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GLOBAL WELL-POSEDNESS OF 3D INHOMOGENEOUS INCOMPRESSIBLE NEMATIC LIQUID CRYSTAL SYSTEMS IN CRITICAL BESOV SPACES WITH INITIAL DENSITY PERTURBED AROUND THE EQUILIBRIUM

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Abstract. In this article, we establish the existence of unique global solutions for three-dimensional inhomogeneous incompressible nematic liquid crystal systems. Our analysis does not assume that the initial density ρ_0 is endowed with any regularity, requiring only that the fluid density satisfy $\|\rho_0 - 1\|_{L^\infty(\mathbb{R}^3)} \leq c$ for a sufficiently small constant c. While the initial velocity u_0 and the gradient of the initial molecular orientation $\nabla \mathrm{d}_0$ belong to the critical space $\dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)$. This work extends the result by Danchin and Wang [15] for the Navier-Stokes equations to the three-dimensional inhomogeneous nematic liquid crystal system.

1. Introduction

This study is devoted to the mathematical analysis of the three-dimensional inhomogeneous incompressible nematic liquid crystal system

$$\begin{gathered}
\partial_t \rho + u\cdot\nabla\rho = 0,\\
\rho(\partial_t u + u\cdot\nabla u) - \mu\Delta u + \nabla\pi = -\lambda\,\mathrm{div}(\nabla \mathrm{d}\odot\nabla \mathrm{d}),\\
\partial_t \mathrm{d} + u\cdot\nabla \mathrm{d} = \theta(\Delta \mathrm{d} + |\nabla \mathrm{d}|^2 \mathrm{d}),\\
\mathrm{div}\ u = 0,\\
|\mathrm{d}| = 1,\\
(\rho, u, \mathrm{d})|_{t=0} = (\rho_0, u_0, \mathrm{d}_0).
\end{gathered} \tag{1.1}$$

In this system, ρ and u denote the fluid density and velocity field respectively, and π represents the pressure field. The molecular orientation of the system (1.1) is described by the director field $\mathrm{d} \in \mathbb{S}^2$, where $\mathbb{S}^2 = \{\mathrm{d} \in \mathbb{R}^3 : |\mathrm{d}| = 1\}$ is the unit sphere in $\mathbb{R}^3$. The stress tensor $\nabla \mathrm{d}\odot\nabla \mathrm{d}$ is a 3×3 matrix with components $(\nabla \mathrm{d}\odot\nabla \mathrm{d})_{ij} = \partial_i \mathrm{d}\cdot\partial_j \mathrm{d}$ for $1 \leq i, j \leq 3$. System (1.1) contains three positive viscosity parameters: μ (shear viscosity), λ (elastic relaxation), and θ (orientational diffusivity). The initial data satisfy the constraints that $\mathrm{div}\, u_0 = 0$ (divergence-free condition), and $|\mathrm{d}_0| \equiv 1$ (unit length constraint).

For the special case of homogeneous molecular alignment where $\mathrm{d}(x,t) \equiv \mathrm{d}_0$ (constant), implying $\nabla \mathrm{d} \equiv 0$, the coupled system (1.1) degenerates to the inhomogeneous Navier-Stokes equations (INS) describing variable-density fluid motion:

$$\begin{gathered}
\partial_t \rho + \mathrm{div}(\rho u) = 0,\\
\rho(\partial_t u + u\cdot\nabla u) - \Delta u + \nabla P = 0,\\
\mathrm{div}\, u = 0,\\
(\rho, u)|_{t=0} = (\rho_0, u_0).
\end{gathered} \tag{1.2}$$

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The notion of *critical spaces* with scaling invariance matching the natural scaling of scaling,

$$u_\lambda(t,x) := \lambda u(\lambda^2 t, \lambda x), \quad \lambda > 0, \tag{1.3}$$

plays a pivotal role in modern analysis. Because of the scaling invariance property of the system (1.2), there has been growing interest in studying their behavior in critical spaces. When the initial density ρ_0 possesses some regularity, Abidi [1] and Danchin [11] obtained a global solution to system (1.2) provided that ρ_0 is a small perturbation of a positive constant in the critical Besov space $\dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)$ with the initial velocity $\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}(\mathbb{R}^d)} \le \varepsilon_0$ for $p \in (1,2d)$. This result was later extended by Danchin and Mucha [13], who showed through Lagrangian coordinate techniques that global existence and uniqueness hold for the system (1.2) when the integral index p belongs to $[1,2d)$ under the condition that the initial density ρ_0 remains close to a constant in the multiplier space $\mathcal{M}(\dot{B}^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d))$ and the initial velocity u_0 keeps in the critical space $\dot{B}^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d)$. These existing results fundamentally assume either density continuity or small perturbations around a positive constant, thus precluding the analysis of discontinuous densities with bounded cases. Zhang [28] pioneered the study of system (1.2) with merely bounded initial density ρ_0 (having a positive lower bound), establishing global existence when the initial datum u_0 has sufficiently small norm in the critical Besov space $\dot{B}^{1/2}_{2,1}(\mathbb{R}^d)$. The uniqueness of Zhang's solutions [28] was subsequently established by Danchin and Wang [15] who showed $\nabla u \in L(0,T;L^\infty(\mathbb{R}^3))$ through careful analysis in Lorentz spaces. Concurrently, the authors [15] proved global existence and uniqueness for the system (1.2) under the specific initial data requirements that $\|\rho_0 - 1\|_{L^\infty(\mathbb{R}^3)} \le c$ for a sufficiently small constant c and u_0 belongs to $\dot{B}^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)$ if $1 < p \le 2$ or u_0 belongs to $\dot{B}^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)$ if $2 < p < 3$. The Lipschitz property of the fluid flow (i.e. $\nabla u \in L(0,T;L^\infty(\mathbb{R}^3))$) plays a pivotal role in establishing the uniqueness. For technical details, we can also refer to the recent work of Danchin [12]. More related developments on the Navier-Stokes equations in critical function spaces are documented in [2, 3, 4, 10, 19, 21, 27].

For the special case of non-vanishing density $\rho \equiv 0$, the coupled system (1.1) reduces to the inhomogeneous incompressible nematic liquid crystal flow equations. There exists an extensive literature concerning the well-posedness of system (1.1) in critical function spaces. In 2015, De Anna [16] proved the global existence of solutions to the system (1.1) with $(\rho_0^{-1}-1, u_0, \nabla \mathrm{d}_0) \in L^\infty(\mathbb{R}^3) \times \dot{B}^{\frac{3}{p}-1}_{p,r}(\mathbb{R}^3) \times \dot{B}^{\frac{3}{p}-1}_{p,r}(\mathbb{R}^3)$ $(1<p<3,\ 1<r<\infty)$ and ρ_0 bounded away from vacuum and infinity, while complete well-posedness (including uniqueness) required the additional regularity. Building on the works of [1] and [11], Zhai, Li and Yan [30] established local well-posedness for the system when the initial data satisfy $(\rho_0^{-1}-1, u_0, \mathrm{d}_0 - \bar{\mathrm{d}}_0) \in \dot{B}^{3/2}_{2,1}(\mathbb{R}^3) \times \dot{B}^{1/2}_{2,1}(\mathbb{R}^3) \times \dot{B}^{3/2}_{2,1}(\mathbb{R}^3)$, where the orientation vector d asymptotically tends to $\bar{\mathrm{d}}_0$ as $|x| \to \infty$. Furthermore, they extended this to global existence and uniqueness by imposing some additional conditions on the initial data that

$$\begin{aligned}
&C\|\rho_0^{-1}-1\|_{\dot{B}^{3/2}_{2,1}(\mathbb{R}^3)}(1 + \|u_0^3\|_{\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)} + (\|u_0^h\|_{\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)} + \|\mathrm{d}_0\|^2_{\dot{B}^{3/2}_{2,1}(\mathbb{R}^3)})) \le 1,\\
&C(\|u_0^h\|_{\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)} + \|\mathrm{d}_0\|_{\dot{B}^{3/2}_{2,1}(\mathbb{R}^3)}(1 + (\|u_0^h\|_{\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)} + \|\mathrm{d}_0\|_{\dot{B}^{3/2}_{2,1}(\mathbb{R}^3)})^2\\
&+ \|u_0^3\|_{\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)} + (\|u_0^h\|_{\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)} + \|\mathrm{d}_0\|_{\dot{B}^{3/2}_{2,1}(\mathbb{R}^3)})^{1/2}) \le 1.
\end{aligned}$$

Building upon the techniques introduced in [13], Hu and Liu [20] successfully generalized the results to system (1.1). In the recent works [8], Chen, Liang, and Ye [8] resolved an open question from [16] by establishing global existence and uniqueness of weak solutions to system (1.1) in the critical Besov space. Their result require $0 < c_0 \le \rho_0 \le C_0 < +\infty$ and small norm $\|(u_0, \nabla \mathrm{d}_0)\|_{\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)}$. Subsequently, Chen et al. [9] relaxed the regularity assumptions, proving well-posedness for $(u_0, \nabla \mathrm{d}_0) \in \dot{H}^{1/2}(\mathbb{R}^3) \times (\dot{H}^{1/2}(\mathbb{R}^3) \cap \dot{B}^{1/2}_{2,\frac{4}{3}}(\mathbb{R}^3)$ and the initial density $\rho_0 \in L^\infty(\mathbb{R}^3)$ permitting vacuum states. This extension substantially improves their earlier result in [8], particularly in handling density degeneracy. Notable progress on critical space methods for nematic liquid crystal systems includes the fundamental contributions of [7, 18, 23, 24, 25, 26].

Inspired by [15], we prove the existence of unique global solutions to the three-dimensional inhomogeneous incompressible nematic liquid crystal system under the regularity assumptions that the initial density ρ_0 satisfies $\|\rho_0 - 1\|_{L^\infty(\mathbb{R}^3)} \le c$ for sufficiently small constant $c > 0$ with no additional regularity required and the initial data $(u_0, \nabla \mathrm{d}_0)$ belongs to the critical Besov space $\dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ for $1 < p \le 2$ and $2 < p < 3$. Now we can formulate our main results.

