\}$, and $\zeta = \frac{(\gamma+1)\,\mathrm{meas}(\Gamma_1)\big(\beta^{-1}(1)\big)^{\frac{\gamma+2}{\gamma+1}}}{\gamma+2}$.

Proof. Using (1.1), we obtain

$$\begin{aligned} N'(t) &= \|u_t\|_2^2 - \mu(t)\| \, |\nabla_g u|_g\|_2^2 + \|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} - \int_{\Gamma_1} q(u_t)u \, d\Gamma \\ &\ge \Big(1+\frac{\theta}{2}\Big)\|u_t\|_2^2 + \mu_0\Big(\frac{\theta}{2}-1\Big)\| \, |\nabla_g u|_g\|_2^2 + \Big(1-\frac{\theta}{\gamma+2}\Big)\|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} \\ &\quad + \theta G(t) - \theta E_1 - \int_{\Gamma_1} q(u_t)u \, d\Gamma, \end{aligned} \tag{6.15}$$

where $\theta = \gamma+2-\epsilon_8$ with $0<\epsilon_8<\min\{1,\gamma\}$.

Now we estimate the last term on the right-hand side of (6.15). We note that

$$\Big|\int_{\Gamma_1} q(u_t)u \, d\Gamma\Big| \le \int_{\Gamma_1} |q(u_t)| \, |u| \, d\Gamma = \int_{|u_t|\le 1} |q(u_t)| \, |u| \, d\Gamma + \int_{|u_t|>1} |q(u_t)| \, |u| \, d\Gamma.$$

By using (2.10) and the imbedding $L^{\gamma+2}(\Gamma_1) \hookrightarrow L^{\rho+2}(\Gamma_1)$, we have

$$\begin{aligned} \int_{|u_t|\le 1} |q(u_t)| \, |u| \, d\Gamma &\le \Big(\int_{|u_t|\le 1} |q(u_t)|^{\frac{\rho+2}{\rho+1}} \, d\Gamma\Big)^{\frac{\rho+1}{\rho+2}} \Big(\int_{|u_t|\le 1} |u|^{\rho+2} \, d\Gamma\Big)^{\frac{1}{\rho+2}} \\ &\le \Big(\int_{|u_t|\le 1} |\beta^{-1}(1)|^{\frac{\rho+2}{\rho+1}} \, d\Gamma\Big)^{\frac{\rho+1}{\rho+2}} \|u\|_{\rho+2,\Gamma_1} \\ &\le \beta^{-1}(1)\big(\mathrm{meas}(\Gamma_1)\big)^{\frac{\gamma+1}{\gamma+2}} \|u\||_{\gamma+2,\Gamma_1} \\ &\le \frac{(\gamma+1)\,\mathrm{meas}(\Gamma_1)\big(\beta^{-1}(1)\big)^{\frac{\gamma+2}{\gamma+1}}}{\epsilon_8(\gamma+2)} + \frac{\epsilon_8^{\gamma+1}}{\gamma+2}\|u\|_{\gamma+2,\Gamma_1}^{\gamma+2}. \end{aligned} \tag{6.16}$$

On the other hand, by using (2.11), we obtain

$$\begin{aligned}
\int_{|u_t|>1} |q(u_t)|\,|u|\,d\Gamma &\le c_4 \int_{|u_t|>1} |u_t|^{\rho+1}|u|\,d\Gamma \\
&\le c_4 \Big(\int_{|u_t|>1} |u_t|^{\rho+2}\,d\Gamma \Big)^{\frac{\rho+1}{\rho+2}} \|u\|_{\rho+2,\Gamma_1} \\
&\le \Big(C(\epsilon_9) \int_{|u_t|>1} |u_t|^{\rho+2}\,d\Gamma + \epsilon_9 \|u\|_{\gamma+2,\Gamma_1}^{(\gamma+2)(\chi+\frac{1}{\gamma+2})(\rho+2)} \Big) \|u\|_{\gamma+2,\Gamma_1}^{-(\gamma+2)\chi},
\end{aligned} \tag{6.17}$$

where $0 < \chi < \frac{\gamma-\rho}{(\rho+2)(\gamma+2)}$ and $C(\epsilon_9)$, ϵ_9 are for positive constants. Moreover $\chi < \frac{\gamma-\rho}{(\rho+2)(\gamma+2)}$ implies that $(\chi + \frac{1}{\gamma+2})(\rho+2) < 1$. Hence we obtain

$$\|u\|_{\gamma+2,\Gamma_1}^{(\gamma+2)(\chi+\frac{1}{\gamma+2})(\rho+2)} \le \begin{cases} \|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} & \text{if } \|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} > 1, \\ G_0^{-1}G_0 & \text{if } \|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} \le 1. \end{cases}$$

