

L^q -REGULARITY ESTIMATES FOR DOUBLE OBSTACLE PROBLEMS WITH QUASILINEAR OPERATORS AND SCHRÖDINGER-TYPE LOWER ORDER TERMS

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ABSTRACT. We derive interior and boundary L^q -regularity estimates for double obstacle problems involving quasilinear operators of p -Laplacian type and lower-order terms with nonnegative potential functions satisfying a reverse Hölder type condition. We prove that the L^q norms of the gradient of a solution to the obstacle problem, as well as the lower-order term, can be estimated by the L^q norms of the data and the gradients of the obstacles. Moreover, the relevant constants in the L^q estimates depend only on the constant in the reverse Hölder type condition for the potential and are independent of the potential.

1. INTRODUCTION

Obstacle problems have long played a central role in mathematical models where physical constraints are present. A classical motivation arises from studying the equilibrium configuration of a stretched membrane that is restricted from below by a fixed obstacle. Mathematically, this leads to constrained minimization problems, where the goal is to find energy-minimizing functions subject to a pointwise lower bound. Such formulations naturally belong to the framework of variational inequalities and establish a strong connection with both the calculus of variations and partial differential equations. These types of problems frequently emerge in real-world scenarios, including fluid flow through porous materials, thermally constrained systems, elasto-plastic models in mechanics, optimal control theory, and pricing models in finance.

While much progress has been made for single obstacle problems, many practical systems involve situations where a solution is constrained between both lower and upper bounds. This gives rise to the double obstacle problem, where the solution is restricted to lie between two obstacle functions. These problems are not only natural generalizations of the single obstacle case but also present new analytical challenges, especially in quasilinear settings and in the presence of lower-order terms, which can significantly affect the behavior and regularity of solutions. Motivated by these challenges, we aim to contribute to the theory by analyzing regularity properties of solutions to double obstacle problems governed by quasilinear operators with additional lower-order terms.

In the calculus of variations, the double obstacle problem is to find a function (solution) that minimizes the energy

$$\int_{\Omega} f(x, v, Dv) \, dx$$

among the functions $v \in W^{1,1}(\Omega)$ with the same boundary value, such that $\psi_1 \leq v \leq \psi_2$ for some obstacle functions ψ_1 and ψ_2 on $\Omega \subset \mathbb{R}^n$. Note that the single obstacle problem is the case $\psi_2 = \infty$. If we assume a suitable p -growth condition with $1 < p < \infty$ on f , then the solution u of the double obstacle problem is in $W^{1,p}(\Omega)$ and satisfies the variational inequality

$$\int_{\Omega} D_{\xi} f(x, u, Du) \cdot D(u - \varphi) + \partial_v f(x, u, Du)(u - \varphi) \, dx \leq 0$$

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for all $\varphi \in W^{1,p}(\Omega)$ such that $\psi_1 \leq \varphi \leq \psi_2$ and $u - \varphi \in W_0^{1,p}(\Omega)$. Here, $D_\xi f(x, v, \xi)$ is the gradient of f with respect to $\xi \in \mathbb{R}^n$, and $f_v(x, v, \xi)$ is the partial derivative of f with respect to $v \in \mathbb{R}$. A fundamental observation in the study of regularity theory for obstacle problems is that the regularity of the solution is strongly governed, often even determined, by how smooth the obstacle function is. Many researchers have explored the regularity properties of obstacle problems with p -growth. Notably, for single obstacle problems, Michael and Ziemer [29] proved that if the obstacle function is Hölder continuous, then the solution is also Hölder continuous. Later, Choe [11] showed that the Hölder continuity of the gradient of the obstacle function ensures the same for the gradient of the solution. Bögelein, Duzaar and Mingione [4] obtained L^q -regularity estimates for the gradient of the solution. For further regularity results on single obstacle problems, we refer to, for example, [3, 6, 12, 19, 26, 27, 33, 38]. On the other hand, for regularity results concerning the double obstacle problems, we refer to [13] for pointwise regularity, [22, 37] for Hölder continuity for the solution, [25, 31] for Hölder continuity for the gradient of the solution, and [7, 8] for L^q -estimates for the gradient of the solution.

We study L^q -regularity theory on variational inequalities when the function f has the form

$$f(x, v, \xi) = g(x, \xi) + \frac{1}{s}V(x)|v|^s - |F(x)|^{p-2}F(x) \cdot \xi, \quad (x, v, \xi) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n,$$

where $1 < p < \infty$, $1 < s < p^*$, $F(x)$ is a given \mathbb{R}^n -valued function, and $V(x)$ is a nonnegative function, often referred to as a potential function, satisfying certain integrability conditions. In this setting, we have that $D_\xi f(x, v, \xi) = D_\xi g(x, \xi) - |F(x)|^{p-2}F(x)$, and $f_v(x, v, \xi) = V(x)|v|^{s-2}v$. More generally, we consider a general \mathbb{R}^n -vector valued function $\mathbf{a}(x, \xi)$ of the p -Laplacian type, replacing $D_\xi g(x, \xi)$. The corresponding quasilinear elliptic equation then takes the form

$$-\operatorname{div} \mathbf{a}(x, Du) + V|u|^{s-2}u = -\operatorname{div}(|F|^{p-2}F). \quad (1.1)$$

In the simplest case when $p = s = 2$ and $\mathbf{a}(x, \xi) \equiv \xi$, this reduces to

$$(-\Delta + V)u = -\Delta u + Vu = -\operatorname{div} F,$$

where $-\Delta + V$ is known as the Schrödinger operator. For this equation, L^q -regularity theory has been studied by Shen [39, 40], and by Auscher and Ben [2]. Specifically, in [40], Shen proved that if $V \in \mathcal{B}_\gamma$ for some $\gamma > n/2$ (see Section 2.1 for the definition of \mathcal{B}_γ), then

$$\|Du\|_{L^{2q}(\mathbb{R}^n)} + \chi_{\{q \leq \gamma\}} \|V^{1/2}u\|_{L^{2q}(\mathbb{R}^n)} \leq c\|F\|_{L^q(\mathbb{R}^n)}$$

for all $(\gamma^*)'/2 \leq q \leq \gamma^*/2$. Here, $\chi_{\{q \leq \gamma\}} = 1$ if $q \leq \gamma$ and $\chi_{\{q \leq \gamma\}} = 0$ otherwise, and the constant $c > 0$ depends only on n, γ, q , and the relevant constant of the \mathcal{B}_γ condition, and is independent of the function V . For regularity results on linear equations associated with the Schrödinger operator, we refer to [5, 10, 14, 20, 21, 35, 47, 48, 50]. In recent years, L^q -regularity estimates have been successfully established for the corresponding nonlinear equations with p -growth in the form of (1.1) by the authors [23, 24]. We refer to [32, 42, 44, 45, 46, 49] for L^q -regularity theory on nonlinear problems with Schrödinger-type lower order terms.

In this article, we derive L^q estimates for $|Du|^p$ and $V|u|^s$, where u is a function satisfying the given variational inequality arising from the double obstacle problem associated with (1.1). Notably, the paper [32] considered double obstacle problems concerned with (1.1) in the case $s = p$, obtaining L^q -regularity estimates. However, their estimates are not sharp, as the right-hand side contains the additional terms involving $V|\psi_1|^p$ and $V|\psi_2|^p$. In contrast, our estimates exclude these terms, thereby providing sharper bounds and closely matching the estimates obtained in [24]. We also note that L^q estimates for the single obstacle problem associated with (1.1) have not been studied; our results can be readily adapted to this simpler case.

Our proof is based on a comparison scheme that reduces the analysis to auxiliary problems for which sharper estimates are available. The key idea is to construct three carefully designed comparison problems: a variational inequality associated with a single obstacle problem, a non-homogeneous equation in which the forcing term contains the super obstacle function, and the corresponding homogeneous equation. We emphasize that all these problems include the lower-order terms, which ultimately contribute to deriving the desired sharp estimates. A crucial step in this scheme involves comparing the solution of a localized double obstacle problem with that

of a suitably chosen single obstacle problem. This comparison effectively reduces the two-sided constraint to a one-sided one, allowing us to circumvent the inherent difficulties of the double obstacle setting. Once this reduction is achieved, the remainder of the argument proceeds in close analogy with the analysis of the single obstacle case. In particular, we derive comparison estimates for the gradient of the solution and the lower-order terms separately. This decoupling enables us to handle them independently in the final stage of establishing the L^q estimates, via the approach introduced by Mingione in [1, 30]. At this stage, we also make essential use of the higher integrability results for homogeneous equations established in [24] (see Section 2.2), which are specifically tailored to nonlinear elliptic equations with Schrödinger-type lower order terms.