Theorem 1.1. *Let $p \in (1,3)$ and $q \in (1,\infty)$ such that $\frac{3}{p} + \frac{2}{q} = 3$. There exist a positive constant c such that if the initial density ρ_0 satisfies*

$$\|\rho_0 - 1\|_{L^\infty(\mathbb{R}^3)} < c, \tag{1.4}$$

and if the initial data $(u_0, \nabla \mathrm{d}_0)$ satisfies

$$\begin{gathered} (u_0, \nabla \mathrm{d}_0) \in (\dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3))^2 \quad (1 < p \le 2), \\ (u_0, \nabla \mathrm{d}_0) \in (\dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))^2 \quad (2 < p < 3), \end{gathered} \tag{1.5}$$

with

$$\|(u_0, \nabla \mathrm{d}_0)\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)} < c, \tag{1.6}$$

then system (1.1) has a unique global-in-time weak solution $(\rho, u, \nabla P, \mathrm{d})$ with

$$\nabla P \in L^{q,1}(\mathbb{R}^+; L^p(\mathbb{R}^3)) \quad \text{and} \quad (u, \nabla \mathrm{d}) \in (\dot{W}_{p,(q,1)}^{2,1}(\mathbb{R}^+ \times \mathbb{R}^3))^2. \tag{1.7}$$

If $p > 2$, then the following estimate holds:

$$\|\rho - 1\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^3)} = \|\rho_0 - 1\|_{L^\infty(\mathbb{R}^3)} < c, \tag{1.8}$$

and, furthermore, the following properties hold:

- *∇u, $\nabla^2 \mathrm{d} \in L^1(\mathbb{R}^+; L^\infty(\mathbb{R}^3))$ and $u, \nabla \mathrm{d} \in L^2(\mathbb{R}^+; L^\infty(\mathbb{R}^3))$;*
- *$tu, t\nabla \mathrm{d} \in W_{m,(s,1)}^{2,1}(\mathbb{R}^+; L^m(\mathbb{R}^3))$ and $t\nabla P \in L^{s,1}(\mathbb{R}^+; L^m(\mathbb{R}^3))$ for $3 < m < \infty$ and $q < s < \infty$ such that $\frac{3}{m} + \frac{2}{s} = 1$;*
- *$tD_t u, tD_t \nabla \mathrm{d} \in \dot{W}_{p,(q,1)}^{2,1}(\mathbb{R}^+ \times \mathbb{R}^3)$;*
- *$u, \nabla \mathrm{d}, tD_t u, tD_t \nabla \mathrm{d} \in L^{s,1}(\mathbb{R}^+; L^m(\mathbb{R}^3))$.*

Remark 1.2. The work in [29] established the existence of a unique global solution for the three-dimensional compressible liquid crystal flow without imposing smallness conditions. It is our belief that this result can be extended to certain classes of large initial data via a decomposition into vertical and horizontal components. A rigorous consideration of this extension will be discussed in our future research.

Remark 1.3. Remarkably, by Besov embeddings $\dot{B}_{p_1,r_1}^{s}(\mathbb{R}^d) \hookrightarrow \dot{B}_{p_2,r_2}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}(\mathbb{R}^d)$ for $1 \le p_1 \le p_2 \le \infty$ and $1 \le r_1 \le r_2 \le \infty$, Theorem 1.1 in [8] imposes milder conditions than our Theorem 1.1 if $1 < p \le 2$ and consequently obtains sharper results. Nevertheless, our work achieves greater generality by extending the admissible integrability range from $p = 2$ to $1 < p < 3$, while [8] restricts initial data to critical Besov spaces $\dot{B}_{2,1}^{1/2}(\mathbb{R}^3)$.

Remark 1.4. The perturbation condition (1.4) is introduced to reformulate the momentum equation into a homogeneous liquid crystal system. This transformation enables us to exploit the smallness of (1.4) during energy estimates, allowing its terms to be absorbed by the left hand.

The following theorem states the uniqueness of the solutions to system (1.1).

Theorem 1.5. *Let $T > 0$ and consider two solutions $(\rho_1, u_1, P_1, \mathrm{d}_1)$ and $(\rho_2, u_2, P_2, \mathrm{d}_2)$ of system (1.1) on $[0,T] \times \mathbb{R}^3$ sharing identical initial data. Furthermore, we assume that*

- *$\sqrt{\rho_1}(u_1 - u_2), \nabla \mathrm{d}_1 - \nabla \mathrm{d}_2 \in L^\infty(0,T; L^2(\mathbb{R}^3))$;*
- *$\nabla u_1 - \nabla u_2, \nabla^2 \mathrm{d}_1 - \nabla^2 \mathrm{d}_2 \in L^2(0,T; L^2(\mathbb{R}^3))$;*
- *$\nabla u_2, \nabla^2 \mathrm{d}_2, \nabla^2 \mathrm{d}_1 \in L^1(0,T; L^\infty(\mathbb{R}^3))$;*
- *$\nabla \mathrm{d}_1, \nabla \mathrm{d}_2 \in L^2(0,T; L^\infty(\mathbb{R}^3))$;*
- *$tD_t u_2 \in L^2(0,T; L^\infty(\mathbb{R}^3))$;*

- $t\nabla D_t u_2 \in L^2(0,T;L^3(\mathbb{R}^3))$;
- $\nabla \mathrm{d}_1 \in \dot{B}^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)$.

Then $(\rho_1, u_1, P_1, \mathrm{d}_1) = (\rho_2, u_2, P_2, \mathrm{d}_2)$ *on* $[0,T]\times\mathbb{R}^3$.

Remark 1.6. Unlike in [15], the proof of uniqueness via energy estimates here requires condition $\nabla \mathrm{d}_1 \in \dot{B}^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)$, as specified in (5.2) and (5.3).

The rest of this article is organized as follows. In Section 2, we present some information on the Besov spaces and Lorentz spaces. Section 3 states the a priori estimates and gives the proof of Theorem 1.1. The existence and uniqueness of solutions are established in Sections 4 and 5.

We conclude this section by establishing some notation that will be used throughout this article. The material derivative operator is defined as $D_t := \partial_t + u\cdot\nabla$. Given a Banach space X and $q \in [1,+\infty)$, the Banach space $L^q(0,T;X)$ consists of all strongly measurable functions $f\colon (0,T)\to X$ such that the mapping $t \mapsto \|f(t)\|_X$ belongs to $L^q(0,T;X)$. The space is equipped with the norm: $\|f\|_{L^q(0,T;X)} := \big(\int_0^T \|f(t)\|_X^q\,dt\big)^{1/q}$. For any pair of measurable functions (f,g) on X, we define the joint norm $\|(f,g)\|_X := \|f\|_X + \|g\|_X$.

2. Preliminaries

In this section, we present the definitions of Besov spaces, Lorentz spaces, and key lemmas, which can be found in the references [5, 6, 14, 15, 17, 22].

First, we introduce the dyadic decomposition and the homogeneous Besov spaces. Given a tempered distribution $u \in \mathcal{S}'_h$, we define

$$\forall j\in\mathbb{Z},\quad \dot{\Delta}_j u := \varphi(2^{-j}\cdot)u,\quad \dot{S}_j u = \sum_{j'\le j-1}\dot{\Delta}_{j'}u = \chi(2^{-j}\cdot u),$$

where $\chi(\tau)$ and $\varphi(\tau)$ are smooth functions such that

$$\operatorname{supp}\varphi \subset \mathcal{C}\quad\text{and}\quad \forall\tau>0, \sum_{j\in\mathbb{Z}}\varphi(2^{-j}\tau) = 1,$$

$$\operatorname{supp}\chi \subset \mathcal{B}\quad\text{and}\quad \forall\tau>0, \chi(\tau) + sum_{j\ge 0}\varphi(2^{-j}\tau) = 1.$$

Here $\mathcal{C} = \{\tau\in\mathbb{R}^d | \frac{3}{4}\le|\tau|\le\frac{8}{3}\}$, $\mathcal{B} = \{\tau\in\mathbb{R}^d | |\tau|\le\frac{4}{3}\}$.

Definition 2.1 ([5])**.** Let (p,r) be in $[1,\infty]^2$ and s be in $\mathbb{R}$. Assume that $u\in\mathcal{S}'_h(\mathbb{R}^d)$, which means that u is in $\mathcal{S}'(\mathbb{R}^d)$ and satisfies $\lim_{j\to-\infty}\|S_j u\|_{L^\infty} = 0$. We set

$$\|u\|_{\dot{B}^s_{p,r}} := \|(2^{js}\|\dot{\Delta}_j u\|_{L^p})_{j\in\mathbb{Z}}\|_{l^r(\mathbb{Z})}.$$

(i) If $s<\frac{d}{p}$(or $s=\frac{d}{p}$ if $r=1$), we define $\dot{B}^s_{p,r} := \{u\in\mathcal{S}'_h(\mathbb{R}^d) | \|u\|_{\dot{B}^s_{p,r}}<\infty\}$.

(ii) If $k\in\mathbb{N}$ and if $s<\frac{d}{p}$(or s=$\frac{d}{p}$ if $r=1$), then we define $\dot{B}^s_{p,r}(\mathbb{R}^d)$ as the subset of u in $\mathcal{S}'_h(\mathbb{R}^d)$ such that $\partial^\beta u$ belongs to $\dot{B}^{s-k}_{p,r}(\mathbb{R}^d)$ whenever $|\beta| = k$.

Lemma 2.2 (Besov embedding [5])**.** *(1) For any* (p,q) *in* $[1,\infty]^2$ *such that* $p\le q$, *we have*

$$\dot{B}^{\frac{d}{p}-\frac{d}{q}}_{p,1}(\mathbb{R}^d)\hookrightarrow L^q(\mathbb{R}^d).$$

(2) Let $1\le p_1\le p_2\le\infty$ *and* $1\le r_1\le r_2\le\infty$. *Then, for any real number* s,

$$\dot{B}^s_{p_1,r_1}(\mathbb{R}^d)\hookrightarrow \dot{B}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}(\mathbb{R}^d).$$

Let us define Lorentz spaces and recall some useful properties.