From (6.10) and (6.11) we have

$$\|u\|_{\gamma+2,\Gamma_1}^{(\gamma+2)(\chi+\frac{1}{\gamma+2})(\rho+2)} \le G_0^{-1} \|u\|_{\gamma+2,\Gamma_1}^{\gamma+2}$$

and, consequently, from (2.11), (6.9), (6.11), and (6.17),

$$\begin{aligned}
\int_{|u_t|>1} |q(u_t)|\,|u|\,d\Gamma &\le \Big(C(\epsilon_9) \int_{|u_t|>1} |u_t|^{\rho+2}\,d\Gamma + \epsilon_9 G_0^{-1} \|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} \Big) \|u\|_{\gamma+2,\Gamma_1}^{-(\gamma+2)\chi} \\
&\le \Big(C(\epsilon_9) G'(t) + \epsilon_9 G_0^{-1} \|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} \Big)\, G^{-\chi}(t) \\
&\le C(\epsilon_9) G'(t) G_0^{\overline{\chi}-\chi} G^{-\overline{\chi}}(t) + \epsilon_9 G_0^{-(\chi+1)} \|u\|_{\gamma+2,\Gamma_1}^{\gamma+2},
\end{aligned} \tag{6.18}$$

for $0 < \overline{\chi} < \chi$. From (6.16) and (6.18), we obtain that

$$\Big| \int_{\Gamma_1} q(u_t) u\,d\Gamma \Big| \le C(\epsilon_9) G'(t) G_0^{\overline{\chi}-\chi} G^{-\overline{\chi}}(t) + \Big(\frac{\epsilon_8^{\gamma+1}}{\gamma+2} + \epsilon_9 G_0^{-(\chi+1)} \Big) \|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} + \frac{\zeta}{\epsilon_8}, \tag{6.19}$$

where $\zeta = \frac{(\gamma+1)\,\mathrm{meas}(\Gamma_1)\big(\beta^{-1}(1)\big)^{\frac{\gamma+2}{\gamma+1}}}{\gamma+2}$.

By replacing (6.19) in (6.15) and choosing ϵ_9 small enough we obtain

$$N'(t) \ge C_{16}\big(\|u_t\|_2^2 + \|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} + G(t) - G'(t) G_0^{\overline{\chi}-\chi} G^{-\overline{\chi}}(t) \big) + \mu_0 \Big(\frac{\theta}{2} - 1 \Big) \| \, |\nabla_g u|_g \|_2^2 - \theta E_1 - \frac{\zeta}{\epsilon_8},$$

where C_{16} is a positive constant. Therefore (6.14) follows. $\square$

The following Lemma estimates the last three terms on the right-hand side of (6.14).

Lemma 6.3. *It holds*

$$\mu_0 \Big(\frac{\theta}{2} - 1 \Big) \| \, |\nabla_g u|_g \|_2^2 - \theta E_1 - \frac{\zeta}{\epsilon_8} > 0 \quad \text{for } \frac{\ell - \sqrt{\ell^2 - 4\varrho\zeta}}{2\varrho} \le \epsilon_8 \le \frac{\ell + \sqrt{\ell^2 - 4\varrho\zeta}}{2\varrho}, \tag{6.20}$$

where $\varrho = \frac{\mu_0 \lambda_0^2}{2} - E_1$ *and* $\ell = \frac{\gamma \mu_0 \lambda_0^2}{2} - (\gamma+2)E_1$.