1.1. Main results. Throughout the paper, let Ω be a bounded domain in \mathbb{R}^n with $n \geq 2$, $1 < p < \infty$, and $1 < s < p^*$ (see (2.1) for the definition of p^*). We always assume that $F \in L^p(\Omega, \mathbb{R}^n)$, and $\psi_1, \psi_2 \in W^{1,p}(\Omega)$ are obstacle functions satisfying that $\psi_1 \leq 0 \leq \psi_2$ a.e. in Ω . Note that the condition for the obstacles is stronger than $\psi_1 \leq \psi_2$ a.e. in Ω and $\max\{\psi_1, 0\}, \max\{-\psi_2, 0\} \in W_0^{1,p}(\Omega)$, which are typically assumed in zero Dirichlet boundary value problems. This stronger condition is crucial for the comparison estimates to handle the lower-order term $V|u|^{s-2}u$, see Section 3. Then we consider the admissible set

$$\mathcal{A}_0(\Omega) = \{\varphi \in W_0^{1,p}(\Omega) : \psi_1 \leq \varphi \leq \psi_2 \text{ a.e. in } \Omega\}.$$

The potential $V : \mathbb{R}^n \rightarrow [0, \infty)$ is assumed to satisfy that $V \in L_{loc}^{\gamma_0}(\mathbb{R}^n)$, where

$$\gamma_0 := \begin{cases} \frac{np}{np-s(n-p)}, & \text{when } 1 < p < n, \\ \text{any number larger than 1,} & \text{when } p = n, \\ 1, & \text{when } p > n. \end{cases} \tag{1.2}$$

and to be taken in an appropriate class \mathcal{B}_γ (see (2.2)). Note that $\gamma_0 = \frac{n}{p}$ if $s = p \in (1, n)$. The nonlinearity $\mathbf{a} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, which is a vector-valued Carathéodory function (i.e., $\mathbf{a}(x, \xi)$ is measurable in the x -variable and continuous in the ξ -variable), is assumed to satisfy that $\mathbf{a}(x, \cdot) \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$ for each $x \in \Omega$, and the growth and ellipticity conditions:

$$|\mathbf{a}(x, \xi)| + |D_\xi \mathbf{a}(x, \xi)| |\xi| \leq L |\xi|^{p-1}, \tag{1.3}$$

$$D_\xi \mathbf{a}(x, \xi) \eta \cdot \eta \geq \nu |\eta|^2 |\xi|^{p-2} \tag{1.4}$$

for a.e. $x \in \mathbb{R}^n$ and any $\xi, \eta \in \mathbb{R}^n$ ($\xi \neq 0$) and for some constants $0 < \nu \leq L$. The prototype of the nonlinearity \mathbf{a} is

$$\mathbf{a}(x, \xi) = (A(x)\xi \cdot \xi)^{\frac{p-2}{2}} A(x)\xi, \tag{1.5}$$

where $A : \mathbb{R}^n \rightarrow \mathbb{R}^{n^2}$ is an $n \times n$ matrix satisfying that

$$\nu |\eta|^2 \leq A(x)\eta \cdot \eta \quad \text{and} \quad |A(x)| \leq L, \quad x, \eta \in \mathbb{R}^n,$$

for some $0 < \nu \leq L$. We remark that condition (1.4) implies the monotonicity condition

$$(\mathbf{a}(x, \xi) - \mathbf{a}(x, \eta)) \cdot (\xi - \eta) \geq \tilde{\nu} (|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2 \tag{1.6}$$

for some $\tilde{\nu} > 0$ depending on p and ν , for any $\xi, \eta \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$.

We deal with the function $u \in \mathcal{A}_0(\Omega)$ that satisfies the variational inequality

$$\int_\Omega \mathbf{a}(x, Du) \cdot D(u - \varphi) \, dx + \int_\Omega V|u|^{s-2}u(u - \varphi) \, dx \leq \int_\Omega |F|^{p-2}F \cdot D(u - \varphi) \, dx \tag{1.7}$$

for all $\varphi \in \mathcal{A}_0(\Omega)$. Notice that under the above setting, the existence and uniqueness of a function satisfying (1.7), or the variational inequalities appearing in this paper, are ensured by general existence theory for monotone operators; see, for instance, [16, 41].

Now we present the main results of this paper. Definitions of additional conditions on the potential V , the nonlinearity \mathbf{a} and the domain Ω , which are necessary for obtaining L^q estimates, will be introduced in the next section. The first result provides local L^q estimates in both the interior and boundary regions. For the definitions of p^*, γ^* and \tilde{p} , see (2.1).

Theorem 1.1. *Let $1 < p < \infty$ and $1 < s < p^*$, and let $\mathbf{a} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfy (1.3) and (1.4), $F \in L^p(\Omega, \mathbb{R}^n)$, $\psi_1, \psi_2 \in W^{1,p}(\Omega)$ be such that $\psi_1 \leq 0 \leq \psi_2$ a.e. in Ω , and $V : \mathbb{R}^n \rightarrow [0, \infty)$ be such that $V \in L_{\text{loc}}^{\gamma_0}(\mathbb{R}^n)$ with γ_0 given in (1.2). For each p, γ and q satisfying specific conditions given below, there exists a small $\delta = \delta(n, p, q, L, \nu, \gamma) > 0$ such that if \mathbf{a} is (δ, R) -vanishing, Ω is a (δ, R) -Reifenberg flat domain for some $R > 0$, and $u \in \mathcal{A}_0(\Omega)$ satisfies the variational inequality (1.7) with $V \in \mathcal{B}_\gamma$, then the following estimates hold for any $x_0 \in \bar{\Omega}$ and $r \in (0, \frac{R}{2}]$: denote*

$$\Psi := |F|^p + |D\psi_1|^p + |D\psi_2|^p. \quad (1.8)$$

(1) *If $1 < p < \infty$, $\tilde{p} < \gamma < \infty$, and $1 < q < \frac{\gamma^*(p-1)}{p}$,*

$$\int_{\Omega_r(x_0)} |Du|^{pq} dx \leq c \left(\int_{\Omega_{2r}(x_0)} |Du|^p dx \right)^q + c \int_{\Omega_{2r}(x_0)} \Psi^q dx. \quad (1.9)$$

(2) *If $p \geq 2$, $1 < \gamma < \infty$, and $1 < q < \gamma$,*

$$\int_{\Omega_r(x_0)} [V|u|^s]^q dx \leq c \left(\int_{\Omega_{2r}(x_0)} V|u|^s dx \right)^q + c \int_{\Omega_{2r}(x_0)} \Psi^q dx. \quad (1.10)$$

(3) *If $1 < p < 2$, $\frac{n}{p} \leq \gamma < \infty$, and $1 < q < \gamma$,*

$$\int_{\Omega_r(x_0)} [V|u|^s]^q dx \leq c \left(\int_{\Omega_{2r}(x_0)} |Du|^p + V|u|^s dx \right)^q + c \int_{\Omega_{2r}(x_0)} \Psi^q dx. \quad (1.11)$$

Here, the constants $c > 0$ depend on $n, p, L, \nu, s, \gamma, b_\gamma$, and q .

For the upper bound γ of the exponent q , we refer to the comments in [24, Remarks 2.6 and 2.7]. The second result provides global L^q estimates. As a direct consequence of Theorem 1.1, we obtain the following global estimates by employing a standard covering argument, as used in the proof of [23, Corollary 2.6]. Therefore we omit the proof.

Theorem 1.2. *Let $1 < p < \infty$ and $1 < s < p^*$, and let $\mathbf{a} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfy (1.3) and (1.4), $F \in L^p(\Omega, \mathbb{R}^n)$, $\psi_1, \psi_2 \in W^{1,p}(\Omega)$ be such that $\psi_1 \leq 0 \leq \psi_2$ a.e. in Ω , and $V : \mathbb{R}^n \rightarrow [0, \infty)$ be such that $V \in L_{\text{loc}}^{\gamma_0}(\mathbb{R}^n)$ with γ_0 given in (1.2). For each p, γ and q satisfying specific conditions given below, there exists a small $\delta = \delta(n, p, q, L, \nu, \gamma) > 0$ such that if \mathbf{a} is (δ, R) -vanishing, Ω is a (δ, R) -Reifenberg flat domain for some $R > 0$, and $u \in \mathcal{A}_0(\Omega)$ satisfies the variational inequality (1.7) with $V \in \mathcal{B}_\gamma$, then the following estimates hold: recall Ψ given in (1.8).*

(1) *If $1 < p < \infty$, $\tilde{p} < \gamma < \infty$ and $1 < q < \frac{\gamma^*(p-1)}{p}$, then*

$$\|Du\|_{L^{pq}(\Omega)} \leq c \left(\frac{\text{diam}(\Omega)}{R} \right)^{n(q-1)} \|\Psi\|_{L^q(\Omega)}.$$

(2) *If $p \geq 2$, $1 < \gamma < \infty$ and $1 < q < \gamma$, or if $1 < p < 2$, $\frac{n}{p} \leq \gamma < \infty$ and $1 < q < \gamma$, then*

$$\|V^{1/p}|u|^{s/p}\|_{L^{pq}(\Omega)} \leq c \left(\frac{\text{diam}(\Omega)}{R} \right)^{n(q-1)} \|\Psi\|_{L^q(\Omega)}.$$

Here, the constants $c > 0$ depend on $n, p, L, \nu, s, \gamma, b_\gamma$, and q .

Remark 1.3. The nonnegative potential V is assumed to satisfy two conditions: $V \in L_{\text{loc}}^{\gamma_0}(\mathbb{R}^n)$ and $V \in \mathcal{B}_\gamma$. The first condition guarantees the existence of a function u satisfying the variational inequality (1.7). Note that if $1 < p < n$, then $\tilde{p} < \gamma_0$ and hence γ may be smaller than γ_0 . In this case, both conditions on V are required for the above results. On the other hand, if $p \geq n$, or if $1 < p < n$ and $\gamma \geq \gamma_0$, then the second condition implies the first.

The remainder of the paper is organized as follows. In Section 2, we present notation, the main assumptions and preliminary lemmas essential for our analysis. Section 3 is devoted to establishing key comparison estimates that play a central role in our approach. In Section 4, we provide the proof of our main theorem.

2. PRELIMINARIES

We begin with standard notation and definitions. For any $y \in \mathbb{R}^n$ and $r > 0$, we denote by $B_r(y)$ the open ball in \mathbb{R}^n centered at y with radius r . Given a domain $\Omega \subset \mathbb{R}^n$, we define the truncated domains by $\Omega_r(y) := B_r(y) \cap \Omega$ and $\partial_w \Omega_r(y) := B_r(y) \cap \partial\Omega$, which represent, respectively, the portion of Ω and its boundary within the ball $B_r(y)$. For simplicity of notation, when the center y is the origin or not relevant, we write $B_r := B_r(y)$ and $\Omega_r := \Omega_r(y)$. For a measurable function $f : U \rightarrow \mathbb{R}$ with $U \subset \mathbb{R}^n$, we use the notation

$$(f)_U := \int_U f \, dx = \frac{1}{|U|} \int_U f \, dx$$

to denote the average value of f over the set U , where $|U|$ stands for the Lebesgue measure of U . Furthermore, we define the positive part of f by $f_+ := \max\{f, 0\}$. For $f \in W^{1,p}(\Omega_r(y))$, the boundary condition “ $f = 0$ on $\partial_w \Omega_r(y)$ ” is understood in the sense that the extension of f by zero to $B_r(y)$ belongs to $W^{1,p}(B_r(y))$. In addition, we use the notation

$$p^* := \begin{cases} \frac{np}{n-p}, & \text{if } 1 < p < n, \\ \infty, & \text{if } p \geq n, \end{cases} \quad \text{and} \quad \tilde{p} := \begin{cases} \frac{np}{np-n+p}, & \text{if } 1 < p < n, \\ 1, & \text{if } p \geq n. \end{cases} \tag{2.1}$$

We use the following standard iteration lemma, its proof can be found in [18].