Definition 2.3 ([17, 22])**.** Given f a measurable function on a measure space (X,ν) and $1\le p,r\le\infty$, we define

$$\|f\|_{L^{p,r}(X,\nu)} = \begin{cases}\big(\int_0^\infty (t^{1/p}f^*(t))^r\frac{dt}{t}\big)^{1/r}, & \text{if } r<\infty,\\ \sup_{t>0} t^{1/p}f^*(t), & \text{if } r=\infty.\end{cases}\tag{2.1}$$

where $f^*(t) := \inf\{s \geq 0 | \nu(|f| > s) \leq t\}$. The set of all f with $\|f\|_{L^{p,r}(X,\nu)}$ is called the Lorentz space with indices p and r.

The following properties of Lorentz spaces will play an important part in the proof of uniqueness.

Lemma 2.4 ([17]). *Let Ω be a measurable set, $\mathbb{R}^+$ be the positive real set and $\mathbb{R}^d$ be the d-dimensional Euclidean space. Then we have the following properties.*

1. *Embedding: $L^{p,r_1}(\Omega) \hookrightarrow L^{p,r_2}(\Omega)$ if $r_1 \leq r_2$, and $L^{p,p}(\Omega) = L^p(\Omega)$;*
2. *the Hölder inequality: For $1 < p, p_1, p_2 < \infty$ and $1 \leq r, r_1, r_2 \leq \infty$, it holds*
$$\|fg\|_{L^{p,r}(\Omega)} \leq C\|f\|_{L^{p_1,r_1}(\Omega)}\|g\|_{L^{p_2,r_2}(\Omega)}$$
with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. This inequality still holds for the pairs $(1,1)$ and (∞,∞) with the convention $L^{1,1}(\Omega) = L^1(\Omega)$ and $L^{\infty,\infty}(\Omega) = L^\infty(\Omega)$;
3. *For any $\alpha > 0$ and nonnegative measurable function f, one has $\|f^\alpha\|_{L^{p,r}(\Omega)} = \|f\|^\alpha_{L^{p\alpha,r\alpha}(\Omega)}$;*
4. *For any $k > 0$, one has $\|x^{-k}1_{\mathbb{R}^+}\|_{L^{\frac{1}{k},\infty}(\Omega)} = 1$.*

Next, we introduce the Besov interpolation.

Lemma 2.5 ([15, Proposition A.4]). *Let $1 \leq q < \infty$, $1 \leq p < r \leq \infty$, and $\theta \in (0,1)$ such that*
$$\frac{1}{r} + \frac{1}{d} - \frac{2\theta}{dq} = \frac{1}{p}.$$
Then, there exists C such that
$$\|\nabla u\|_{L^r(\mathbb{R}^d)} \leq C\|\nabla^2 u\|^\theta_{L^p(\mathbb{R}^d)}\|u\|^{1-\theta}_{\dot{B}^{2-\frac{2}{q}}_{p,\infty}(\mathbb{R}^d)}.$$

The following two maximal regularity estimates for Navier-Stokes equations play a pivotal role in the proof of our main results.

Lemma 2.6 ([15, Proposition A.5]). *Let $1 < p, q < \infty$ and $1 \leq r \leq \infty$. Then for any initial data $u_0 \in \dot{B}^{2-\frac{2}{q}}_{p,r}(\mathbb{R}^d)$ with $\operatorname{div} u_0 = 0$, and any external force $f \in L^{q,r}(0,T;L^p(\mathbb{R}^d))$, the Stokes system*
$$\begin{gathered}\partial_t u - \Delta u + \nabla P = f, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d,\\ \operatorname{div} u = 0,\\ u|_{t=0} = u_0.\end{gathered}$$
in
$$\dot{W}^{2,1}_{p,(q,r)}((0,T)\times\mathbb{R}^d) := \{u \in C([0,T];\dot{B}^{2-\frac{2}{q}}_{p,r}(\mathbb{R}^d)) : u_t, \nabla^2 u \in L^{q,r}(0,T;L^p(\mathbb{R}^d))\}$$
admits a unique solution $(u,\nabla P)$ satisfying $\nabla P \in L^{q,r}(0,T;L^p(\mathbb{R}^d))$ and the maximal regularity estimate
$$\begin{aligned}&\|u\|_{L^\infty(0,T;\dot{B}^{2-\frac{2}{q}}_{p,r}(\mathbb{R}^d))} + \|(u_t,\nabla^2 u,\nabla P)\|_{L^{q,r}(0,T;L^p(\mathbb{R}^d))}\\ &\leq C\big(\|u_0\|_{\dot{B}^{2-\frac{2}{q}}_{p,r}(\mathbb{R}^d)} + \|f\|_{L^{q,r}(0,T;L^p(\mathbb{R}^d))}\big),\end{aligned}$$
where the constant $C > 0$ is independent of T.

Next we have a maximal regularity estimate for heat equation.

Lemma 2.7 ([14, Proposition 2.1]). *Let $1 < p, q < \infty$ and $1 \leq r \leq \infty$. Then for any initial data $u_0 \in \dot{B}^{2-\frac{2}{q}}_{p,r}(\mathbb{R}^d)$ and any source term $f \in L^{q,r}(0,T;L^p(\mathbb{R}^d))$, the heat equation*
$$\begin{gathered}\partial_t u - \Delta u = f, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d,\\ u|_{t=0} = u_0.\end{gathered}$$
in
$$\dot{W}^{2,1}_{p,(q,r)}(\mathbb{R}^+\times\mathbb{R}^d) := \{u \in C_b(\mathbb{R}^+;\dot{B}^{2-\frac{2}{q}}_{p,r}(\mathbb{R}^d)) : u_t, \nabla^2 u \in L^{q,r}(\mathbb{R}^+;L^p(\mathbb{R}^d))\}$$
admits a unique solution u satisfying the maximal regularity estimate
$$\|u\|_{L^\infty(\mathbb{R}^+;\dot{B}^{2-\frac{2}{q}}_{p,r}(\mathbb{R}^d))} + \|(u_t,\nabla^2 u)\|_{L^{q,r}(\mathbb{R}^+;L^p(\mathbb{R}^d))} \leq C\big(\|u_0\|_{\dot{B}^{2-\frac{2}{q}}_{p,r}(\mathbb{R}^d)} + \|f\|_{L^{q,r}(\mathbb{R}^+;L^p(\mathbb{R}^d))}\big).$$

3. Proof of Theorem 1.1

A priori estimates.

Proposition 3.1. *Let* (ρ, u, d) *be a smooth solution of system* (1.1) *on* $[0,T]\times\mathbb{R}^3$, *with* u *and* $\nabla\mathrm{d}$ *sufficiently decaying at infinity and* ρ *satisfying*

$$\sup_{t\in[0,T]}\|\rho(t)-1\|_{L^\infty(\mathbb{R}^3)}\le c<<1, \tag{3.1}$$

Then, for all $1<m,p,q,s<\infty$ *such that*

$$\frac{3}{p}+\frac{2}{q}=3 \quad \text{and} \quad \frac{3}{m}+\frac{2}{s}=1 \quad \text{with} \quad p<m<\infty \quad \text{and} \quad q<s<\infty, \tag{3.2}$$

it follows that

$$\begin{aligned}
&\|(u,\nabla\mathrm{d})\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}+\|(u_t,D_tu,\nabla^2u,\nabla P,\nabla\mathrm{d}_t,D_t\nabla\mathrm{d},\nabla^3\mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&+\|(u,\nabla\mathrm{d})\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\\
&\le C\|(u_0,\nabla\mathrm{d}_0)\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)},
\end{aligned}\tag{3.3}$$

and

$$\|(u,\nabla\mathrm{d})\|_{L^2(0,T;L^\infty(\mathbb{R}^3))}\le C\|(u_0,\nabla\mathrm{d}_0)\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}. \tag{3.4}$$

Proof. First, we can rewrite $(1.1)_2$ as

$$u_t-\Delta u+\nabla P=(1-\rho)u_t-\rho u\cdot\nabla u-\operatorname{div}(\nabla\mathrm{d}\odot\nabla\mathrm{d}). \tag{3.5}$$

From the Hölder inequality in Lorentz spaces and the maximal regularity estimates for the Navier-Stokes equations, it follows that

$$\begin{aligned}
&\|u\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}+\|(u_t,\nabla^2u,\nabla P)\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}+\|u\|_{L^{s,1}(0,T;L^p(\mathbb{R}^3))}\\
&\le C(\|u_0\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}+\|(1-\rho)u_t\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}+\|\rho u\cdot\nabla u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\quad+\|\operatorname{div}(\nabla\mathrm{d}\odot\nabla\mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))})\\
&\le C(\|u_0\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}+\|\rho-1\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\|u_t\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\quad+\|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\|u\cdot\nabla u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}+\|\operatorname{div}(\nabla\mathrm{d}\odot\nabla\mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}).
\end{aligned}$$

It follows from condition (3.1) that the second term on the right-hand side of the above inequality can be absorbed by the left-hand side, and $\|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}$ is a bounded quantity.