Proof. From Lemma 5.1 and the definition of θ, we have

$$\begin{aligned}
&\mu_0 \Big(\frac{\theta}{2} - 1 \Big) \| \, |\nabla_g u|_g \|_2^2 - \theta E_1 - \frac{\zeta}{\epsilon_8} > \mu_0 \Big(\frac{\theta}{2} - 1 \Big) \lambda_0^2 - \theta E_1 - \frac{\zeta}{\epsilon_8} \\
&= \Big(E_1 - \frac{\mu_0 \lambda_0^2}{2} \Big) \epsilon_8 - \frac{\zeta}{\epsilon_8} + \frac{\gamma \mu_0 \lambda_0^2}{2} - (\gamma+2)E_1 \\
&= -\frac{\varrho \epsilon_8^2 - \ell \epsilon_8 + \zeta}{\epsilon_8} := P(\epsilon_8).
\end{aligned} \tag{6.21}$$

We note that

$$\varrho = \frac{\mu_0 \lambda_0^2}{2} - E_1 > \frac{\mu_0 \lambda_0^2}{2} - d_0 = \frac{1}{\gamma+2} K_0^{\gamma+2} \lambda_0^{\gamma+2} > 0,$$

$$\ell = \frac{\gamma\mu_0\lambda_0^2}{2} - (\gamma+2)E_1 > \frac{\gamma\mu_0\lambda_0^2}{2} - (\gamma+2)d_0 = 0.$$

Since (2.19) holds, we have $\ell^2 - 4\varrho\zeta \geq 0$. Therefore, $P(\epsilon_8)$ represents a curve connecting horizontal axis points $\frac{\ell-\sqrt{\ell^2-4\varrho\zeta}}{2\varrho}$ and $\frac{\ell+\sqrt{\ell^2-4\varrho\zeta}}{2\varrho}$, and

$$P(\epsilon_8) \geq 0 \quad \text{for} \quad \frac{\ell-\sqrt{\ell^2-4\varrho\zeta}}{2\varrho} \leq \epsilon_8 \leq \frac{\ell+\sqrt{\ell^2-4\varrho\zeta}}{2\varrho}.$$

Thus we obtain

$$\mu_0\Big(\frac{\theta}{2}-1\Big)\|\,|\nabla_g u|_g\|_2^2 - \theta E_1 - \frac{\zeta}{\epsilon_8} > 0 \quad \text{for } \frac{\ell-\sqrt{\ell^2-4\varrho\zeta}}{2\varrho} \leq \epsilon_8 \leq \frac{\ell+\sqrt{\ell^2-4\varrho\zeta}}{2\varrho}. \qquad \square$$

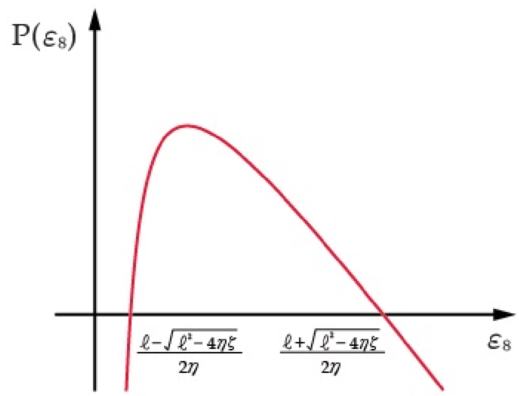

FIGURE 3. The figure of $P(\epsilon_8)$

Combining (6.13), (6.14), (6.20) and then choosing $0 < \overline{\chi} < \min\{\frac{1}{2}, \chi\}$ and τ small enough, we obtain

$$M'(t) \geq C_{17}\big(\|u_t\|_2^2 + \|u\|_{\gamma+2,\Gamma_1}^{\gamma+2} + G(t)\big),$$

where C_{17} is a positive constant, which implies that $M(t)$ is a positive increasing function. By the same arguments on [16, p.333], we have

$$M'(t) \geq C_{18} M^{\frac{1}{1-\overline{\chi}}}(t) \quad \text{for all } t \geq 0,$$

where C_{18} is a positive constant and $1 < \frac{1}{1-\overline{\chi}} < 2$. Then we conclude that $M(t)$ blows up in finite time and u also blows up in finite time. Thus this is a contradiction, consequently, the proof is complete.

Acknowledgments. This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (RS-2022-NR075641).

REFERENCES

[1] L. Bociu; *Local and global wellposedness of weak solutions for the wave equation with nonlinear boundary and interior sources of supercritical exponents and damping*, Nonlinear Anal., **71** (2009), 560–e575.

[2] L. Bociu, I. Lasiecka; *Uniqueness of weak solutions for the semilinear wave equations with supercritical boundary/interior sources and damping*, Discrete Contin. Dyn. Syst., **22** (2008), 835–860.