Lemma 2.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded nonnegative function. Suppose that for any τ_1, τ_2 with $0 \leq a \leq \tau_1 < \tau_2 \leq b$,*

$$f(\tau_1) \leq \tau f(\tau_2) + \frac{C_1}{(\tau_2 - \tau_1)^\beta} + C_2$$

where $C_1, C_2 \geq 0, \beta > 0$ and $0 \leq \tau < 1$. Then we have

$$f(\tau_1) \leq c \left(\frac{C_1}{(\tau_2 - \tau_1)^\beta} + C_2 \right)$$

for some constant $c = c(\beta, \tau) > 0$.

2.1. Main assumptions. We consider a nonnegative function $V : \mathbb{R}^n \rightarrow [0, \infty)$, which serves as a potential in the problem under consideration. We assume that V belongs to the class \mathcal{B}_γ for some $\gamma > 1$, denoted by $V \in \mathcal{B}_\gamma$, meaning that $V \in L^{\tilde{\gamma}}_{\text{loc}}(\mathbb{R}^n)$ and there exists a constant $b_\gamma > 0$ such that the following reverse Hölder inequality holds:

$$\left(\int_B V^\gamma \, dx \right)^{1/\gamma} \leq b_\gamma \int_B V \, dx \tag{2.2}$$

for every ball $B \subset \mathbb{R}^n$. The class \mathcal{B}_γ , which includes a broad range of functions such as all nonnegative polynomials, was introduced independently by Muckenhoupt [28] in the study of weighted norm inequalities and by Gehring [15] in the theory of quasiconformal mappings. A representative example of a function in this class is $V(x) = |x|^{-n/\gamma}$ which actually belongs to the $\mathcal{B}_{\tilde{p}}$ class for every $\tilde{p} < \gamma$. Moreover, the class \mathcal{B}_γ is closely related to the Muckenhoupt class A_p . We refer the reader to [17, Chapter 9] for a detailed account of the properties of the class \mathcal{B}_γ and its relationship with Muckenhoupt weights.

The following two definitions describe the structural assumptions imposed on the nonlinearity \mathbf{a} and the geometry of the domain Ω .

Definition 2.2. Let $\delta, R > 0$, and $\mathbf{a} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfy (1.3) and (1.4). We say that \mathbf{a} is (δ, R) -vanishing if

$$\sup_{0 < \rho \leq R} \sup_{y \in \mathbb{R}^n} \int_{B_\rho(y)} |\Theta(\mathbf{a}, B_\rho(y))(x)| \, dx \leq \delta,$$

where

$$\Theta(\mathbf{a}, B_\rho(y))(x) := \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|\mathbf{a}(x, \xi) - (\mathbf{a}(\cdot, \xi))_{B_\rho(y)}|}{|\xi|^{p-1}}$$

and

$$(\mathbf{a}(\cdot, \xi))_{B_\rho(y)} := \int_{B_\rho(y)} \mathbf{a}(x, \xi) \, dx.$$

The condition in the above definition implies that the map $x \mapsto \mathbf{a}(x, \xi)/|\xi|^{p-1}$ belongs to the local bounded mean oscillation (BMO) space with the BMO semi-norm controlled by δ for all $\xi \in \mathbb{R}^n$. Consequently, the nonlinearity \mathbf{a} is allowed to be discontinuous in the x -variable. In particular, in the model case (1.5), the definition implies that $A(\cdot)$ is locally BMO.

Definition 2.3. Let $\delta \in (0, 1/8)$ and $R > 0$. A domain $\Omega \subset \mathbb{R}^n$ is said to be a (δ, R) -Reifenberg flat domain if for every $x \in \partial\Omega$ and every $\rho \in (0, R]$, there exists a coordinate system $\{y_1, y_2, \dots, y_n\}$ which may depend on ρ and x , such that in this coordinate system $x = 0$ and that

$$B_\rho(0) \cap \{y_n > \delta\rho\} \subset B_\rho(0) \cap \Omega \subset B_\rho(0) \cap \{y_n > -\delta\rho\}.$$

The restriction $\delta < 1/8$ in the above definition is standard in the literature and arises naturally in connection with Sobolev embedding results (see, e.g., [43]). In our setting, this restriction is not essential, as we will only consider sufficiently small values of δ . Lipschitz domains with Lipschitz constants at most δ are special cases of (δ, R) -Reifenberg flat domains for some $R > 0$. Notably, the Reifenberg class allows for more general geometries that are not necessarily graph-defined. Furthermore, the (δ, R) -Reifenberg flat domain Ω satisfies the measure density condition

$$1 \leq \sup_{0 < \rho \leq R} \sup_{y \in \bar{\Omega}} \frac{|B_\rho(y)|}{|\Omega_\rho(y)|} \leq \left(\frac{2}{1-\delta}\right)^n \leq \left(\frac{16}{7}\right)^n. \quad (2.3)$$

We refer to [9, 34, 36, 43] for further information on Reifenberg flat domains and their applications.

2.2. Estimates for homogeneous equations. We consider the homogeneous Dirichlet problem

$$\begin{aligned} -\operatorname{div} \mathbf{a}(x, Dh) + V|h|^{s-2}h &= 0 \quad \text{in } \Omega_{2r}, \\ h &= 0 \quad \text{on } \partial_w \Omega_{2r}, \quad \text{if } B_{2r} \not\subset \Omega, \end{aligned} \quad (2.4)$$

where $\Omega_{2r} = \Omega_{2r}(x_0)$ for some $x_0 \in \bar{\Omega}$, $\mathbf{a} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies

$$|\mathbf{a}(x, \xi)| \leq L|\xi|^{p-1} \quad \text{and} \quad \mathbf{a}(x, \xi) \cdot \xi \geq \nu|\xi|^p, \quad x, \xi \in \mathbb{R}^n, \quad (2.5)$$

for some $0 < \nu \leq L$, and $V : \mathbb{R}^n \rightarrow [0, \infty)$ does $V \in L_{\text{loc}}^{\gamma_0}(\mathbb{R}^n)$. Note that the assumptions (1.3) and (1.4) imply (2.5).

We recall the following two reverse Hölder type higher integrability results for the homogeneous equation (2.4), see [24, Theorems 3.6 and 3.7] for their proofs.

Theorem 2.4. Assume $\mathbf{a} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies (2.5), and $V : \mathbb{R}^n \rightarrow [0, \infty)$ is such that $V \in L_{\text{loc}}^{\gamma_0}(\mathbb{R}^n)$ with γ_0 given in (1.2) and $V \in \mathcal{B}_\gamma$ for some $\gamma > 1$. If $h \in W^{1,p}(\Omega_{2r})$ is a weak solution to (2.4), then

$$\left(\frac{1}{|B_r|} \int_{\Omega_r} [V|h|^s]^\gamma dx\right)^{1/\gamma} \leq \frac{c}{|B_{2r}|} \int_{\Omega_{2r}} V|h|^s dx$$

for some $c = c(n, p, \nu, L, s, \gamma, b_\gamma) > 0$.

Theorem 2.5. Assume $\mathbf{a} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies (1.3) and (1.4), and $V : \mathbb{R}^n \rightarrow [0, \infty)$ is such that $V \in L_{\text{loc}}^{\gamma_0}(\mathbb{R}^n)$ with γ_0 given in (1.2) and $V \in \mathcal{B}_\gamma$ for some $\gamma \in (\tilde{p}, n)$ with \tilde{p} given in (2.1). Then there exists a small $\delta = \delta(n, p, L, \nu, \gamma) > 0$ such that the following holds: if \mathbf{a} is (δ, R) -vanishing, Ω is a (δ, R) -Reifenberg flat domain for some $R > 0$, and $h \in W^{1,p}(\Omega_{2r})$ is a weak solution to (2.4), then

$$\left(\int_{\Omega_r} |Dh|^{\gamma^*(p-1)} dx\right)^{\frac{p}{\gamma^*(p-1)}} \leq c \int_{\Omega_{2r}} |Dh|^p dx$$

for some $c = c(n, p, \nu, L, s, \gamma, b_\gamma) > 0$.

Remark 2.6. In the above theorems, it may occur that $\gamma < \gamma_0$. In this case, since $V \in L^{\gamma_0}$, we have $V|h|^s \in L^{\gamma_0}(\Omega_r(x_0))$ in Theorem 2.4, and $Dh \in L^{\gamma_0^*(p-1)}(\Omega_r(x_0), \mathbb{R}^n)$ in Theorem 2.5. However, the reverse Hölder-type estimates, where the constant c depends only on b_γ for V , can still be obtained with the exponent γ , owing to the assumption $V \in \mathcal{B}_\gamma$.

3. COMPARISON ESTIMATES

In this section, we derive comparison estimates with associated reference problems. Throughout the following lemmas and proposition, let $\mathbf{a} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfy (1.3) and (1.6) for some $L, \tilde{\nu} > 0$, and $V : \mathbb{R}^n \rightarrow [0, \infty)$ belong to $L^{\gamma_0}_{\text{loc}}(\mathbb{R}^n)$ with γ_0 given in (1.2). Furthermore, assume that $F \in L^p(\Omega, \mathbb{R}^n)$ and $\psi_1, \psi_2 \in W^{1,p}(\Omega)$ with $\psi_1 \leq 0 \leq \psi_2$ a.e. in Ω .