• Estimate for $\|u\cdot\nabla u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}$. By the embedding

$$\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3) \tag{3.6}$$

and

$$\dot W^{1,p}(\mathbb{R}^3)\hookrightarrow L^{p*}(\mathbb{R}^3) \quad \text{with} \quad \frac{1}{p*}=\frac{1}{p}-\frac{1}{3}, \tag{3.7}$$

one has

$$\begin{aligned}
\|u\cdot\nabla u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}&\le C\|u\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\|\nabla u\|_{L^{q,1}(0,T;L^{p^*}(\mathbb{R}^3))}\\
&\le C\|u\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\|\nabla^2u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}.
\end{aligned}$$

• Estimate for $\|\operatorname{div}(\nabla\mathrm{d}\odot\nabla\mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}$

$$\begin{aligned}
\|\operatorname{div}(\nabla\mathrm{d}\odot\nabla\mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}&\le C\|\nabla\mathrm{d}\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\|\nabla^2\mathrm{d}\|_{L^{q,1}(0,T;L^{p^*}(\mathbb{R}^3))}\\
&\le C\|\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\|\nabla^3\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}.
\end{aligned}$$

Applying ∇ to $(1.1)_3$, we obtain

$$\nabla\mathrm{d}_t+\nabla u^T\nabla\mathrm{d}+u\cdot\nabla(\nabla\mathrm{d})=\Delta\nabla\mathrm{d}+\nabla(|\nabla\mathrm{d}|^2)\mathrm{d}+|\nabla\mathrm{d}|^2\nabla\mathrm{d},$$

which can be written as

$$\nabla \mathrm{d}_t - \Delta\nabla \mathrm{d} = -\nabla u^T\nabla \mathrm{d} - u\cdot\nabla(\nabla \mathrm{d}) + \nabla(|\nabla \mathrm{d}|^2)\mathrm{d} + |\nabla \mathrm{d}|^2\nabla \mathrm{d}. \tag{3.8}$$

On the other hand, the maximal regularity estimate for the heat equation yields

$$\begin{aligned}
&\|\nabla \mathrm{d}\|_{L^\infty(0,T;\dot B_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3))} + \|(\nabla \mathrm{d}_t, \nabla^3 \mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + \|\nabla \mathrm{d}\|_{L^{s,1}(0,T;L^p(\mathbb{R}^3))} \\
&\le C(\|\nabla \mathrm{d}_0\|_{\dot B_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)} + \|\nabla u^T\nabla \mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + \|u\cdot\nabla(\nabla \mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} \\
&\quad + \|\nabla(|\nabla \mathrm{d}|^2)\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + \||\nabla \mathrm{d}|^2\nabla \mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}).
\end{aligned}$$

- Estimate for $\|\nabla u^T\nabla \mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}$.

$$\begin{aligned}
\|\nabla u^T\nabla \mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} &\le C\|\nabla \mathrm{d}\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\|\nabla u\|_{L^{q,1}(0,T;L^{p^*}(\mathbb{R}^3))} \\
&\le C\|\nabla \mathrm{d}\|_{L^\infty(0,T;\dot B_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3))}\|\nabla^2 u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}.
\end{aligned}$$

- Estimate for $\|u\cdot\nabla(\nabla \mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}$.

$$\begin{aligned}
\|u\cdot\nabla(\nabla \mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} &\le C\|u\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\|\nabla^2 \mathrm{d}\|_{L^{q,1}(0,T;L^{p^*}(\mathbb{R}^3))} \\
&\le C\|u\|_{L^\infty(0,T;\dot B_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3))}\|\nabla^3 \mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}.
\end{aligned}$$

- Estimate for $\|\nabla(|\nabla \mathrm{d}|^2)\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}$.

$$\begin{aligned}
\|\nabla(|\nabla \mathrm{d}|^2)\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} &\le C\|\nabla \mathrm{d}\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\|\nabla^2 \mathrm{d}\|_{L^{q,1}(0,T;L^{p^*}(\mathbb{R}^3))}\|\mathrm{d}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))} \\
&\le C\|\nabla \mathrm{d}\|_{L^\infty(0,T;\dot B_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3))}\|\nabla^3 \mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}.
\end{aligned}$$

- Estimate for $\||\nabla \mathrm{d}|^2\nabla \mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}$. From the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
\||\nabla \mathrm{d}|^2\nabla \mathrm{d}\|_{L^p(\mathbb{R}^3)} \le C\|\nabla \mathrm{d}\|^3_{L^{3p}(\mathbb{R}^3)} &\le C(\|\nabla \mathrm{d}\|^{\frac{2}{3}}_{L^3(\mathbb{R}^3)}\|\nabla^3 \mathrm{d}\|^{\frac{1}{3}}_{L^p(\mathbb{R}^3)})^3 \\
&\le C\|\nabla \mathrm{d}\|^2_{L^3(\mathbb{R}^3)}\|\nabla^3 \mathrm{d}\|_{L^p(\mathbb{R}^3)}.
\end{aligned}$$

Then, employing the Hölder inequality in Lorentz spaces, it holds

$$\begin{aligned}
\||\nabla \mathrm{d}|^2\nabla \mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} &\le C\|\nabla \mathrm{d}\|^2_{L^\infty(0,T;L^3(\mathbb{R}^3))}\|\nabla^3 \mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} \\
&\le C\|\nabla \mathrm{d}\|^2_{L^\infty(0,T;\dot B_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3))}\|\nabla^3 \mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}.
\end{aligned}$$

Next, we denote

$$E_0 := \|(u_0, \nabla \mathrm{d}_0)\|_{\dot B_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)}, \quad T^* := \sup\Big\{\ t < T :\ \|(u,\nabla \mathrm{d})\|_{L^\infty(0,t;\dot B_{P,1}^{-1+\frac{3}{p}}(\mathbb{R}^3))} \le c_1\Big\},$$

$$\begin{aligned}
E_1(t) := &\|(u,\nabla \mathrm{d})\|_{L^\infty(0,T;\dot B_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3))} + \|(u_t, D_t u, \nabla^2 u, \nabla P, \nabla \mathrm{d}_t, D_t\nabla \mathrm{d}, \nabla^3 \mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} \\
&+ \|(u,\nabla \mathrm{d})\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}.
\end{aligned}$$

We conclude that $E_1(t) \le CE_0 + Cc_1E_1(t)$. By selecting c_1 sufficiently small to ensure $Cc_1 \le \frac{1}{2}$, we obtain

$$E_1(t) \le 2CE_0.$$

By taking ε_0 sufficiently small to ensure $2C\varepsilon_0 \le c_1/2$, then from the standard continuous argument, we can find that

$$E_1(t) \le CE_0.$$

On the other hand, by the Hölder inequality in Lorentz spaces, the Gagliardo-Nirenberg inequality and the embedding, one has

$$\begin{aligned}
&\|(D_t u, D_t\nabla \mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} \\
&\le C\|(u_t,\nabla \mathrm{d}_t)\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + C\|u\|_{L^\infty(0,T;\dot B_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3))}\|(\nabla^2 u, \nabla^3 \mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}
\end{aligned}$$

$$\le CE_1(t)+C(E_1(t))^2 \le C\|(u_0,\nabla \mathrm{d}_0)\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}.$$

Finally, by using the Gagliardo-Nirenberg inequality and the embedding

$$\|f\|_{L^\infty(\mathbb{R}^3)} \le C\|f\|^{1-\frac{q}{2}}_{L^3(\mathbb{R}^3)}\|\nabla^2 f\|^{q/2}_{L^p(\mathbb{R}^3)} \le C\|f\|^{1-\frac{q}{2}}_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}\|\nabla^2 f\|^{q/2}_{L^p(\mathbb{R}^3)},$$

we have

$$\begin{aligned}\int_0^T \|(u,\nabla \mathrm{d})\|^2_{L^\infty(\mathbb{R}^3)}\,d\tau &\le C\int_0^T \|(u,\nabla \mathrm{d})\|^{2-q}_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}\|(\nabla^2 u,\nabla^3 \mathrm{d})\|^q_{L^p(\mathbb{R}^3)}\,d\tau\\ &\le C\|(u,\nabla \mathrm{d})\|^{2-q}_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\|(\nabla^2 u,\nabla^3 \mathrm{d})\|^q_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\ &\le C(E_1(t))^2 \le C\|(u_0,\nabla \mathrm{d}_0)\|^2_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}.\end{aligned}$$

□

Proposition 3.2. *Under the hypotheses of Proposition 3.1, we obtain*

$$\begin{aligned}&\|t(u,\nabla \mathrm{d})\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))} + \|((tu)_t,\nabla^2(tu),\nabla(tP),(t\nabla \mathrm{d})_t,\nabla^3(t\mathrm{d}))\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\\ &\le C\|(u_0,\nabla \mathrm{d}_0)\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}.\end{aligned} \tag{3.9}$$

Furthermore, it follows that

$$\int_0^T \|(\nabla u,\nabla^2 \mathrm{d})\|_{L^\infty(\mathbb{R}^3)}dt + \Big(\int_0^T t\|(\nabla u,\nabla^2 \mathrm{d})\|^2_{L^\infty(\mathbb{R}^3)}dt\Big)^{1/2} \le C\|(u_0,\nabla \mathrm{d}_0)\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}.$$

Proof. Multiplying both sides of (3.5) by time t yields

$$(tu)_t-\Delta(tu)+\nabla(tP)=(1-\rho)(tu)_t+\rho u - t\rho u\cdot\nabla u - t\,\mathrm{div}(\nabla \mathrm{d}\odot\nabla \mathrm{d}).$$

From the Hölder inequality in Lorentz spaces and the maximal regularity estimates for the Navier-Stokes equations, it follows that

$$\begin{aligned}&\|tu\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))}+\|((tu)_t,\nabla^2(tu),\nabla(tP))\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\\ &\le C\|((1-\rho)(tu)_t+\rho u - t\rho u\cdot\nabla u - t\,\mathrm{div}(\nabla \mathrm{d}\odot\nabla \mathrm{d}))\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\\ &\le C\|\rho-1\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\|(tu)_t\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\\ &\quad + C\|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\big(\|u\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}+\|tu\cdot\nabla u\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\big)\\ &\quad + C\|t\,\mathrm{div}(\nabla \mathrm{d}\odot\nabla \mathrm{d})\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}.\end{aligned} \tag{3.10}$$

It follows from condition (3.1) that the first term on the right-hand side of the above inequality can be absorbed by the left-hand side, and $\|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}$ is a bounded quantity.