[3] L. Bociu, M. Rammaha, D. Toundykov; *On a wave equation with supercritical interior and boundary sources and damping terms*, Math. Nachr., **284** (16) (2011), 2032–2064.

[4] Y. Boukhatem, B. Benabderramane; *Existence and decay of solutions for a viscoelastic wave equation with acoustic boundary conditions*, Nonlinear Anal., **97** (2014), 191–209.

[5] S. Brenner, L. R. Scott; *The Mathematical Theory of Finite Element Methods*, Springer Verlag, 1994.

[6] X. Cao, P. Yao; *General decay rate estimates for viscoelastic wave equation with variable coefficients*, J. Syst. Sci. Complex, **27** (2014), 836–852.

[7] M. M. Cavalcanti, V. N. Domingos Cavalcanti, I. Lasiecka; *Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping-source interaction*, J, Differential Equations, **236** (2007) 407–459.
[8] M. M. Cavalcanti, V. N. Domingos Cavalcanti, P. Martinez; *Existence and decay rate estimates for the wave equation with nonlinear boundary damping and source term*, J. Differential Equations, **203** (2004) 119–158.
[9] M. M. Cavalcanti, A. Khemmoudj, M. Medjden; *Uniform stabilization of the damped Cauchy-Ventcel problem with variable coefficients and dynamic boundary conditions*, J. Math. Anal. Appl. **328** (2007), 900–930.
[10] I. Chueshov, M. Eller, I. Lasiecka; *On the attractor for a semilinear wave equation with critical exponent and nonlinear boundary dissipation*, Comm. Partial Differential Equations, **27** (9-10) (2002) 1901–1951.
[11] Y. Guo and M. A. Rammaha; *Global existence and decay of energy to systems of wave equations with damping and supercritical sources*, Z. Angew. Math. Phys., **64** (3) (2013) 621–658.
[12] Y. Guo, M. A. Rammaha, S. Sakuntasathien; *Blow-up of a hyperbolic equation of viscoelasticity with supercritical nonlinearities*, J. Differential Equations, **262** (2017), 1956–1979.
[13] Y. Guo, M. A. Rammaha, S. Sakuntasathien, E. S. Titi, D. Toudykov; *Hadamard well-posedness for a hyperbolic equation of viscoelasticity with supercritical sources and damping*, J. Differetinal Equations, **257** (2014), 3778–3812.
[14] T. G. Ha; *Asymptotic stability of the semilinear wave equation with boundary damping and source term*, C. R. Math. Acad. Sci. Paris Ser. I, **352** (2014), 213–218.
[15] T. G. Ha; *Asymptotic stability of the viscoelastic equation with variable coefficients and the Balakrishnan-Taylor damping*, Taiwanese J. Math., **22** (4) (2018), 931–948.
[16] T. G. Ha; *Blow-up for semilinear wave equation with boudnary damping and source terms*, J. Math. Anal. Appl., **390** (2012), 328-334.
[17] T. G. Ha; *Blow-up for wave equation with weak boundary damping and source terms*, Appl. Math. Lett., **49** (2015), 166-172.
[18] T. G. Ha; *Energy decay for the wave equation of variable coefficients with acoustic boundary conditions in domains with nonlocally reacting boundary*, Appl. Math. Lett., **76** (2018), 201–207.
[19] T. G. Ha; *Energy decay rate for the wave equation with variable coefficients and boundary source term*, Appl. Anal., **100** (11) (2021), 2301–2314.
[20] T. G. Ha; *General decay estimates for the wave equation with acoustic boundary conditions in domains with nonlocally reacting boundary*, Appl. Math. Lett., **60** (2016), 43–49.
[21] T. G. Ha; *General decay rate estimates for viscoelastic wave equation with Balakrishnan-Taylor damping*, Z. Angew. Math. Phys., **67** (2) (2016) Art. 32, 17 pp.
[22] T. G. Ha; *Global existence and general decay estimates for the viscoelastic equation with acoustic boundary conditions*, Discrete Contin. Dyn. Syst., **36** (2016), 6899–6919.
[23] T. G. Ha; *Global solutions and blow-up for the wave equation with variable coefficients: I. interior supercritical source*, Appl. Math. Optim. **84** (suppl. 1) (2021), 767–803.
[24] T. G. Ha; *On the viscoelastic equation with Balakrishnan-Taylor damping and acoustic boundary conditions*, Evol. Equ. Control Theory, **7** (2) (2018), 281–291.
[25] T. G. Ha; *On viscoelastic wave equation with nonlinear boundary damping and source term*, Commun. Pur. Appl. Anal., **9** (6) (2010), 1543–1576.
[26] T. G. Ha; *Stabilization for the wave equation with variable coefficients and Balakrishnan-Taylor damping*, Taiwanese J. Math., **21** (2017), 807–817.
[27] T. G. Ha, D. Kim, I. H. Jung; *Global existence and uniform decay rates for the semi-linear wave equation with damping and source terms*, Comput. Math. Appl., **67** (2014), 692–707.
[28] V. Komornik, E. Zuazua; *A direct method for boundary stabilization of the wave equation*, J. Math. Pures Appl., **69** (1990), 33–54.
[29] I. Lasiecka; *Mathematical Control Theory of Coupled PDE's, CBMS-SIAM Lecture Notes*, SIAM, Philadelphia, 2002.
[30] I. Lasiecka, D. Tataru; *Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping*, Differential and Integral Equations, **6** (3) (1993), 507–533.
[31] I. Lasiecka, D. Toundykov; *Energy decay rates for the semilinear wave equation with nonlinear localized damping and source terms*, Nonlinear Anal., **64** (2006), 1757–1797.
[32] I. Lasiecka, R. Triggiani, P. F. Yao; *Inverse/observability estimates for second-order hyperbolic equations with variable coefficients*, J. Math. Anal. Appl., **235** (1999), 13–57.
[33] J. Li, S. Chai; *Energy decay for a nonlinear wave equation of variable coefficients with acoustic boundary conditions and a time-varying delay in the boundary feedback*, Nonlinear Anal., **112** (2015), 105–117.
[34] L. Lu, S. Li; *Higher order energy decay for damped wave equations with variable coefficients*, J. Math. Anal. Appl., **418** (2014), 64–78.
[35] P. Martinez; *A new method to obtain decay rate estimates for dissipative systems*, ESAIM: Control, Optimisation Calc. Var., **4** (1999), 419–444.
[36] J. Y. Park, T. G. Ha; *Energy decay for nondissipative distributed systems with boundary damping and source term*, Nonlinear Anal., **70** (2009), 2416–2434.
[37] J. Y. Park, T. G. Ha; *Existence and asymptotic stability for the semilinear wave equation with boundary damping and source term*, J. Math. Phys., **49** (2008), 053511.