Before proceeding, we recall the following elementary inequalities, which will be used frequently in the sequel. For every $1 < q < \infty$ and every $\xi_1, \xi_2 \in \mathbb{R}^d$ with $d \in \mathbb{N}$, it holds

$$(|\xi_1|^2 + |\xi_2|^2)^{\frac{q-2}{2}} |\xi_2 - \xi_1|^2 \leq c(q, d) (|\xi_2|^{q-2} \xi_2 - |\xi_1|^{q-2} \xi_1) \cdot (\xi_2 - \xi_1) \tag{3.1}$$

for some constant $c(q, d) > 0$. Moreover, if $q \geq 2$,

$$|\xi_2 - \xi_1|^q \leq (|\xi_1|^2 + |\xi_2|^2)^{\frac{q-2}{2}} |\xi_2 - \xi_1|^2, \tag{3.2}$$

and, if $1 < q < 2$, we also have that for any $\kappa \in (0, 1)$,

$$\begin{aligned} |\xi_2 - \xi_1|^q &= |\xi_2 - \xi_1|^q (|\xi_2|^2 + |\xi_1|^2)^{\frac{q(q-2)}{4}} (|\xi_2|^2 + |\xi_1|^2)^{\frac{q(2-q)}{4}} \\ &\leq \kappa (|\xi_2|^2 + |\xi_1|^2)^{q/2} + c(\kappa, q) (|\xi_2|^2 + |\xi_1|^2)^{\frac{q-2}{2}} |\xi_2 - \xi_1|^2 \end{aligned} \tag{3.3}$$

for some constant $c(\kappa, q) > 0$.

We start by recalling the following comparison lemma. For its proof, we refer, for instance, to [6, Lemma 3.5].

Lemma 3.1. *Let U be a bounded open set in \mathbb{R}^n . Suppose that $v_1, v_2 \in W^{1,p}(U)$ satisfy that $(v_1 - v_2)_+ \in W^{1,p}_0(U)$ and*

$$\int_U (\mathbf{a}(x, Dv_1) - \mathbf{a}(x, Dv_2)) \cdot D(v_1 - v_2)_+ dx \leq 0.$$

Then $v_1 \leq v_2$ a.e. in U .

We derive comparison estimates. We first compare the function $u \in \mathcal{A}_0(\Omega)$, which satisfies the variational inequality (1.7) corresponding to a double obstacle problem, with a function that satisfies a variational inequality corresponding to a single obstacle problem in a local region. Fix any $x_0 \in \bar{\Omega}$ and write $\Omega_r = \Omega_r(x_0)$.

Lemma 3.2. *Let $u \in \mathcal{A}_0(\Omega)$ satisfy the variational inequality (1.7). If $k \in \tilde{\mathcal{A}}_u(\Omega_r) := \{f \in u + W^{1,p}_0(\Omega_r) : f \geq \psi_1 \text{ a.e. in } \Omega_r\}$ satisfies the following variational inequality*

$$\int_{\Omega_r} \mathbf{a}(x, Dk) \cdot D(k - \varphi) dx + \int_{\Omega_r} V|k|^{s-2} k(k - \varphi) dx \leq \int_{\Omega_r} \mathbf{a}(x, D\psi_2) \cdot D(k - \varphi) dx \tag{3.4}$$

for all $\varphi \in \tilde{\mathcal{A}}_u(\Omega_r)$. Then we have the following estimates:

(i) If $p \geq 2$, then

$$\int_{\Omega_r} |Du - Dk|^p dx \leq c \int_{\Omega_r} |F|^p + |D\psi_2|^p dx \tag{3.5}$$

and, for every $\epsilon \in (0, 1)$,

$$\int_{\Omega_r} V|u - k|^s dx \leq \epsilon \int_{\Omega_r} V|u|^s dx + c(\epsilon) \int_{\Omega_r} |F|^p + |D\psi_2|^p dx. \tag{3.6}$$

(ii) If $1 < p < 2$, then for every $\epsilon \in (0, 1)$,

$$\int_{\Omega_r} |Du - Dk|^p dx \leq \epsilon \int_{\Omega_r} |Du|^p dx + c(\epsilon) \int_{\Omega_r} |F|^p + |D\psi_2|^p dx \tag{3.7}$$

and

$$\int_{\Omega_r} V|u - k|^s dx \leq \epsilon \int_{\Omega_r} |Du|^p + V|u|^s dx + c(\epsilon) \int_{\Omega_r} |F|^p + |D\psi_2|^p dx. \tag{3.8}$$

Here, $c > 0$ depends on $n, p, L, \tilde{\nu}$, and s , while $c(\epsilon) > 0$ depends on $n, p, L, \tilde{\nu}, s$, and ϵ .

Proof. We note that $\varphi := \min\{k, \psi_2\} = k - (k - \psi_2)_+ \in \tilde{\mathcal{A}}_u(\Omega_r)$. Plugging this φ into (3.4) yields

$$\int_{\Omega_r} (\mathbf{a}(x, Dk) - \mathbf{a}(x, D\psi_2)) \cdot D(k - \psi_2)_+ dx \leq - \int_{\Omega_r} V|k|^{p-2}k(k - \psi_2)_+ dx \leq 0,$$

where, in the last inequality, we have used the fact that $k \geq \psi_2 \geq 0$ when $(k - \psi_2)_+ \geq 0$. Hence, by Lemma 3.1, $k \leq \psi_2$ a.e. in Ω_r . Moreover, since $k - u \in W_0^{1,p}(\Omega_r)$, we extend k to $\Omega \setminus \Omega_r$ by u . Then $k \in \mathcal{A}_0(\Omega)$, since $u \in \mathcal{A}_0(\Omega)$. Now we take the test functions $\varphi := k$ and $\varphi := u$ into (1.7) and (3.4), respectively, and combine the two resulting estimates, to obtain

$$\begin{aligned} & \int_{\Omega_r} (\mathbf{a}(x, Du) - \mathbf{a}(x, Dk)) \cdot D(u - k) dx + \int_{\Omega_r} (V|u|^{s-2}u - V|k|^{s-2}k)(u - k) dx \\ & \leq \int_{\Omega_r} (\mathbf{a}(x, D\psi_2) - |F|^{p-2}F) \cdot D(k - u) dx \\ & \leq c \int_{\Omega_r} (|F|^{p-1} + |D\psi_2|^{p-1})|Du - Dk| dx, \end{aligned} \quad (3.9)$$

where we have used (1.3) in the last inequality.

We first consider the case $p \geq 2$. Then we derive from the monotonicity condition (1.6) and the elementary inequalities (3.1) and (3.2) that

$$\begin{aligned} & \int_{\Omega_r} |Du - Dk|^p dx + \int_{\Omega_r} V(|u|^2 + |k|^2)^{\frac{s-2}{2}}|u - k|^2 dx \\ & \leq \sigma_1 \int_{\Omega_r} |Du - Dk|^p dx + c(\sigma_1) \int_{\Omega_r} |F|^p + |D\psi_2|^p dx \end{aligned}$$

for any $\sigma_1 \in (0, 1)$, by using Young's inequality. Taking $\sigma_1 = 1/2$, we obtain

$$\int_{\Omega_r} |Du - Dk|^p dx + \int_{\Omega_r} V(|u|^2 + |k|^2)^{\frac{s-2}{2}}|u - k|^2 dx \leq c \int_{\Omega_r} |F|^p + |D\psi_2|^p dx. \quad (3.10)$$

Note that this directly implies (3.5) and, using (3.2), also implies (3.6) when $s \geq 2$. We next prove (3.6) when $1 < s < 2$. By using (3.3) and (3.10), we derive that

$$\begin{aligned} \int_{\Omega_r} V|u - k|^s dx & \leq \sigma_2 \int_{\Omega_r} V(|u|^2 + |k|^2)^{\frac{s}{2}} dx + c(\sigma_2) \int_{\Omega_r} V(|u|^2 + |k|^2)^{\frac{s-2}{2}}|u - k|^2 dx \\ & \leq \sigma_2 \int_{\Omega_r} V(|u|^s + |k|^s) dx + c(\sigma_2) \int_{\Omega_r} |F|^p + |D\psi_2|^p dx, \end{aligned} \quad (3.11)$$

for any $\sigma_2 \in (0, 1)$. We first choose $\sigma_2 = 1/2$, so that

$$\begin{aligned} \int_{\Omega_r} V|k|^s dx & \leq \int_{\Omega_r} V|u|^s dx + \int_{\Omega_r} V|u - k|^s dx \\ & \leq \frac{1}{2} \int_{\Omega_r} V|k|^s dx + \frac{3}{2} \int_{\Omega_r} V|u|^s dx + c \int_{\Omega_r} |F|^p + |D\psi_2|^p dx, \end{aligned}$$

which implies that

$$\int_{\Omega_r} V|k|^s dx \leq c \int_{\Omega_r} V|u|^s + |F|^p + |D\psi_2|^p dx.$$

We insert this into (3.11), and obtain

$$\int_{\Omega_r} V|u - k|^s dx \leq c\sigma_2 \int_{\Omega_r} V|u|^s dx + c(\sigma_2) \int_{\Omega_r} |F|^p + |D\psi_2|^p dx,$$

which implies the estimate (3.6) when $1 < s < 2$.

