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GLOBAL BIFURCATION FOR SEMILINEAR EIGENVALUE PROBLEMS INVOLVING NONLOCAL TERMS

QINGBO LIU, LAN ZHAO

Abstract. We investigate global bifurcation phenomenon for a class of semilinear eigenvalue problems involving nonlocal terms. Under certain assumptions, we demonstrate the existence of a global continuum emanating from the first eigenvalue of the unperturbed problem. As an application of this result, we identify the parameter interval for which positive solutions exist in the problem with general nonlinearities f, where f exhibits asymptotic $(q-1)$-linear behavior both near zero and at infinity. To study the global structure of bifurcation branch, we also establish some properties of the first eigenvalue for a semilinear eigenvalue problem.

1. Introduction

In this article, we study the global bifurcation phenomenon for the problem:

$$\begin{gathered} -\Delta u = \lambda\Big(\int_\Omega |u|^q dx\Big)^{\frac{2}{q}-1}|u|^{q-2}u + h(x,u,\lambda) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with a smooth boundary $\partial\Omega$, and λ is a bifurcation parameter. The exponent q satisfies $1 \leq q \leq 2$. The function $h : \Omega \times \mathbb{R} \times \mathbb{R}$ satisfies the Carathéodory condition in the first two variable.

Problem (1.1) is related to the so-called semilinear eigenvalue problem of the Dirichlet Laplacian

$$\begin{gathered} -\Delta u = \lambda\Big(\int_\Omega |u|^q dx\Big)^{\frac{2}{q}-1}|u|^{q-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where the exponent $q \in [1, 2^*)$. The first eigenvalue of semilinear problem (1.2),

$$\lambda_{1,q}(\Omega) = \inf\Big\{\frac{\int_\Omega |\nabla u|^2 \, dx}{\big(\int_\Omega |u|^q \, dx\big)^{2/q}} : u \in W_0^{1,2}(\Omega), u \not\equiv 0\Big\},$$

is significant and has been extensively studies in relation to the geometry and function theory of Ω, as well as in the context of mathematical physics issues encountered in engineering. In particular, it is associated with the sharp constant in the Sobolev embedding $W_0^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$:

$$\mathcal{S}_q(\Omega) = \frac{1}{\sqrt{\lambda_{1,q}(\Omega)}} = \sup\Big\{\frac{\big(\int_\Omega |u|^q \, dx\big)^{1/q}}{\big(\int_\Omega |\nabla u|^2 \, dx\big)^{1/2}} : u \in W_0^{1,2}(\Omega), u \not\equiv 0\Big\}.$$

$\mathcal{S}_1(\Omega) = \lambda_{1,1}(\Omega)$ is usually referred to as the torsional rigidity of the set Ω, which is a key parameter that measures the ability of rod-like structures (such as beams, shafts, or columns) to resist torsional deformation within the frameworks of elasticity theory and structural engineering. For a given area, the disk (or ball) maximizes $\mathcal{S}_1(\Omega)$ (Pólya-Szegö theorem). $\lambda_{1,2}(\Omega)$ is the principal frequency of the membrane(more generally, the bottom of the spectrum of the Laplacian). For a given area, the disk (or ball) minimizes $\lambda_{1,2}(\Omega)$(Faber-Krahn inequality). In accordance with

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equation (1.2), $\lambda_{1,q}(\Omega)$ interpolates between the torsional rigidity of a domain and its principal frequency as q ranges from 1 to 2 (see [3]).

Rabinowitz [9] studied problem (1.1) with $q = 2$ by using the topological degree argument, to be precise, the Leray-Schauder degree. He proved the existence of two distinct continua of nontrivial solutions, positive and negative ones, that bifurcate at the point $(\lambda_{1,2}, 0)$. In this article, we hope to obtain analogous results. Unfortunately, problem (1.1) is not linear, so Leray-Schauder degree argument does not work directly. Upon recognizing that problem (1.1) is homogeneous, and drawing inspiration from the work of Petr Girg and Peter Takáč [7] and Pavel Drábek [5], we employ the Browder-Petryshyn degree instead of the Leray-Schauder degree to address this challenge.