[38] J. Y. Park, T. G. Ha; *Well-posedness and uniform decay rates for the Klein-Gordon equation with damping term and acoustic boundary conditions*, J. Math. Phys., **50** (2009), 013506.

[39] J. Y. Park, T. G. Ha, Y. H. Kang; *Energy decay rates for solutions of the wave equation with boundary damping and source term*, Z. Angew. Math. Phys., **61** (2010), 235–265.

[40] P. Radu; *Weak solutions to the Cauchy problem of a semilinear wave equation with damping and source terms*, Adv. Differential Equations, **10** (11) (2005), 1261–1300.

[41] I. E. Segal; *Non-linear semigroups*, Ann. Math., **78** (1963), 339–364.

[42] E. Vitillaro; *On the wave equation with hyperbolic dynamical boundary conditions, interior and boundary damping and supercritical sources*, J. Differential Equations, **265** (2018), 4873–4941.

[43] E. Vitillaro; *On the wave equation with hyperbolic dynamical boundary conditions, interior and boundary damping and source*, Arch. Ration. Mech. Anal., **223** (2017) (3), 1183–1237.

[44] J. Wu; *Uniform energy decay of a variable coefficient wave equation with nonlinear acoustic boundary conditions*, J. Math. Anal. Appl., **399** (2013), 369–377.

[45] P. Yao; *Energy decay for the Cauchy problem of the linear wave equation of variable coefficients with dissipation*, Chin. Ann. Math. Ser. B, **31** (1) (2010), 59–70.

[46] P. F. Yao; *On the observability inequality for exact controllablility of wave equations with variable coefficients*, SIAM J. Control Optim., **37** (1999), 1568–1599.

Tae Gab Ha

Department of Mathematics, and Institute of Pure and Applied Mathematics, Jeonbuk National University, Jeonju 54896, Korea

Email address: `tgha@jbnu.ac.kr`