We next consider the case $1 < p < 2$. From (3.9) with (1.6), (3.1), (3.3) and Young's inequality, we then derive that

$$\int_{\Omega_r} |Du - Dk|^p dx + \int_{\Omega_r} V(|u|^2 + |k|^2)^{\frac{s-2}{2}}|u - k|^2 dx$$

$$\begin{aligned}
 &\leq \sigma_3 \int_{\Omega_r} (|Du|^2 + |Dk|^2)^{\frac{s}{2}} dx + c(\sigma_3) \int_{\Omega_r} (|Du|^2 + |Dk|^2)^{\frac{s-2}{2}} |Du - Dk|^2 dx \\
 &\quad + \int_{\Omega_r} V(|u|^2 + |k|^2)^{\frac{s-2}{2}} |u - k|^2 dx \\
 &\leq \sigma_3 \int_{\Omega_r} |Du|^p + |Dk|^p dx + c(\sigma_3) \int_{\Omega_r} (|F|^{p-1} + |D\psi_2|^{p-1}) |Du - Dk| dx \\
 &\leq \sigma_3 \int_{\Omega_r} |Du|^p + |Dk|^p dx + \sigma_1 \int_{\Omega_r} |Du - Dk|^p dx + c(\sigma_1, \sigma_3) \int_{\Omega_r} |F|^p + |D\psi_2|^p dx
 \end{aligned}$$

for any $\sigma_3, \sigma_1 \in (0, 1)$. By taking $\sigma_1 = 1/2$, we have

$$\begin{aligned}
 &\int_{\Omega_r} |Du - Dk|^p dx + \int_{\Omega_r} V(|u|^2 + |k|^2)^{\frac{s-2}{2}} |u - k|^2 dx \\
 &\leq c\sigma_3 \int_{\Omega_r} |Du|^p + |Dk|^p dx + c(\sigma_3) \int_{\Omega_r} |F|^p + |D\psi_2|^p dx.
 \end{aligned}$$

We first select $\sigma_3 \in (0, 1)$ small, to obtain

$$\int_{\Omega_r} |Dk|^p dx \leq c \int_{\Omega_r} |Du|^p + |F|^p + |D\psi_2|^p dx.$$

Then inserting this into the previous estimate, we obtain

$$\begin{aligned}
 &\int_{\Omega_r} |Du - Dk|^p dx + \int_{\Omega_r} V(|u|^2 + |k|^2)^{\frac{s-2}{2}} |u - k|^2 dx \\
 &\leq c\sigma_3 \int_{\Omega_r} |Du|^p dx + c(\sigma_3) \int_{\Omega_r} |F|^p + |D\psi_2|^p dx
 \end{aligned} \tag{3.12}$$

for any small $\sigma_3 \in (0, 1)$. This implies the estimate (3.7) and, using (3.2), also implies (3.8) when $s \geq 2$. It remains to prove (3.8) when $1 < s < 2$. Using (3.3), we have from (3.12) that for any $\sigma_4 \in (0, 1)$,

$$\begin{aligned}
 &\int_{\Omega_r} V|u - k|^s dx \\
 &\leq \sigma_4 \int_{\Omega_r} V(|u|^2 + |k|^2)^{\frac{s}{2}} dx + c(\sigma_4) \int_{\Omega_r} V(|u|^2 + |k|^2)^{\frac{s-2}{2}} |u - k|^2 dx \\
 &\leq c\sigma_4 \int_{\Omega_r} V(|u|^s + |k|^s) dx + c(\sigma_4)\sigma_3 \int_{\Omega_r} |Du|^p dx + c(\sigma_3, \sigma_4) \int_{\Omega_r} |F|^p + |D\psi_2|^p dx.
 \end{aligned} \tag{3.13}$$

We first select $\sigma_4 > 0$ so small and $\sigma_3 = 1/2$, to obtain

$$\int_{\Omega_r} V|k|^s dx \leq c \int_{\Omega_r} V|u|^s + |Du|^p + |F|^p + |D\psi_2|^p dx.$$

Inserting this into (3.13), we obtain

$$\begin{aligned}
 &\int_{\Omega_r} V|u - k|^s dx \\
 &\leq c\sigma_4 \int_{\Omega_r} V|u|^s dx + \sigma_3 c(\sigma_4) \int_{\Omega_r} |Du|^p dx + c(\sigma_3, \sigma_4) \int_{\Omega_r} |F|^p + |D\psi_2|^p dx
 \end{aligned}$$

for any small $\sigma_3, \sigma_4 \in (0, 1)$. This implies the estimate (3.8) when $1 < s < 2$. □

We next compare the function k in the above lemma with a weak solution to a homogeneous quasilinear equation. Let $v \in W^{1,p}(\Omega_r)$ be a weak solution to

$$\begin{aligned}
 &-\operatorname{div} \mathbf{a}(x, Dv) + V|v|^{s-2}v = -\operatorname{div} \mathbf{a}(x, D\psi_1) \quad \text{in } \Omega_r, \\
 &v = k \quad \text{on } \partial\Omega_r,
 \end{aligned} \tag{3.14}$$

and $h \in W^{1,p}(\Omega_r)$ be a weak solution to

$$\begin{aligned} -\operatorname{div} \mathbf{a}(x, Dh) + V|h|^{s-2}h &= 0 \quad \text{in } \Omega_r, \\ h &= v \quad \text{on } \partial\Omega_r. \end{aligned} \quad (3.15)$$

Lemma 3.3. *Let $k \in \tilde{\mathcal{A}}_u(\Omega_r)$ satisfy the variational inequality (3.4) given in Lemma 3.2, and $h \in W^{1,p}(\Omega_r)$ be a weak solution to (3.15). Then we have the following estimates:*

(i) If $p \geq 2$,

$$\int_{\Omega_r} |Dk - Dh|^p dx \leq c \int_{\Omega_r} |D\psi_1|^p + |D\psi_2|^p dx, \quad (3.16)$$

and, for every $\epsilon \in (0, 1)$,

$$\int_{\Omega_r} V|k - h|^s dx \leq \epsilon \int_{\Omega_r} V|k|^s dx + c(\epsilon) \int_{\Omega_r} |D\psi_1|^p + |D\psi_2|^p dx. \quad (3.17)$$

(ii) If $1 < p < 2$, for every $\epsilon \in (0, 1)$,

$$\int_{\Omega_r} |Dk - Dh|^p dx \leq \epsilon \int_{\Omega_r} |Dk|^p dx + c(\epsilon) \int_{\Omega_r} |D\psi_1|^p + |D\psi_2|^p dx, \quad (3.18)$$

and

$$\int_{\Omega_r} V|k - h|^s dx \leq \epsilon \int_{\Omega_r} |Dk|^p + V|k|^s dx + c(\epsilon) \int_{\Omega_r} |D\psi_1|^p + |D\psi_2|^p dx. \quad (3.19)$$

Here, $c > 0$ depends on n, p, L, \tilde{v} , and s , while $c(\epsilon) > 0$ depends on n, p, L, \tilde{v}, s , and ϵ .

Proof. Since $v = k = u$ on $\partial\Omega_r$ in the trace sense, we have $(\psi_1 - v)_+ \in W_0^{1,p}(\Omega_r)$. Then by adopting the test function $(\psi_1 - v)_+$ in the weak formulation of (3.14), we have

$$\int_{\Omega_r} (\mathbf{a}(x, D\psi_1) - \mathbf{a}(x, Dv)) \cdot D(\psi_1 - v)_+ dx = \int_{\Omega_r} V|v|^{s-2}v(\psi_1 - v)_+ dx \leq 0,$$

where, in the last inequality, we have used the fact that $v \leq \psi_1 \leq 0$ when $(\psi_1 - v)_+ \geq 0$. Then by Lemma 3.1 with $v_1 := \psi_1$ and $v_2 := v$, we have $v \geq \psi_1$ a.e. in Ω_r . Therefore, $v \in \tilde{\mathcal{A}}_u(\Omega_r)$ and so we take v as a test function into (3.4) in order to infer that

$$\begin{aligned} &\int_{\Omega_r} \mathbf{a}(x, Dk) \cdot D(k - v) dx + \int_{\Omega_r} V|k|^{s-2}k(k - v) dx \\ &\leq \int_{\Omega_r} \mathbf{a}(x, D\psi_2) \cdot D(k - v) dx. \end{aligned}$$

Furthermore, we take $v - k \in W_0^{1,p}(\Omega_r)$ as a test function in the weak formulation of (3.14) to discover that

$$\begin{aligned} &\int_{\Omega_r} \mathbf{a}(x, Dv) \cdot D(v - k) dx + \int_{\Omega_r} V|v|^{s-2}v(v - k) dx \\ &= \int_{\Omega_r} \mathbf{a}(x, D\psi_1) \cdot D(v - k) dx. \end{aligned}$$

Combining the above two estimates and using (1.3), we obtain

$$\begin{aligned} &\int_{\Omega_r} (\mathbf{a}(x, Dk) - \mathbf{a}(x, Dv)) \cdot D(k - v) dx + \int_{\Omega_r} (V|k|^{s-2}k - V|v|^{s-2}v)(k - v) dx \\ &\leq c \int_{\Omega_r} (|D\psi_1|^{p-1} + |D\psi_2|^{p-1})|Dk - Dv| dx. \end{aligned}$$

We notice that this estimate is the same as (3.9) with u, k , and $|F|^{p-1} + |D\psi_2|^{p-1}$ replaced by k, v , and $|D\psi_1|^{p-1} + |D\psi_2|^{p-1}$, respectively. Therefore, by repeating the same arguments used in the proof of Lemma 3.2 for (3.9), we obtain the following estimates: for $\epsilon \in (0, 1)$: if $p \geq 2$, then

$$\int_{\Omega_r} |Dk - Dv|^p dx \leq c \int_{\Omega_r} |D\psi_1|^p + |D\psi_2|^p dx \quad (3.20)$$

and

$$\int_{\Omega_r} V|k - v|^s dx \leq \epsilon \int_{\Omega_r} V|k|^s dx + c(\epsilon) \int_{\Omega_r} |D\psi_1|^p + |D\psi_2|^p dx; \tag{3.21}$$

and if $1 < p < 2$, then

$$\int_{\Omega_r} |Dk - Dv|^p dx \leq \epsilon \int_{\Omega_r} |Dk|^p dx + c(\epsilon) \int_{\Omega_r} |D\psi_1|^p + |D\psi_2|^p dx \tag{3.22}$$

and

$$\int_{\Omega_r} V|k - v|^s dx \leq \epsilon \int_{\Omega_r} |Dk|^p + V|k|^s dx + c(\epsilon) \int_{\Omega_r} |D\psi_1|^p + |D\psi_2|^p dx. \tag{3.23}$$