Another distinguishing feature of problem (1.1) is that the first equation contains a nonlocal term $\int_\Omega |u|^q dx$ and hence the equation is no longer a pointwise identity. This also raises some essential difficulties to study this kind of problems. To tackle the nonlocal term, we partially employ the approach outlined in [4], which investigates a comparable nonlocal equation, namely a Kirchhoff-type equation. It is crucial to highlight that the nonlocal term in [4] is formulated using $\int_\Omega |\nabla u|^2\, dx$, which differs from our specific formulation. Therefore, our methodology does not represent a full replication of the techniques described in [4].

Finally, we establish analogous results as discussed in [7, 9]. We believe that problem (1.1) has not been considered earlier by bifurcation arguments, thus our results with some innovative features are extension of those existed results.

In section 2, we investigate the first eigenvalue of eigenvalue problem (1.2) and give its basic properties. This is also a contribution of this article. The remaining section is devoted to some a priori estimates about convergence analysis and the equivalence of norms near $(\lambda_1, 0)$ and (λ_1, ∞). This part is essential for bifurcation analysis.

In section 3, we establish the global bifurcation result for (1.1). Let $\lambda_1 \equiv \lambda_{1,q}$ denote the first eigenvalue of (1.2). Assume that

(H1) $h : \Omega \times \mathbb{R} \times \mathbb{R}$ satisfies the Carathéodory condition in the first two variable and is locally Hölder continuous in the second variable. There exists a constant $C \in (0, \infty)$ and $p \in (2, 2^*)$ such that
$$|h(x, u; \lambda)| \le C|u|^{p-1}$$
for a.e. $x \in \Omega$ and all $(u, \lambda) \in \mathbb{R} \times \mathbb{R}$.

(H2) (for bifurcations from zero)
$$\lim_{s\to 0} \frac{h(x, s, \lambda)}{s} = 0$$
uniformly for almost every $x \in \Omega$ and λ on bounded sets.

(H3) (for bifurcations from infinity)
$$\lim_{s\to +\infty} \frac{h(x, s, \lambda)}{s} = 0$$
uniformly for almost every $x \in \Omega$ and λ on bounded sets.

Let us point out that the advantage of working with the class locally Hölder continuous is that we can get classical solutions of elliptic equations we deal with by means of Schauder estimates, not merely weak solutions. The first main result of this article is the following theorem.

Theorem 1.1. *Let $q \in [1, 2]$ and h satisfy* (H1), (H2). *Then the pair $(\lambda_1, 0)$ is a bifurcation point of* (1.1). *Moreover, there is a component $\mathscr{C}$ of the set of nontrivial solutions of* (1.1) *in $\mathbb{R} \times W_0^{1,2}(\Omega)$ whose closure contains $(\lambda_1, 0)$ and it is either unbounded or contains a pair $(\bar\lambda, 0)$ for some $\bar\lambda$, an eigenvalue of* (1.2) *with $\bar\lambda \ne \lambda_1$.*

Let us reformulate Theorem 1.1 in terms of bifurcation from infinity for problem (1.1) at (λ_1, ∞). Then the second main result of this paper is stated below. We only state the result for bifurcation from infinity without providing a proof, as the proof can be obtained by combining Theorem 1.1 and [10, Theorem 1.6].

Theorem 1.2. *Let $q \in [1,2]$ and h satisfies* (H1), (H3). *Then the pair (λ_1, ∞) is a bifurcation point of* (1.1). *Moreover, there is a component $\mathscr{D}$ of the set of nontrivial solutions of* (1.1) *in $\mathbb{R} \times W_0^{1,2}(\Omega)$ which meets (λ_1, ∞). If $\Lambda \subset \mathbb{R}$ is an interval such that $\Lambda \cap r(L) = \{\lambda_1\}$ where $r(L)$ is the set of real eigenvalue values of* (1.2) *and $\mathscr{M}$ is a neighborhood of (λ_1, ∞) whose projection on $\mathbb{R}$ lies in Λ and whose projection on $W_0^{1,2}(\Omega)$ is bounded away from 0, then either*