- Estimate for $\|tu\cdot\nabla u\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}$. By the Hölder inequality in Lorentz spaces and the embedding

$$\dot B^{3/m}_{m,1}(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3), \tag{3.11}$$

one has

$$\begin{aligned}\|tu\cdot\nabla u\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))} &\le C\|u\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\|t\nabla u\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\\ &\le C\|u\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\|tu\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))}.\end{aligned}$$

- Estimate for $\|t\,\mathrm{div}(\nabla \mathrm{d}\odot\nabla \mathrm{d})\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}$. Similarly, one has

$$\begin{aligned}\|t\,\mathrm{div}(\nabla \mathrm{d}\odot\nabla \mathrm{d})\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))} &\le C\|\nabla \mathrm{d}\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\|t\nabla^2 \mathrm{d}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\\ &\le C\|\nabla \mathrm{d}\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\|t\nabla \mathrm{d}\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))}.\end{aligned}$$

On the other hand, multiplying both sides of (3.8) by time t yields

$$(t\nabla \mathrm{d})_t - t\Delta\nabla \mathrm{d} = \nabla \mathrm{d} - t\nabla u^T\nabla \mathrm{d} - tu\cdot\nabla(\nabla \mathrm{d}) + t\nabla(|\nabla \mathrm{d}|^2)\mathrm{d} + t|\nabla \mathrm{d}|^2\nabla \mathrm{d}.$$

The maximal regularity estimate for the heat equation yields

$$\begin{aligned}
&\|t\nabla \mathrm{d}\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))} + \|((t\nabla \mathrm{d})_t, \nabla^3(t\nabla \mathrm{d}))\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\\
&\le C\|\nabla \mathrm{d}\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))} + C\|t\nabla u^T\nabla \mathrm{d}\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\\
&\quad + C\|tu\cdot\nabla(\nabla \mathrm{d})\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))} + C\|t\nabla(|\nabla \mathrm{d}|^2)\mathrm{d}\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\\
&\quad + C\|t|\nabla \mathrm{d}|^2\nabla \mathrm{d}\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}.
\end{aligned} \tag{3.12}$$

• Estimate for $\|t\nabla u^T\nabla \mathrm{d}\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}$. By the Hölder inequality in Lorentz spaces and the embedding (3.11), one has

$$\begin{aligned}
\|t\nabla u^T\nabla \mathrm{d}\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))} &\le C\|\nabla \mathrm{d}\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\|t\nabla u\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\\
&\le C\|\nabla \mathrm{d}\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\|tu\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))}.
\end{aligned}$$

• Estimate for $\|tu\cdot\nabla(\nabla \mathrm{d})\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}$. Similarly, one has

$$\begin{aligned}
\|tu\cdot\nabla(\nabla \mathrm{d})\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))} &\le C\|u\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\|t\nabla^2 \mathrm{d}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\\
&\le C\|u\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\|t\nabla \mathrm{d}\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))}.
\end{aligned}$$

• Estimate for $\|t\nabla(|\nabla \mathrm{d}|^2)\mathrm{d}\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}$. Similarly, one has

$$\begin{aligned}
\|t\nabla(|\nabla \mathrm{d}|^2)\mathrm{d}\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))} &\le C\|\nabla \mathrm{d}\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\|t\nabla^2 \mathrm{d}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\|\mathrm{d}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\\
&\le C\|\nabla \mathrm{d}\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\|t\nabla \mathrm{d}\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))}.
\end{aligned}$$

• Estimate for $\|t|\nabla \mathrm{d}|^2\nabla \mathrm{d}\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}$. From the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
\||\nabla \mathrm{d}|^2\nabla \mathrm{d}\|_{L^m(\mathbb{R}^3)} \le C\|\nabla \mathrm{d}\|^3_{L^{3m}(\mathbb{R}^3)} &\le C(\|\nabla \mathrm{d}\|^{\frac{2}{3}}_{L^3(\mathbb{R}^3)}\|\nabla^3 \mathrm{d}\|^{\frac{1}{3}}_{L^m(\mathbb{R}^3)})^3\\
&\le C\|\nabla \mathrm{d}\|^2_{L^3(\mathbb{R}^3)}\|\nabla^3 \mathrm{d}\|_{L^m(\mathbb{R}^3)}.
\end{aligned}$$

Similarly, one has

$$\begin{aligned}
\|t|\nabla \mathrm{d}|^2\nabla \mathrm{d}\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))} &\le C\|\nabla \mathrm{d}\|^2_{L^\infty(0,T;L^3(\mathbb{R}^3))}\|t\nabla^3 \mathrm{d}\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\\
&\le C\|\nabla \mathrm{d}\|^2_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\|t\nabla^3 \mathrm{d}\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}.
\end{aligned}$$

Now let us denote

$$E_2(t) := \|t(u,\nabla \mathrm{d})\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))} + \|((tu)_t, \nabla^2(tu), \nabla(tP), (t\nabla \mathrm{d})_t, \nabla^3(t\mathrm{d}))\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}.$$

Then we have

$$E_2(t) \le C(1+E_2(t))\cdot(E_1(t)+(E_1(t))^2).$$

Applying the theory of bootstrapping again, one has

$$E_2(t) \le CE_0.$$

Bounding $\nabla u, \nabla^2\mathrm{d}$ relies on the following interpolation inequality (as (3.2) implies that $p < 3 < m$),

$$\|f\|_{L^\infty(\mathbb{R}^3)} \le \|\nabla f\|^{\frac{p(m-3)}{3(m-p)}}_{L^p(\mathbb{R}^3)}\|\nabla f\|^{\frac{m(3-p)}{3(m-p)}}_{L^m(\mathbb{R}^3)}. \tag{3.13}$$

Applying the Hölder inequality in Lorentz spaces with exponents:

$$(p_1,r_1) = \Big(\frac{3(m-p)}{m(3-p)},\infty\Big),\quad (p_2,r_2) = \Big(\frac{3q(m-p)}{p(m-3)},\frac{p_2}{q}\Big),\quad (p_3,r_3) = \Big(\frac{3s(m-p)}{m(3-p)},\frac{p_3}{s}\Big),$$

and using that $t^{-\alpha} \in L^{\frac{1}{\alpha},\infty}(\mathbb{R}^+)$ with $\alpha = \frac{m(3-p)}{3(m-p)}$, (3.3) and the first inequalitv of Proposition 3.1 we end up with

$$\begin{aligned}
\int_0^T \|(\nabla u,\nabla^2\mathrm{d})\|_{L^\infty(\mathbb{R}^3)}dt &\le C\int_0^T t^{-\frac{m(3-p)}{3(m-p)}}\|(\nabla^2 u,\nabla^3\mathrm{d})\|^{\frac{p(m-3)}{3(m-p)}}_{L^p(\mathbb{R}^3)}\|t(\nabla^2 u,\nabla^3\mathrm{d})\|^{\frac{m(3-p)}{3(m-p)}}_{L^m(\mathbb{R}^3)}dt\\
&\le C\|(\nabla^2 u,\nabla^3\mathrm{d})\|^{\frac{p(m-3)}{3(m-p)}}_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\|t(\nabla^2 u,\nabla^3\mathrm{d})\|^{\frac{m(3-p)}{3(m-p)}}_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}
\end{aligned}$$

$$\le C\|(u_0,\nabla \mathrm{d}_0)\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}.$$

Morever, from the Hölder inequality and the embedding (3.11), one has

$$\begin{aligned}\int_0^T t\|(\nabla u,\nabla^2\mathrm{d})\|^2_{L^\infty(\mathbb{R}^3)}dt &\le \int_0^T t\|(\nabla u,\nabla^2\mathrm{d})\|_{\dot B^{3/m}_{m,1}(\mathbb{R}^3)}\|(\nabla u,\nabla^2\mathrm{d})\|_{L^\infty(\mathbb{R}^3)}dt\\ &\le \|t(u,\nabla\mathrm{d})\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))}\int_0^T \|(\nabla u,\nabla^2\mathrm{d})\|_{L^\infty(\mathbb{R}^3)}dt\\ &\le C\|(u_0,\nabla\mathrm{d}_0)\|^2_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}.\end{aligned}$$

□

To prove the uniqueness, we need the following time weighted estimate.

Proposition 3.3. *Under the hypotheses of Proposition 3.1, we obtain*

$$\begin{aligned}&\|t(D_tu,D_t\nabla\mathrm{d})\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\\ &+\|((tD_tu)_t,t\nabla^2D_tu,(tD_t\nabla\mathrm{d})_t,t\nabla^2D_t\nabla\mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\ &+\|t(D_tu,D_t\nabla\mathrm{d})\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\\ &\le C\|(u_0,\nabla\mathrm{d}_0)\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}.\end{aligned}\tag{3.14}$$

Furthermore,

$$\begin{aligned}&\|t(\nabla D_tu,\nabla D_t\nabla\mathrm{d})\|_{L^2(0,T;L^3(\mathbb{R}^3))}+\|t(D_tu,D_t\nabla\mathrm{d})\|_{L^2(0,T;L^\infty(\mathbb{R}^3))}\\ &\le C\|(u_0,\nabla\mathrm{d}_0)\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}.\end{aligned}\tag{3.15}$$

Proof. Applying D_t to $(1.1)_2$, we obtain

$$D_t(\rho D_tu)-D_t(\Delta u)+D_t(\nabla P)=-D_t(\mathrm{div}(\nabla\mathrm{d}\odot\nabla\mathrm{d})).$$

By the definition of commutator, we have

$$[D_t;\nabla]f:=-\nabla u\cdot\nabla f,\quad [D_t;\Delta]f:=-\Delta u\cdot\nabla f-2\nabla u\cdot\nabla^2 f.$$

Therefore,

$$\begin{gathered}D_t(\rho D_tu)=D_t\rho\, D_tu+\rho D_tD_tu=\rho D_tD_tu=\rho D_t^2u,\\ \rho D_t^2u-\Delta D_tu+\nabla D_tP=f,\end{gathered}\tag{3.16}$$

where

$$f:=-\Delta u\cdot\nabla u-2\nabla u\cdot\nabla^2u+\nabla u\cdot\nabla P-D_t(\mathrm{div}(\nabla\mathrm{d}\odot\nabla\mathrm{d})).$$

Multiplying both sides of (3.16) by time t yields

$$\rho(tD_tu)_t-\Delta(tD_tu)+\nabla(tD_tP)=-t\rho u\cdot\nabla D_tu+\rho D_tu+tf.$$