We next test the equations (3.14) and (3.15) with the test function $\varphi := v - h \in W_0^{1,p}(\Omega_r)$ to derive

$$\int_{\Omega_r} \mathbf{a}(x, Dv) \cdot D(v - h) dx + \int_{\Omega_r} V|v|^{s-2}v(v - h) dx = \int_{\Omega_r} \mathbf{a}(x, D\psi_1) \cdot D(v - h) dx$$

and

$$\int_{\Omega_r} \mathbf{a}(x, Dh) \cdot D(v - h) dx + \int_{\Omega_r} V|h|^{s-2}h(v - h) dx = 0.$$

Combining these estimates and using (1.3), we have

$$\begin{aligned} & \int_{\Omega_r} (\mathbf{a}(x, Dv) - \mathbf{a}(x, Dh)) \cdot D(v - h) dx + \int_{\Omega_r} (V|v|^{s-2}v - V|h|^{s-2}h)(v - h) dx \\ & \leq c \int_{\Omega_r} |D\psi_1|^{p-1} |Dv - Dh| dx. \end{aligned}$$

We observe that this estimate has the same structure as (3.9) with u, k , and $|F|^{p-1} + |D\psi_2|^{p-1}$ replaced by v, h , and $|D\psi_1|^{p-1}$, respectively. Therefore, by applying the same argument used for (3.9) in the proof of Lemma 3.2, we obtain the following estimate: for $\epsilon \in (0, 1)$, if $p \geq 2$, then

$$\int_{\Omega_r} |Dv - Dh|^p dx \leq c \int_{\Omega_r} |D\psi_1|^p dx \tag{3.24}$$

and

$$\begin{aligned} \int_{\Omega_r} V|v - h|^s dx & \leq \epsilon \int_{\Omega_r} V|v|^s dx + c(\epsilon) \int_{\Omega_r} |D\psi_1|^p dx \\ & \leq c\epsilon \left(\int_{\Omega_r} V|k - v|^s dx + \int_{\Omega_r} V|k|^s dx \right) + c(\epsilon) \int_{\Omega_r} |D\psi_1|^p dx; \end{aligned} \tag{3.25}$$

and if $1 < p < 2$, then

$$\begin{aligned} \int_{\Omega_r} |Dv - Dh|^p dx & \leq \epsilon \int_{\Omega_r} |Dv|^p dx + c(\epsilon) \int_{\Omega_r} |D\psi_1|^p dx \\ & \leq c\epsilon \left(\int_{\Omega_r} |Dk - Dv|^p dx + \int_{\Omega_r} |Dk|^p dx \right) + c(\epsilon) \int_{\Omega_r} |D\psi_1|^p dx, \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} \int_{\Omega_r} V|v - h|^s dx & \leq \epsilon \int_{\Omega_r} |Dv|^p + V|v|^s dx + c(\epsilon) \int_{\Omega_r} |D\psi_1|^p dx \\ & \leq c\epsilon \int_{\Omega_r} |Dk - Dv|^p + V|k - v|^s dx \\ & \quad + c\epsilon \int_{\Omega_r} |Dk|^p + V|k|^s dx + c(\epsilon) \int_{\Omega_r} |D\psi_1|^p dx. \end{aligned} \tag{3.27}$$

Therefore, combining (3.20)–(3.23) and (3.24)–(3.27), we arrive at the desired estimates. □

Combining Lemmas 3.2 and 3.3, we obtain the following comparison estimates.

Proposition 3.4. *Let $u \in \mathcal{A}_0(\Omega)$ satisfy the variational inequality (1.7), and $h \in W^{1,p}(\Omega_r)$ be the weak solution to (3.15). Then we have the estimates:*

(i) (Energy estimate)

$$\int_{\Omega_r} |Dh|^p dx \leq c \int_{\Omega_r} |Du|^p + |F|^p + |D\psi_1|^p + |D\psi_2|^p dx, \quad (3.28)$$

and

$$\int_{\Omega_r} V|h|^s dx \leq c \int_{\Omega_r} V|u|^s + |F|^p + |D\psi_1|^p + |D\psi_2|^p + |Du|^p \chi_{\{p < 2\}} dx, \quad (3.29)$$

where $\chi_{\{p < 2\}} = 1$ if $1 < p < 2$ and $\chi_{\{p < 2\}} = 0$ if $p \geq 2$.

(ii) If $p \geq 2$,

$$\int_{\Omega_r} |Du - Dh|^p dx \leq c \int_{\Omega_r} |F|^p + |D\psi_1|^p + |D\psi_2|^p dx, \quad (3.30)$$

and, for every $\epsilon \in (0, 1)$,

$$\int_{\Omega_r} V|u - h|^s dx \leq \epsilon \int_{\Omega_r} V|u|^s dx + c(\epsilon) \int_{\Omega_r} |F|^p + |D\psi_1|^p + |D\psi_2|^p dx. \quad (3.31)$$

(iii) If $1 < p < 2$, for every $\epsilon \in (0, 1)$,

$$\int_{\Omega_r} |Du - Dh|^p dx \leq \epsilon \int_{\Omega_r} |Du|^p dx + c(\epsilon) \int_{\Omega_r} |F|^p + |D\psi_1|^p + |D\psi_2|^p dx, \quad (3.32)$$

and

$$\int_{\Omega_r} V|u - h|^s dx \leq \epsilon \int_{\Omega_r} |Du|^p + V|u|^s dx + c(\epsilon) \int_{\Omega_r} |F|^p + |D\psi_1|^p + |D\psi_2|^p dx. \quad (3.33)$$

Here, $c > 0$ depends on $n, p, L, \tilde{\nu}$, and s , while $c(\epsilon) > 0$ depends on $n, p, L, \tilde{\nu}, s$, and ϵ .

Proof. We first observe from (3.5) and (3.16), when $p \geq 2$, and from (3.7) and (3.18), when $1 < p < 2$, that

$$\int_{\Omega_r} |Dk|^p dx \leq c \int_{\Omega_r} |Du|^p + |F|^p + |D\psi_2|^p dx, \quad (3.34)$$

and hence

$$\begin{aligned} \int_{\Omega_r} |Dh|^p dx &\leq c \int_{\Omega_r} |Dk|^p + |D\psi_1|^p + |D\psi_2|^p dx \\ &\leq c \int_{\Omega_r} |Du|^p + |F|^p + |D\psi_1|^p + |D\psi_2|^p dx. \end{aligned}$$

Note that the last inequality verifies (3.28). Moreover, from (3.6) and (3.17), when $p \geq 2$, and from (3.8) and (3.19), when $1 < p < 2$, that

$$\int_{\Omega_r} V|k|^s dx \leq c \int_{\Omega_r} V|u|^s + |F|^p + |D\psi_2|^p + |Du|^p \chi_{\{p < 2\}} dx, \quad (3.35)$$

and hence

$$\begin{aligned} \int_{\Omega_r} V|h|^s dx &\leq c \int_{\Omega_r} V|k|^s + |D\psi_1|^p + |D\psi_2|^p + |Dk|^p \chi_{\{p < 2\}} dx \\ &\leq c \int_{\Omega_r} V|u|^s + |F|^p + |D\psi_1|^p + |D\psi_2|^p + |Du|^p \chi_{\{p < 2\}} dx. \end{aligned}$$

Note that the last inequality verifies (3.29).

Finally, by plugging (3.34) and (3.35) into the estimates in Lemma 3.3 and combining these with the estimates in Lemma 3.2, we obtain the desired estimates (3.30)-(3.33). \square

Remark 3.5. By carefully following the proofs of the above lemmas in the case $s \geq 2$, we can actually obtain the following sharper comparison estimates:

$$\int_{\Omega_r} V|u - h|^s dx \leq c \int_{\Omega_r} |F|^p + |D\psi_1|^p + |D\psi_2|^p dx,$$

when $p \geq 2$, and

$$\int_{\Omega_r} V|u - h|^s dx \leq \epsilon \int_{\Omega_r} |Du|^p dx + c(\epsilon) \int_{\Omega_r} |F|^p + |D\psi_1|^p + |D\psi_2|^p dx,$$

for every $\epsilon \in (0, 1)$, when $1 < p < 2$. Note that these bounds are sharper than the estimates (3.31) and (3.33), respectively. However, they do not yield any additional advantage in the proof of Theorem 1.1 in the next section. For this reason, we proceed with the estimates (3.31) and (3.33), which are valid for both the cases $1 < s < 2$ and $2 \leq s < p^*$.

4. L^q ESTIMATES

We are now ready to prove our main results.

Proof of Theorem 1.1. We follow the approach introduced by Mingione in [1, 30]. The overall proof strategy is essentially the same as in [24, Theorem 2.3], so we omit some of the technical details.

Step 1. (Setting) Assume $\mathbf{a} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies (1.3) and (1.4) and $V : \mathbb{R}^n \rightarrow [0, \infty)$ is such that $V \in L^{\gamma_0}_{\text{loc}}(\mathbb{R}^n)$ with γ_0 given in (1.2). Furthermore, we assume that $\mathbf{a} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is (δ, R) -vanishing and Ω is (δ, R) -Reifenberg flat for some $R > 0$, where $\delta \in (0, 1)$ will be chosen sufficiently small later in Step 3 (see Remark 4.2), and that $V \in \mathcal{B}_\gamma$, where the range of γ is given in (1)–(3) of Theorem 1.1. Finally, let $F \in L^p(\Omega, \mathbb{R}^n)$ and $\psi_1, \psi_2 \in W^{1,p}(\Omega)$ with $\psi_1 \leq 0 \leq \psi_2$ a.e. in Ω . Fix any $x_0 \in \bar{\Omega}$ and $r > 0$ satisfying $r \leq R/2$.