(i) *$\mathscr{D} - \mathscr{M}$ is bounded in $\mathbb{R} \times W_0^{1,2}(\Omega)$ in which case $\mathscr{D} - \mathscr{M}$ meets $\mathscr{R} = \{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$ or*
(ii) *$\mathscr{D} - \mathscr{M}$ is unbounded.*

If (ii) occurs and $\mathscr{D} - \mathscr{M}$ has a bounded projection on $\mathbb{R}$, then $\mathscr{D} - \mathscr{M}$ meets $(\widehat{\lambda}, \infty)$ where $\lambda_1 \neq \widehat{\lambda} \in r(L)$.

In Section 4, as a extension and application of the results in Section 3, we consider the problem

$$
\begin{gathered}
-\Delta u = \lambda\Big(\int_\Omega |u|^q dx\Big)^{\frac{2}{q}-1} f(u) \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{1.3}
$$

We assume that f satisfies the following conditions:

(H4) $f : \mathbb{R}^+ = [0, +\infty) \to \mathbb{R}^+$ is a locally Hölder continuous function. There exists a positive constant τ such that $f(\tau) = 0$, $f(s)s > 0$ for $s \in (0, \tau) \cup (\tau, +\infty)$ and there exists a constant $\kappa > 0$ such that
$$\lim_{s \to \tau^-} \frac{f(s)}{\tau - s} = \kappa.$$
(H5) There exists $f_0 \in (0, +\infty)$ such that
$$\lim_{s \to 0^+} \frac{f(s)}{|s|^{q-2}s} = f_0.$$
(H6) There exists $f_\infty \in (0, +\infty)$ such that
$$\lim_{s \to +\infty} \frac{f(s)}{|s|^{q-2}s} = f_\infty.$$

The third main result reads as follows.

Theorem 1.3. *Assume that $q \in [1,2]$. f satisfies* (H4)–(H6) *and $f_0 \neq f_\infty$. Then*

(i) *if $\lambda \in \big(\min\{\lambda_1/f_\infty, \lambda_1/f_0\}, \max\{\lambda_1/f_\infty, \lambda_1/f_0\}\big]$, then* (1.3) *has at least one positive solution;*
(i) *if $\lambda \in \big(\max\{\lambda_1/f_\infty, \lambda_1/f_0\}, +\infty\big)$, then* (1.3) *has at least two positive solutions.*

See Figure 1,

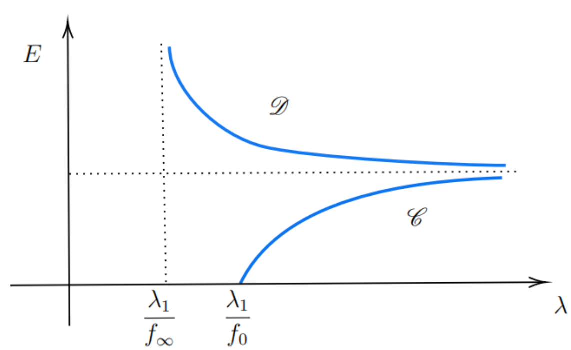

Figure 1. Schematic diagram of $f_0, f_\infty \in (0, +\infty)$ and $f_0 \neq f_\infty$.

Theorem 1.3 cannot be derived directly. In fact, Theorem 1.1 is not used in the derivation of Theorem 1.3. By expanding $f(u)$ at both zero and infinity, it becomes evident that the perturbation of Equation (1.3) does contain nonlocal terms, whereas the perturbation of Equation (1.1) does not include nonlocal terms. Consequently, in Section 4, we will formulate a new bifurcation theorem that is related to (1.3).

In the appendix, we give some preliminary results concerning Browder-Petryshyn degree for perturbations of monotone operators. The definition and basic properties can be found in [11, 12].