Since $\mathrm{div}(tD_tu)\neq 0$, we cannot directly apply the maximal regularity estimates for the Navier-Stokes equations. However, the maximal regularity estimates for the heat equation [14, Proposition 2.1] need not $\mathrm{div}(tD_tu)=0$. But we need to eliminate $\nabla(tD_tP)$. Now, we introduce the definition of Helmholtz decomposition

$$\mathbb{P}:=\mathrm{Id}+\nabla(-\Delta)^{-1}\mathrm{div}\,,\quad \mathbb{Q}:=-\nabla(-\Delta)^{-1}\mathrm{div}\,,$$

where $\mathbb{P}$ stands for Leray projector, while $\mathbb{Q}$ represents the orthogonal complement of $\mathbb{P}$. Meanwhile, we observe that

$$\nabla(tD_tP)=\mathbb{Q}\Big(-t\rho u\cdot\nabla D_tu+\rho D_tu+tf-\rho(tD_tu)_t+\Delta(tD_tu)\Big).$$

Since $\mathrm{div}\, u=0$, one has

$$\begin{gathered}\mathrm{div}\, D_tu=\sum_{1\le i,j\le 3}\partial_iu^j\partial_ju^i=\mathrm{Tr}(\nabla u\cdot\nabla u),\\ \mathbb{Q}(t\Delta D_tu)=t\nabla\mathrm{Tr}(\nabla u\cdot\nabla u),\\ \mathbb{Q}(u\cdot\nabla u_t)=\mathbb{Q}(u_t\cdot\nabla u),\end{gathered}$$

$$\mathbb{Q}\big(\rho(tD_tu)_t\big) = \mathbb{Q}\Big((\rho-1)(tD_tu)_t + D_tu + tu\cdot\nabla u_t + tu_t\cdot\nabla u\Big),$$

and

$$\begin{aligned}(tD_tu)_t - \Delta(tD_tu) = \mathbb{P}[(1-\rho)(tD_tu)_t - t\rho u\cdot\nabla D_tu + \rho D_tu + tf]\\ + \mathbb{Q}(D_tu + 2tu_t\cdot\nabla u) - t\nabla\mathrm{Tr}(\nabla u\cdot\nabla u).\end{aligned} \tag{3.17}$$

Applying the maximal regularity estimates for the heat equation [14, Proposition 2.1], the Hölder inequality in Lorentz spaces and t that the Helmholtz projectors $\mathbb{P}$ and $\mathbb{Q}$ are bounded operators on $L^p(\mathbb{R}^d)$, it follows that

$$\begin{aligned}&\|tD_tu\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))} + \|((tD_tu)_t, t\nabla^2D_tu)\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + \|tD_tu\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\\
&\le C(\|(1-\rho)(tD_tu)_t\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + \|t\rho u\cdot\nabla D_tu\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\quad + \|\rho D_tu\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + \|tf\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + \|D_tu\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\quad + \|tu_t\cdot\nabla u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + \|t\nabla\mathrm{Tr}(\nabla u\cdot\nabla u)\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))})\\
&\le C(\|\rho-1\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\|(tD_tu)_t\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + \|t\rho u\cdot\nabla D_tu\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\quad + \|\rho D_tu\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + \|tf\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + \|D_tu\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\quad + \|tu_t\cdot\nabla u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + \|t\nabla\mathrm{Tr}(\nabla u\cdot\nabla u)\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}).\end{aligned}$$

It follows from condition (3.1) that the first term on the right-hand side of the above inequality can be absorbed by the left-hand side, and $\|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}$ is a bounded quantity.

• Estimate for $\|tf\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}$ and $\|t\nabla\mathrm{Tr}(\nabla u\cdot\nabla u)\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}$. By the Hölder inequality in Lorentz spaces and (3.11), we have

$$\begin{aligned}&\|-t\Delta u\cdot\nabla u - 2t\nabla u\cdot\nabla^2u + t\nabla u\cdot\nabla P\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + \|t\nabla\mathrm{Tr}(\nabla u\cdot\nabla u)\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\le C\|(\nabla^2u,\nabla P)\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\|t\nabla u\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\\
&\le C\|(\nabla^2u,\nabla P)\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\|tu\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))}\\
&\le C\|(u_0,\nabla \mathrm{d}_0)\|^2_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)},\end{aligned}$$

and

$$\begin{aligned}&\|tD_t(\mathrm{div}(\nabla\mathrm{d}\odot\nabla\mathrm{d}))\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\le C\|tD_t\nabla\mathrm{d}\cdot\nabla^2\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + C\|tD_t\nabla^2\mathrm{d}\cdot\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\le C\|tD_t\nabla\mathrm{d}\cdot\nabla^2\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + C\|t\nabla D_t\nabla\mathrm{d}\cdot\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\quad + C\|t\nabla u\cdot\nabla(\nabla\mathrm{d})\cdot\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\le C\|tD_t\nabla\mathrm{d}\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\|\nabla^2\mathrm{d}\|_{L^{q,1}(0,T;L^{p*}(\mathbb{R}^3))}\\
&\quad + C\|\nabla\mathrm{d}\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\|t\nabla D_t\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^{p*}(\mathbb{R}^3))}\\
&\quad + C\|t\nabla^2\mathrm{d}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\|\nabla u\|_{L^{q,1}(0,T;L^{p*}(\mathbb{R}^3))}\|\nabla\mathrm{d}\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\\
&\le C\|tD_t\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\|\nabla^3\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\quad + C\|\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\|t\nabla^2D_t\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\quad + C\|t\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))}\|\nabla^2u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\|\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\\
&\le C\|(u_0,\nabla\mathrm{d}_0)\|^2_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)} + C\|(u_0,\nabla\mathrm{d}_0)\|^3_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}\\
&\quad + C\|(u_0,\nabla\mathrm{d}_0)\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}\|t\nabla^2D_t\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}.\end{aligned}$$

• Estimate for $\|\rho D_tu\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}$ and $\|D_tu\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}$. From Proposition 3.1, we obtain

$$\|\rho D_tu\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + \|D_tu\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} \le C\|(u_0,\nabla\mathrm{d}_0)\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}.$$

• Estimate for $\|t\rho u\cdot\nabla D_t u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}$ and $\|tu_t\cdot\nabla u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}$. From the Hölder inequality in Lorentz spaces, Proposition 3.1-3.2, the embedding (3.6), (3.7) and (3.11), one has

$$\begin{aligned}
&\|t\rho u\cdot\nabla D_t u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}+\|tu_t\cdot\nabla u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\le C\|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\|t\nabla D_t u\|_{L^{q,1}(0,T;L^{p*}(\mathbb{R}^3))}\|u\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\\
&\quad+C\|u_t\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\|t\nabla u\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\\
&\le C\|t\nabla^2 D_t u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\|u\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\\
&\quad+C\|u_t\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\|tu\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))}\\
&\le C\|t\nabla^2 D_t u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\|(u_0,\nabla \mathrm{d}_0)\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}+C\|(u_0,\nabla \mathrm{d}_0)\|^2_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}.
\end{aligned}\tag{3.18}$$

In summary, we have

$$\begin{aligned}
&\|tD_t u\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}+\|((tD_t u)_t,t\nabla^2 D_t u)\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}+\|tD_t u\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\\
&\le C\|(u_0,\nabla \mathrm{d}_0)\|^2_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}+C\|(u_0,\nabla \mathrm{d}_0)\|^3_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}\\
&\quad+C\|(u_0,\nabla \mathrm{d}_0)\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}(\|t\nabla^2 D_t u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}+\|t\nabla^2 D_t\nabla \mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}+1).
\end{aligned}\tag{3.19}$$

Next, applying D_t to $(1.1)_3$, we obtain

$$D_t^2\nabla \mathrm{d}-\Delta D_t\nabla \mathrm{d}=-D_t(\nabla u^T\nabla \mathrm{d})+D_t(\nabla(|\nabla \mathrm{d}|^2)\mathrm{d})-\Delta u\cdot\nabla(\nabla \mathrm{d})-2\nabla u\cdot\nabla^2(\nabla \mathrm{d}).\tag{3.20}$$

Multiplying both sides of (3.20) by time t yields

$$\begin{aligned}
(tD_t\nabla \mathrm{d})_t-t\Delta D_t\nabla \mathrm{d}&=-tD_t(\nabla u^T\nabla \mathrm{d})+tD_t(\nabla(|\nabla \mathrm{d}|^2)\mathrm{d})-tu\cdot\nabla D_t\nabla \mathrm{d}\\
&\quad+D_t\nabla \mathrm{d}-t\Delta u\cdot\nabla(\nabla \mathrm{d})-2t\nabla u\cdot\nabla^2(\nabla \mathrm{d}).
\end{aligned}$$

By the maximal regularity estimate for the heat equation, one has

$$\begin{aligned}
&\|tD_t\nabla \mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}+\|((tD_t\nabla \mathrm{d})_t,t\nabla^2 D_t\nabla \mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&+\|tD_t\nabla \mathrm{d}\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\\
&\le C(\|tD_t(\nabla u^T\nabla \mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}+\|tD_t(\nabla(|\nabla \mathrm{d}|^2)\mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\quad+\|tu\cdot\nabla D_t\nabla \mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}+\|D_t\nabla \mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\quad+\|t\Delta u\cdot\nabla(\nabla \mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}+\|t\nabla u\cdot\nabla^2(\nabla \mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}).
\end{aligned}\tag{3.21}$$

• Estimate for $\|tD_t(\nabla u^T\nabla \mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}$. By the Hölder inequality in Lorentz spaces and (3.11), we obtain

$$\begin{aligned}
&\|tD_t(\nabla u^T\nabla \mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\le C\|t(\nabla D_t u-\nabla u\cdot\nabla u)\nabla \mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\le C\|\nabla \mathrm{d}\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\|t\nabla D_t u\|_{L^{q,1}(0,T;L^{p*}(\mathbb{R}^3))}\\
&\quad+C\|t\nabla u\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\|\nabla \mathrm{d}\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\|\nabla u\|_{L^{q,1}(0,T;L^{p*}(\mathbb{R}^3))}\\
&\le C\|\nabla \mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\|t\nabla^2 D_t u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\quad+C\|tu\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))}\|\nabla \mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\|\nabla^2 u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\le C\|(u_0,\nabla \mathrm{d}_0)\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}\|t\nabla^2 D_t u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}+C\|(u_0,\nabla \mathrm{d}_0)\|^3_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}.
\end{aligned}$$