We prove the estimates in (1)–(3) of Theorem 1.1, specifically (1.9)–(1.11), at one time by defining the function $\Phi(w; x)$ for $w \in W^{1,p}(\Omega)$ and the constant γ_1 differently in each case as follows:

Case 1: Estimation of (1.9). Let $1 < p < \infty$ and $\tilde{p} < \gamma < \infty$ with \tilde{p} given in (2.1). We fix $q \in (1, \frac{\gamma^*(p-1)}{p})$ with γ^* given in (2.1), and denote

$$\Phi(w) = \Phi(w; x) := |Dw(x)|^p \quad \text{and} \quad \gamma_1 := \begin{cases} \frac{\gamma^*(p-1)}{p} & \text{if } \gamma < n, \\ q + 1 & \text{if } \gamma \geq n. \end{cases}$$

Note that, when $\gamma \geq n$, there exists $\gamma_2 \in (1, n)$ such that $q + 1 = \frac{\gamma_2^*(p-1)}{p}$ since $\frac{(q+1)p}{p-1} > 2$.

Case 2: Estimation of (1.10). Let $p \geq 2$ and $1 < \gamma < \infty$. We fix any $q \in (1, \gamma)$, and denote

$$\Phi(w) = \Phi(w; x) := V(x)|w(x)|^s \quad \text{and} \quad \gamma_1 := \gamma.$$

Case 3: Estimation of (1.11). Let $1 < p < 2$ and $n/p \leq \gamma < \infty$. We fix any $q \in (1, \gamma)$, and denote

$$\Phi(w) = \Phi(w; x) := |Dw(x)|^p + V(x)|w(x)|^s \quad \text{and} \quad \gamma_1 := \gamma.$$

Since $\gamma_1 \leq \gamma$, we note that $V \in \mathcal{B}_{\gamma_1}$ in all the above cases. With the function Φ defined above and $u \in \mathcal{A}_0(\Omega)$ satisfying the variational inequality (1.7), we define

$$E(\lambda, \rho) := \{x \in \Omega_\rho : \Phi(u; x) > \lambda\} \quad \text{for } \lambda > 0,$$

and

$$\lambda_0 := \int_{\Omega_{2r}} \Phi(u) dx + \frac{1}{\delta_1} \int_{\Omega_{2r}} \Psi dx, \tag{4.1}$$

where $\delta_1 \in (0, 1)$ will be chosen sufficiently small later in Step 4, and Ψ is given in (1.8). Finally, fix any τ_1, τ_2 such that $1 \leq \tau_1 < \tau_2 \leq 2$, and then it follows that $\Omega_r \subset \Omega_{\tau_1 r} \subset \Omega_{\tau_2 r} \subset \Omega_{2r}$.

Step 2. (Covering lemma) We consider $\lambda > 0$ large enough such that

$$\lambda > \alpha \lambda_0, \quad \text{where } \alpha := \left(\frac{16}{7}\right)^n \left(\frac{20}{\tau_2 - \tau_1}\right)^n. \tag{4.2}$$

Note that $(\frac{16}{7})^n$ is originated from (2.3). Then we have the following lemma. For the proof we refer to [24, Lemma 4.3].

Lemma 4.1. *Given $\lambda > \alpha\lambda_0$, there exists a disjoint family $\{\Omega_{\rho_i}(y^i)\}_{i=1}^\infty$ with $y^i \in E(\lambda, \tau_1 r)$ and $\rho_i \in (0, \frac{(\tau_2 - \tau_1)r}{10})$ such that*

$$E(\lambda, \tau_1 r) \setminus \mathcal{N} \subset \bigcup_{i=1}^\infty \Omega_{5\rho_i}(y^i)$$

for some measure zero set \mathcal{N} ,

$$\int_{\Omega_{\rho_i}(y^i)} \Phi(u) \, dx + \frac{1}{\delta_1} \int_{\Omega_{\rho_i}(y^i)} \Psi \, dx = \lambda, \tag{4.3}$$

and for any $\rho \in (\rho_i, (\tau_2 - \tau_1)r]$,

$$\int_{\Omega_\rho(y^i)} \Phi(u) \, dx + \frac{1}{\delta_1} \int_{\Omega_\rho(y^i)} \Psi \, dx < \lambda. \tag{4.4}$$

Furthermore, from (4.3) we have

$$|\Omega_{\rho_i}(y^i)| \leq \frac{2}{\lambda} \left(\int_{\Omega_{\rho_i}(y^i) \cap \{\Phi(u) > \frac{\lambda}{4}\}} \Phi(u) \, dx + \frac{1}{\delta_1} \int_{\Omega_{\rho_i}(y^i) \cap \{\Psi > \frac{\delta_1 \lambda}{4}\}} \Psi \, dx \right). \tag{4.5}$$

Step 3. (Comparison estimates) For each i , we note from (4.4) in Lemma 4.1 that

$$\int_{\Omega_{10\rho_i}(y^i)} \Phi(u) \, dx + \frac{1}{\delta_1} \int_{\Omega_{10\rho_i}(y^i)} \Psi \, dx < \lambda.$$

Applying Proposition 3.4 (ii) and (iii) with $r = 10\rho_i$, we have that for any $\epsilon \in (0, 1)$, there exists a small $\delta_1 = \delta_1(\epsilon, n, p, L, \nu, s) \in (0, 1)$ such that

$$\begin{aligned} \int_{\Omega_{10\rho_i}(y^i)} \Phi(u - h_i) \, dx &\leq \epsilon \int_{\Omega_{10\rho_i}(y^i)} \Phi(u) \, dx + c(\epsilon) \int_{\Omega_{10\rho_i}(y^i)} \Psi \, dx \\ &\leq \epsilon\lambda + c(\epsilon)\delta_1\lambda \leq 2\epsilon\lambda \end{aligned} \tag{4.6}$$

(In fact, when $p \geq 2$ in Case 1, the constant $c(\epsilon)$ in (4.6) can be chosen as a constant independent of ϵ), where $h_i \in W^{1,p}(\Omega_{10\rho_i}(y^i))$ is a weak solution to

$$\begin{aligned} -\operatorname{div} \mathbf{a}(x, Dh_i) + V|h_i|^{s-2}h_i &= 0 \quad \text{in } \Omega_{10\rho_i}(y^i), \\ h_i &= 0 \quad \text{on } \partial_w \Omega_{10\rho_i}(y^i) \quad \text{if } B_{10\rho_i}(x_0) \not\subset \Omega. \end{aligned}$$

Furthermore, applying Theorem 2.4 to Case 2 and Case 3, and Theorem 2.5 to Case 1 and Case 3 with $\gamma = \gamma_1$, where γ_1 defined in each case, and also applying Proposition 3.4 (i), we have

$$\left(\int_{\Omega_{5\rho_i}(y^i)} \Phi(h_i)^{\gamma_1} \, dx \right)^{1/\gamma_1} \leq c \int_{\Omega_{10\rho_i}(y^i)} \Phi(h_i) \, dx \leq c \int_{\Omega_{10\rho_i}(y^i)} \Phi(u) + \Psi \, dx \leq c\lambda, \tag{4.7}$$

for some $c = c(n, p, L, \nu, s, \gamma, b_\gamma) > 0$.

Remark 4.2. At this stage, the constant δ is fixed as the one in Theorem 2.5 with $\gamma = \gamma_1$. Therefore, δ depends on n, p, L, ν, γ when $\gamma < n$ or n, p, L, ν, q when $\gamma \geq n$. Moreover, in Case 2, Theorem 2.5 is not used; instead, only Theorem 2.4 is applied. Hence, the (δ, R) -vanishing and (δ, R) -Reifenberg flat assumptions on \mathbf{a} and Ω , respectively, are not needed to obtain (1.10).

Remark 4.3. In Case 3, we assume $\gamma_1 = \gamma \geq \frac{n}{p}$, which is equivalent to $\gamma_1 \leq \gamma_1^* \frac{p-1}{p}$. Therefore, we can apply both Theorem 2.5 and Theorem 2.4 with exponent γ_1 in this case.

Let $x \in \Omega_{5\rho_i}(y^i)$ such that $\Phi(u; x) > K\lambda$, where the constant $K > 1$ will be chosen sufficiently large later in Step 4. For this x , it follows from simple calculation that

$$\begin{aligned} \Phi(u; x) &\leq 2^{\max\{p,s\}-1} (\Phi(u - h_i; x) + \Phi(h_i; x)) \\ &\leq 2^{\max\{p,s\}} \Phi(u - h_i; x) + \frac{2^{\gamma_1 \max\{p,s\}}}{(K\lambda)^{\gamma_1-1}} \Phi(h_i; x)^{\gamma_1}. \end{aligned}$$

Then we apply (4.6), (4.7), and (2.3) to derive

$$\begin{aligned} \int_{\Omega_{5\rho_i}(y^i) \cap E(K\lambda, \tau_2 r)} \Phi(u) \, dx &\leq c \int_{\Omega_{5\rho_i}(y^i)} \Phi(u - h_i) \, dx + \frac{c}{(K\lambda)^{\gamma_1-1}} \int_{\Omega_{5\rho_i}(y^i)} \Phi(h_i)^{\gamma_1} \, dx \\ &\leq c \left(\epsilon\lambda + \frac{\lambda^{\gamma_1}}{(K\lambda)^{\gamma_1-1}} \right) |\Omega_{5\rho_i}(y^i)| \\ &\leq c\lambda (\epsilon + K^{1-\gamma_1}) |\Omega_{\rho_i}(y^i)| \\ &= c\tilde{\epsilon}\lambda |\Omega_{\rho_i}(y^i)| \end{aligned}$$

for a constant $c = c(n, p, L, \nu, s, \gamma, b_\gamma) > 0$, where

$$\tilde{\epsilon} := \epsilon + K^{1-\gamma_1}. \tag{4.8}$$

By inserting (4.5) into the preceding estimate, we obtain that

$$\int_{\Omega_{5\rho_i}(y^i) \cap E(K\lambda, \tau_2 r)} \Phi(u) \, dx \leq c\tilde{\epsilon} \left(\int_{\Omega_{\rho_i}(y^i) \cap \{\Phi(u) > \frac{\lambda}{4}\}} \Phi(u) \, dx + \frac{1}{\delta_1} \int_{\Omega_{\rho_i}(y^i) \cap \{\Psi > \frac{\delta_1 \lambda}{4}\}} \Psi \, dx \right).$$