• Estimate for $\|tD_t(\nabla(|\nabla \mathrm{d}|^2)\mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}$. Similarly, by the estimate for $\|tD_t(\mathrm{div}(\nabla \mathrm{d}\odot\nabla \mathrm{d}))\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}$, we obtain

$$\begin{aligned}
&\|tD_t(\nabla(|\nabla \mathrm{d}|^2)\mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\le C\|tD_t(\nabla^2\mathrm{d}\cdot\nabla \mathrm{d}\cdot \mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}+C\|tD_t(|\nabla \mathrm{d}|^2\cdot\nabla \mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}
\end{aligned}$$

$$
\begin{aligned}
&\le C\|tD_t\nabla^2 \mathrm{d}\cdot\nabla \mathrm{d}\cdot \mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + C\|t\nabla^2\mathrm{d}\cdot D_t\nabla\mathrm{d}\cdot\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\quad + C\|t\nabla^2\mathrm{d}\cdot\nabla\mathrm{d}\cdot D_t\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + C\|tD_t\nabla\mathrm{d}\cdot\nabla\mathrm{d}\cdot\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\le C\|tD_t\nabla^2\mathrm{d}\cdot\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\|\mathrm{d}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\\
&\quad + C\|t\nabla^2\mathrm{d}\cdot D_t\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\|\mathrm{d}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\\
&\quad + C\|t\nabla^2\mathrm{d}\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\|\nabla\mathrm{d}\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\|D_t\mathrm{d}\|_{L^{q,1}(0,T;L^{p*}(\mathbb{R}^3))}\\
&\quad + C\|tD_t\nabla\mathrm{d}\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\|\nabla\mathrm{d}\cdot\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^{p*}(\mathbb{R}^3))}\\
&\le C\|tD_t\nabla^2\mathrm{d}\cdot\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + C\|t\nabla^2\mathrm{d}\cdot D_t\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\quad + C\|t\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))}\|\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\|\nabla D_t\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\quad + C\|tD_t\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\|\nabla^2\mathrm{d}\cdot\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\le C\|tD_t\nabla^2\mathrm{d}\cdot\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + C\|t\nabla^2\mathrm{d}\cdot D_t\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\quad + C\|t\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))}\|\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\\
&\quad \times\big(\|D_t\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + \|\nabla u\cdot\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\big)\\
&\quad + C\|tD_t\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\|\nabla^2\mathrm{d}\cdot\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))},
\end{aligned}
$$

where

$$
\begin{aligned}
&C\|tD_t\nabla\mathrm{d}\cdot\nabla^2\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + C\|tD_t\nabla^2\mathrm{d}\cdot\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\le C\|(u_0,\nabla\mathrm{d}_0)\|^2_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)} + C\|(u_0,\nabla\mathrm{d}_0)\|^3_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}\\
&\quad + C\|(u_0,\nabla\mathrm{d}_0)\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}\|t\nabla^2 D_t\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))},
\end{aligned}
$$

$$
\begin{aligned}
&C\|t\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))}\|\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\\
&\times\big(\|D_t\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + \|\nabla u\cdot\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\big)\\
&\le C\|t\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))}\|\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\\
&\quad\times\big(\|D_t\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + \|\nabla u\|_{L^{q,1}(0,T;L^{p*}(\mathbb{R}^3))}\|\nabla\mathrm{d}\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\big)\\
&\le C\|t\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))}\|\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\\
&\quad\times\big(\|D_t\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} + \|\nabla^2 u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\|\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\big)\\
&\le C\|(u_0,\nabla\mathrm{d}_0)\|^3_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)},
\end{aligned}
$$

and

$$
\begin{aligned}
&C\|tD_t\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\|\nabla^2\mathrm{d}\cdot\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\
&\le C\|tD_t\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\|\nabla^2\mathrm{d}\|_{L^{q,1}(0,T;L^{p*}(\mathbb{R}^3))}\|\nabla\mathrm{d}\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\\
&\le C\|tD_t\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\|\nabla^3\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\|\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\\
&\le C\|(u_0,\nabla\mathrm{d}_0)\|^2_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}\|tD_t\nabla\mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}.
\end{aligned}
$$

- Estimate for $\|tu\cdot\nabla D_t\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}$. Similarly, we obtain

$$
\begin{aligned}
\|tu\cdot\nabla D_t\nabla\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))} &\le C\|u\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}\|t\nabla D_t u\|_{L^{q,1}(0,T;L^{p*}(\mathbb{R}^3))}\\
&\le C\|u\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\|t\nabla^2 D_t u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}
\end{aligned}
$$

$$\le C\|(u_0,\nabla \mathrm{d}_0)\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}\|t\nabla^2 D_t u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}.$$

- Estimate for $\|D_t\nabla \mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}$. From Proposition 3.1, we obtain

$$\|D_t\nabla \mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\le C\|(u_0,\nabla \mathrm{d}_0)\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}.$$

- Estimate for $\|t\Delta u\cdot\nabla(\nabla \mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}$. Similarly, we obtain

$$\begin{aligned}\|t\Delta u\cdot\nabla(\nabla \mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}&\le C\|t\nabla^2\mathrm{d}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\|\nabla^2 u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\&\le C\|t\nabla \mathrm{d}\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))}\|\nabla^2 u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\&\le C\|(u_0,\nabla \mathrm{d}_0)\|^2_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}.\end{aligned}$$

- Estimate for $\|t\nabla u\cdot\nabla^2(\nabla \mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}$. Similarly, we obtain

$$\begin{aligned}\|t\nabla u\cdot\nabla^2(\nabla \mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}&\le C\|t\nabla u\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\|\nabla^3\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\&\le C\|tu\|_{L^\infty(0,T;\dot B^{1+\frac{3}{m}}_{m,1}(\mathbb{R}^3))}\|\nabla^3\mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\&\le C\|(u_0,\nabla \mathrm{d}_0)\|^2_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}.\end{aligned}$$

In summary, we have

$$\begin{aligned}&\|tD_t\nabla \mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}+\|((tD_t\nabla \mathrm{d})_t,t\nabla^2 D_t\nabla \mathrm{d})\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}\\&+\|tD_t\nabla \mathrm{d}\|_{L^{s,1}(0,T;L^m(\mathbb{R}^3))}\\&\le C\|(u_0,\nabla \mathrm{d}_0)\|^2_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}+C\|(u_0,\nabla \mathrm{d}_0)\|^3_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}\\&\quad+C\|(u_0,\nabla \mathrm{d}_0)\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}(\|t\nabla^2 D_t u\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}+\|t\nabla^2 D_t\nabla \mathrm{d}\|_{L^{q,1}(0,T;L^p(\mathbb{R}^3))}+1)\\&\quad+C\|(u_0,\nabla \mathrm{d}_0)\|^2_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}\|tD_t\nabla \mathrm{d}\|_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}.\end{aligned}\tag{3.22}$$

Thus, combining (3.19) with (3.22), since $\|(u_0,\nabla \mathrm{d}_0)\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}$ is small, we obtain (3.14).

In the end, as $p\in(\frac{3}{2},3)$ (which yields $1<q<2$), we have

$$\begin{aligned}\|(D_t u,D_t\nabla \mathrm{d})\|_{L^\infty(\mathbb{R}^3)}&\le C\|(D_t u,D_t\nabla \mathrm{d})\|^{1-\frac{q}{2}}_{L^3(\mathbb{R}^3)}\|(\nabla^2 D_t u,\nabla^2 D_t\nabla \mathrm{d})\|^{q/2}_{L^p(\mathbb{R}^3)}\\&\le C\|(D_t u,D_t\nabla \mathrm{d})\|^{1-\frac{q}{2}}_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}\|(\nabla^2 D_t u,\nabla^2 D_t\nabla \mathrm{d})\|^{q/2}{}_{L^p(\mathbb{R}^3)}.\end{aligned}$$

Thus, one has

$$\begin{aligned}&\|t(D_t u,D_t\nabla \mathrm{d})\|_{L^2(0,T;L^\infty(\mathbb{R}^3))}\\&\le C\|t(D_t u,D_t\nabla \mathrm{d})\|^{1-\frac{q}{2}}_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\|(\nabla^2(tD_t u),\nabla^2(tD_t\nabla \mathrm{d}))\|^{q/2}_{L^q(0,T;L^p(\mathbb{R}^3))}\\&\le C\|(u_0,\nabla \mathrm{d}_0)\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}.\end{aligned}$$

Then, applying (3.14) gives the second part of (3.15).

To complete the proof of (3.15), it suffices to apply Lemma 2.5 with $r=3$ to $tD_t u$ (keeping in mind that $-1+\frac{3}{p}=2-\frac{2}{q}$), then from the Hölder inequality with respect to the time variable, one has

$$\begin{aligned}&\|t(\nabla D_t u,\nabla D_t\nabla \mathrm{d})\|_{L^2(0,T;L^3(\mathbb{R}^3))}\\&\le C\|t(D_t u,D_t\nabla \mathrm{d})\|^{\theta}_{L^\infty(0,T;\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))}\|t(\nabla^2 D_t u,\nabla^2 D_t\nabla \mathrm{d})\|^{1-\theta}_{L^q(0,T;L^p(\mathbb{R}^3))}\\&\le C\|(u_0,\nabla \mathrm{d}_0)\|_{\dot B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}.\end{aligned}$$

where $\theta = \frac{2p-3}{3p-3}$. Then, applying the first part (3.14) of proposition 3.3 gives the desired result. The case $1 < p \leq \frac{3}{2}$ reduces to the case we treated before since $\dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3) \hookrightarrow \dot{B}_{p_1,1}^{-1+\frac{3}{p_1}}(\mathbb{R}^3)$ for some $p_1 \in (\frac{3}{2}, 3)$. □

4. Existence part in Theorem 1.1

This section is dedicated to constructing global solutions under the stated conditions in Theorem 1.1. First, we smooth out the initial data $(\rho_0, u_0, \nabla \mathrm{d}_0)$ to be

$$\rho_0^n := \rho_0 * j_n, u_0^n := u_0 * j_n, \quad \mathrm{d}_0^n := \mathrm{d}_0 * j_n,$$

where $j_n(|x|) = n^2 j(|x|/n)$ is the classical Friedrich's mollifier. Following the classical theory for inhomogeneous incompressible nematic liquid crystal systems (see [30] for instance), we construct a sequence of smooth approximate solutions $(\rho^n, u^n, \nabla \mathrm{d}^n)_{n\in\mathbb{N}}$ to system (1.1). By the a priori estimates established in Proposition 3.1-3.3, we conclude that the solution sequence $(\rho^n, u^n, \nabla \mathrm{d}^n)_{n\in\mathbb{N}}$ is globally defined and uniformly bounded in the specified function spaces. To pass to the limit, we employ compactness arguments. However, a significant difficulty emerges due to the non-reflexivity of the Lorentz spaces, which prevents direct application of classical tools such as the Aubin-Lions lemma. To resolve this issue, we analyze the approximate solutions within an expanded functional framework

$$\dot{W}_{p,r}^{2,1}(\mathbb{R}^+ \times \mathbb{R}^3) := \{u \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}_{p,r}^{2-\frac{2}{r}}(\mathbb{R}^3)) : u_t, \nabla^2 u \in L^r(\mathbb{R}^+; L^p(\mathbb{R}^3)), 1 < r < \infty\},$$

which is a slightly more inclusive reflexive space where standard compactness arguments become applicable. The subsequent analysis follows the framework established in [15], where we incorporate the director field gradient $\nabla \mathrm{d}$ into the velocity estimates while preserving the original estimates for both u and ρ.

5. Uniqueness part in Theorem 1.1

This section is devoted to establishing the uniqueness of solutions constructed in Theorem 1.1. The detailed procedure follows the framework established in [8]. For the subsequent analysis, we recall several energy functionals and related estimates from [8].

$$X(t) = \sup_{\tau\in(0,t]} \tau^{-1} \|\delta\rho\|_{\dot{H}^{-1}(\mathbb{R}^3)},$$

$$Y(t) = (\sup_{\tau\in(0,t]} \|\sqrt{\rho_1}\delta u\|_{L^2(\mathbb{R}^3)}^2 + \sup_{\tau\in(0,t]} \|\nabla\delta \mathrm{d}\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \delta u\|_{L^2(0,t;L^2(\mathbb{R}^3))}^2 + \|\nabla^2 \delta \mathrm{d}\|_{L^2(0,t;L^2(\mathbb{R}^3))}^2)^{1/2},$$

with the following relationships

$$X(t) \leq C(Y(t) + \int_0^t gX\,d\tau), \quad X(t) \leq CY(t) e^{C\int_0^t g\,d\tau}. \tag{5.1}$$

However, unlike [8], the term

$$\int_{\mathbb{R}^3} \nabla(|\nabla \mathrm{d}_1|^2 \cdot \delta \mathrm{d} + \nabla\delta \mathrm{d} \cdot (\nabla \mathrm{d}_1 + \nabla \mathrm{d}_2) \cdot \mathrm{d}_2) \cdot \nabla \delta \mathrm{d}\, dx$$

requires modified estimates, which we now derive.

• Estimate for $\int_{\mathbb{R}^3} \nabla(|\nabla \mathrm{d}_1|^2 \cdot \delta \mathrm{d} + \nabla\delta \mathrm{d} \cdot (\nabla \mathrm{d}_1 + \nabla \mathrm{d}_2) \cdot \mathrm{d}_2) \cdot \nabla \delta \mathrm{d}\, dx$. Applying integration by parts followed by the Hölder inequality and the Young inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} \nabla(|\nabla \mathrm{d}_1|^2 \cdot \delta \mathrm{d} + \nabla\delta \mathrm{d} \cdot ((\nabla \mathrm{d}_1 + \nabla \mathrm{d}_2) \cdot \mathrm{d}_2)) \cdot \nabla \delta \mathrm{d}\, dx \\
&= \int_{\mathbb{R}^3} \nabla(|\nabla \mathrm{d}_1|^2 \cdot \delta \mathrm{d}) \cdot \nabla\delta \mathrm{d} - \nabla\delta \mathrm{d} \cdot (\nabla \mathrm{d}_1 + \nabla \mathrm{d}_2) \cdot \mathrm{d}_2) \cdot \Delta\delta \mathrm{d}\, dx \\
&= \int_{\mathbb{R}^3} 2\nabla^2 \mathrm{d}_1 \cdot \nabla \mathrm{d}_1 \cdot \delta \mathrm{d} \cdot \nabla\delta \mathrm{d}\, dx + \int_{\mathbb{R}^3} |\nabla \mathrm{d}_1|^2 \cdot \nabla\delta \mathrm{d} \cdot \nabla\delta \mathrm{d}\, dx \\
&\quad - \int_{\mathbb{R}^3} \nabla\delta \mathrm{d} \cdot (\nabla \mathrm{d}_1 + \nabla \mathrm{d}_2) \cdot \mathrm{d}_2 \cdot \Delta\delta \mathrm{d}\, dx
\end{aligned}$$

$$
\begin{aligned}
&\leq C\|\nabla^2 \mathrm{d}_1\|_{L^\infty(\mathbb{R}^3)}\|\nabla \mathrm{d}_1\|_{L^3(\mathbb{R}^3)}\|\delta \mathrm{d}\|_{L^6(\mathbb{R}^3)}\|\nabla\delta \mathrm{d}\|_{L^2(\mathbb{R}^3)} + C\|\nabla \mathrm{d}_1\|^2_{L^\infty(\mathbb{R}^3)}\|\nabla\delta \mathrm{d}\|^2_{L^2(\mathbb{R}^3)}\\
&\quad + C\|\nabla\delta \mathrm{d}\|_{L^2(\mathbb{R}^3)}(\|\nabla \mathrm{d}_1\|_{L^\infty(\mathbb{R}^3)} + \|\nabla \mathrm{d}_2\|_{L^\infty(\mathbb{R}^3)})\|\mathrm{d}_2\|_{L^\infty(\mathbb{R}^3)}\|\Delta\delta \mathrm{d}\|_{L^2(\mathbb{R}^3)}\\
&\leq \frac{1}{4}\|\nabla^2\delta \mathrm{d}\|^2_{L^2(\mathbb{R}^3)} + C\|\nabla \mathrm{d}_1\|_{\dot{B}^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}\|\nabla^2 \mathrm{d}_1\|_{L^\infty(\mathbb{R}^3)}\|\nabla\delta \mathrm{d}\|^2_{L^2(\mathbb{R}^3)}\\
&\quad + C\|\nabla \mathrm{d}_1\|^2_{L^\infty(\mathbb{R}^3)}\|\nabla\delta \mathrm{d}\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{4}\|\nabla^2\delta \mathrm{d}\|^2_{L^2(\mathbb{R}^3)}\\
&\quad + C(\|\nabla \mathrm{d}_1\|_{L^\infty(\mathbb{R}^3)} + \|\nabla \mathrm{d}_2\|_{L^\infty(\mathbb{R}^3)})^2\|\nabla\delta \mathrm{d}\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}
$$

By using estimates (84)-(90) from [8], we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}|\delta u|^2dx + \int_{\mathbb{R}^3}|\nabla\delta u|^2dx + \frac{d}{dt}\int_{\mathbb{R}^3}|\nabla\delta \mathrm{d}|^2dx + \int_{\mathbb{R}^3}|\nabla^2\delta \mathrm{d}|^2dx\\
&\leq C\big\|(\nabla u_2, \nabla^2 \mathrm{d}_2, \nabla^2 \mathrm{d}_1)\|_{L^\infty(\mathbb{R}^3)}(1 + \|\nabla \mathrm{d}_1\|_{L^\infty(0,t;\dot{B}^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))})\\
&\quad + \|(\nabla \mathrm{d}_1, \nabla \mathrm{d}_2)\|^2_{L^\infty(\mathbb{R}^3)}\big)\big(\|(\delta u, \nabla\delta \mathrm{d})\|^2_{L^2(\mathbb{R}^3)}\big)\\
&\quad + C^2X^2(\tau\|\nabla D_t u_2\|_{L^3(\mathbb{R}^3)} + \tau\|D_t u_2\|_{L^\infty(\mathbb{R}^3)})^2.
\end{aligned}
\tag{5.2}
$$

Integrating over the time interval $[0,t]$ and applying the Gronwall inequality yields

$$
\begin{aligned}
Y^2(t) &\leq C(\|\sqrt{\rho_0}\delta u_0\|_{L^2} + \|\nabla\delta \mathrm{d}_0\|_{L^2})e^{C(\|g\|_{L^1(0,t)}(\|h\|_{L^\infty(0,t)}+1)+\|f\|^2_{L^2(0,t)})}\\
&\quad \times e^{C\|f\|^2_{L^2(0,t)}e^{4C(\|g\|_{L^1(0,t)}(\|h\|_{L^\infty(0,t)}+1)+\|f\|^2_{L^2(0,t)})}},
\end{aligned}
\tag{5.3}
$$

where

$$
\begin{gathered}
f(t) = (t\|\nabla D_t u_2\|_{L^3(\mathbb{R}^3)} + t\|D_t u_2\|_{L^\infty(\mathbb{R}^3)}) + \|(\nabla \mathrm{d}_1, \nabla \mathrm{d}_2)\|_{L^\infty(\mathbb{R}^3)},\\
g(t) = \|(\nabla u_2, \nabla^2 \mathrm{d}_2, \nabla^2 \mathrm{d}_1)\|_{L^\infty(\mathbb{R}^3)}, h(t) = \|\nabla \mathrm{d}_1\|_{\dot{B}^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)}.
\end{gathered}
$$

Combining this with estimate (5.1) completes the proof of the uniqueness part in Theorem 1.1.

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