From Lemma 4.1, we observe that the sets $\Omega_{\rho_i}(y^i)$ are pairwise disjoint and the following inclusion relations hold

$$E(K\lambda, \tau_1 r) \setminus \mathcal{N} \subset E(\lambda, \tau_1 r) \setminus \mathcal{N} \subset \cup_{i=1}^\infty \Omega_{5\rho_i}(y^i) \subset \Omega_{\tau_2 r},$$

since $K > 1$. Therefore,

$$\begin{aligned} \int_{E(K\lambda, \tau_1 r)} \Phi(u) \, dx &\leq \sum_{i=1}^\infty \int_{\Omega_{5\rho_i}(y^i) \cap E(K\lambda, \tau_1 r)} \Phi(u) \, dx \\ &\leq c\tilde{\epsilon} \left(\int_{\Omega_{\tau_2 r} \cap \{\Phi(u) > \frac{\lambda}{4}\}} \Phi(u) \, dx + \frac{1}{\delta_1} \int_{\Omega_{\tau_2 r} \cap \{\Psi > \frac{\delta_1 \lambda}{4}\}} \Psi \, dx \right) \end{aligned} \tag{4.9}$$

for a constant $c = c(n, p, L, \nu, s, \gamma, b_\gamma) > 0$.

Step 4. (Proof of (1.9)–(1.11)) It remains to conclude the proof via a truncation argument employing Fubini’s theorem. Suppose that

$$\int_{\Omega_{2r}} \Psi^q \, dx < \infty.$$

For $k > 0$, let us define

$$\Phi_k(u) = \Phi_k(u; x) := \min \{ \Phi(u; x), k \},$$

and consider the super-level set with respect to $\Phi_k(u)$ as

$$E_k(\tilde{\lambda}, \rho) := \{ x \in \Omega_\rho : \Phi_k(u; x) > \tilde{\lambda} \} \quad \text{for } \tilde{\lambda}, \rho > 0.$$

Then, since $E_k(\tilde{\lambda}, \rho) = \emptyset$ when $k \leq \tilde{\lambda}$ and $E_k(\tilde{\lambda}, \rho) = E(\tilde{\lambda}, \rho)$ when $k > \tilde{\lambda}$, it follows from (4.9) that

$$\int_{E_k(K\lambda, \tau_1 r)} \Phi(u) \, dx \leq c\tilde{\epsilon} \left(\int_{E_k(\frac{\lambda}{4}, \tau_2 r)} \Phi(u) \, dx + \frac{1}{\delta_1} \int_{\Omega_{\tau_2 r} \cap \{\Psi > \frac{\delta_1 \lambda}{4}\}} \Psi \, dx \right).$$

Multiplying both sides by λ^{q-2} , integrating with respect to λ over $(\alpha\lambda_0, \infty)$, and using Fubini’s theorem, we derive that

$$\begin{aligned} &\int_{E_k(K\alpha\lambda_0, \tau_1 r)} \Phi(u) \left[\int_{\alpha\lambda_0}^{\Phi_k(u)/K} \lambda^{q-2} \, d\lambda \right] \, dx \\ &= \int_{\alpha\lambda_0}^\infty \lambda^{q-2} \int_{E_k(K\lambda, \tau_1 r)} \Phi(u) \, dx \, d\lambda \\ &\leq c\tilde{\epsilon} \left(\int_{\alpha\lambda_0}^\infty \lambda^{q-2} \int_{E_k(\frac{\lambda}{4}, \tau_2 r)} \Phi(u) \, dx \, d\lambda + \int_{\alpha\lambda_0}^\infty \lambda^{q-2} \int_{\Omega_{\tau_2 r} \cap \{\frac{\Psi}{\delta_1} > \frac{\lambda}{4}\}} \frac{\Psi}{\delta_1} \, dx \, d\lambda \right) \\ &\leq c\tilde{\epsilon} \left(\int_{E_k(\frac{\alpha\lambda_0}{4}, \tau_2 r)} \Phi(u) \left[\int_{\alpha\lambda_0}^{4\Phi_k(u)} \lambda^{q-2} \, d\lambda \right] \, dx \right) \end{aligned}$$

$$\begin{aligned}
 &+ \int_{\Omega_{\tau_2 r} \cap \left\{ \frac{\Psi}{\delta_1} > \frac{\alpha \lambda_0}{4} \right\}} \frac{\Psi}{\delta_1} \left[\int_{\alpha \lambda_0}^{4\Psi/\delta_1} \lambda^{q-2} d\lambda \right] dx \\
 &\leq c\tilde{\epsilon} \left(\int_{\Omega_{\tau_2 r}} \Phi(u)\Phi_k(u)^{q-1} dx + \int_{\Omega_{\tau_2 r}} \left[\frac{\Psi}{\delta_1} \right]^q dx \right).
 \end{aligned}$$

Then

$$\begin{aligned}
 &\int_{E_k(K\alpha\lambda_0, \tau_1 r)} \Phi(u)\Phi_k(u)^{q-1} dx \\
 &= (q-1)K^{q-1} \left\{ (\alpha\lambda_0)^{q-1} \int_{E_k(K\alpha\lambda_0, \tau_1 r)} \Phi(u) dx + \frac{(\alpha\lambda_0)^{q-1}}{q-1} \int_{E_k(K\alpha\lambda_0, \tau_1 r)} \Phi(u) dx \right\} \\
 &\leq (K\alpha\lambda_0)^{q-1} \int_{\Omega_{\tau_1 r}} \Phi(u) dx + c\tilde{\epsilon}K^{q-1} \left(\int_{\Omega_{\tau_2 r}} \Phi(u)\Phi_k(u)^{q-1} dx + \int_{\Omega_{\tau_2 r}} \left[\frac{\Psi}{\delta_1} \right]^q dx \right).
 \end{aligned}$$

It can also be seen that

$$\int_{\Omega_{\tau_1 r} \setminus E_k(K\alpha\lambda_0, \tau_1 r)} \Phi(u)\Phi_k(u)^{q-1} dx \leq (K\alpha\lambda_0)^{q-1} \int_{\Omega_{\tau_1 r}} \Phi(u) dx.$$

As a result of the previous two estimates, we obtain

$$\begin{aligned}
 &\int_{\Omega_{\tau_1 r}} \Phi(u)\Phi_k(u)^{q-1} dx \\
 &\leq c(K\alpha\lambda_0)^{q-1} \int_{\Omega_{\tau_1 r}} \Phi(u) dx + c_2\tilde{\epsilon}K^{q-1} \left(\int_{\Omega_{\tau_2 r}} \Phi(u)\Phi_k(u)^{q-1} dx + \int_{\Omega_{\tau_2 r}} \left[\frac{\Psi}{\delta_1} \right]^q dx \right)
 \end{aligned}$$

for a constant $c_2 = c_2(n, p, L, \nu, s, \gamma, b_\gamma, q) > 0$. At this stage, we recall the definition of $\tilde{\epsilon}$ given in (4.8), and then choose a sufficiently large $K > 1$ and a sufficiently small $\epsilon \in (0, 1)$ depending on $n, p, L, \nu, s, \gamma, b_\gamma, q$ such that

$$K \geq (4c_2)^{\frac{1}{\gamma_1 - q}} \quad \text{and} \quad \epsilon \leq \frac{1}{4c_2 K^{q-1}},$$

which implies

$$c_2\tilde{\epsilon}K^{q-1} = c_2(\epsilon K^{q-1} + K^{q-\gamma_1}) \leq \frac{1}{2},$$

hence $\delta_1 = \delta_1(n, p, L, \nu, s, \gamma, b_\gamma, q) \in (0, 1)$ is definitively determined. Recalling the definition of α in (4.2), we thus conclude that

$$\begin{aligned}
 &\int_{\Omega_{\tau_1 r}} \Phi(u)\Phi_k(u)^{q-1} dx \\
 &\leq \frac{1}{2} \int_{\Omega_{\tau_2 r}} \Phi(u)\Phi_k(u)^{q-1} dx + \frac{c\lambda_0^{q-1}}{(\tau_2 - \tau_1)^n} \int_{\Omega_{2r}} \Phi(u) dx + c \int_{\Omega_{2r}} \Psi^q dx.
 \end{aligned}$$

Applying Lemma 2.1, we obtain

$$\int_{\Omega_r} \Phi(u)\Phi_k(u)^{q-1} dx \leq c\lambda_0^{q-1} \int_{\Omega_{2r}} \Phi(u) dx + c \int_{\Omega_{2r}} \Psi^q dx$$

for any large $k > 0$. Finally, invoking Lebesgue’s monotone convergence theorem along with Hölder’s inequality, Young’s inequality, and the definition of λ_0 given in (4.1), we conclude that

$$\begin{aligned}
 \int_{\Omega_r} \Phi(u)^q dx &= \lim_{k \rightarrow \infty} \int_{\Omega_r} \Phi(u)\Phi_k(u)^{q-1} dx \leq c\lambda_0^{q-1} \int_{\Omega_{2r}} \Phi(u) dx + c \int_{\Omega_{2r}} \Psi^q dx \\
 &\leq c \left(\int_{\Omega_{2r}} \Phi(u) dx \right)^q + c \int_{\Omega_{2r}} \Psi^q dx.
 \end{aligned}$$

This yields the desired estimates (1.9)–(1.11), upon recalling the definition of $\Phi(u)$ in **Cases 1–3**.

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