We now introduce some notation conventions which will be used later in this paper. Let X be the usual Sobolev space $W_0^{1,2}(\Omega)$ with the norm $\|u\|_X = \big(\int_\Omega |\nabla u|^2\big)^{1/2}$ and X' be its dual space. Sometimes, we omit dx in the integral symbol. Denote by $\langle \cdot, \cdot \rangle$ the duality pairing between X and X'. We write $u_n \rightharpoonup u$ and $u_n \to u$ for the weak and strong convergence of the sequence $\{u_n\}$ in X, respectively. Let L^p be the usual Lebesgue space with the norm $\|u\|_p = \big(\int_\Omega |u|^p dx\big)^{1/p}$. For a measurable set A of $\mathbb{R}^n$ we denote its measure by $|A|$. Also, we denote by c, c_i, C, and C_i, $i \in \mathbb{N}$, general positive constants the exact value may be different from line to line.

The rest of this paper is arranged as follows. In Section 2, we give some preliminaries which will be used later in this paper. In Section 3, we give the proof of Theorem 1.1. In Section 4, we give the proof of Theorem 1.3.

2. Preliminaries

2.1. Properties of the first eigenvalue λ_1.

By the compactness of the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$, there exists a minimizer of λ_1 and λ_1 is well defined. We are going to study the properties of

$$\lambda_1 = \inf\Big\{ \frac{\int_\Omega |\nabla u|^2 \, dx}{\big(\int_\Omega |u|^q \, dx\big)^{2/q}} : u \in W_0^{1,2}(\Omega), u \not\equiv 0 \Big\}.$$

These properties are important in the study of global bifurcation phenomena.

Lemma 2.1. *Let λ_1 is the first eigenvalue of (1.2) and φ_1 is an eigenfunction corresponding to λ_1. Then $\varphi_1 \in C^{1,\alpha}(\bar\Omega)$ for some $\alpha \in (0,1)$ and $\partial\varphi_1/\partial\nu < 0$ if φ_1 is nonnegative, where ν is the outer unit normal at $x \in \partial\Omega$.*

Proof. In fact, u belongs $C^2(\overline{\Omega})$. The method of proof is so-called the bootstrap argument, we refer readers to [2, Theorem 1.16]. Furthermore, if $\varphi_1 \geq 0$, by the strong maximum principle(see [8]), $\partial\varphi_1/\partial\nu < 0$ for all $x \in \partial\Omega$. □

Lemma 2.2. *Let φ_1 be an eigenfunction associated with λ_1, then either $\varphi_1 > 0$ or $\varphi_1 < 0$ in Ω, i.e. λ_1 is the principal eigenvalue of (1.2).*

Proof. We notice that if φ_1 is an eigenfunction, so is $v := |\varphi_1|$. Without loss of generality, we shall assume that $\|v\|_q = 1$. So we have

$$\begin{gathered} -\Delta v = \lambda_1 v^{q-1} \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

By the strong maximum principle [8], we know that $v > 0$ in the whole domain. By the continuity of φ_1, either φ_1 or $-\varphi_1$ is positive in the whole domain. □

Remark 2.3. Eigenfunction φ_1 can be normalized by $\varphi_1 > 0$ in Ω and $\int_\Omega \varphi_1^q = 1$.

Lemma 2.4. *λ_1 is simple.*

Proof. We only discuss $q \in [1,2)$ and $q = 2$ is easy. Consider the auxiliary problem

$$\begin{gathered} -\Delta u = u^{q-1} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.1}$$

Reference [1, Theorem 8.4.1] states that the positive solution of (2.1) is unique with $q \in [1,2)$. Let u, v be two eigenfunctions associated with λ_1. Then

$$\lambda_1^{2-q} \frac{u}{\|u\|_q^{1/q}} \text{ and } \lambda_1^{2-q} \frac{v}{\|v\|_q^{1/q}}$$

are both solutions to (2.1). Hence

$$u = \Big(\frac{\|u\|_q}{\|v\|_q}\Big)^{1/q} v,$$

which shows that λ_1 is simple. $\square$

Lemma 2.5. (1.2) *has a positive solution if and only if* $\lambda = \lambda_1$.

Proof. Suppose on the contrary that (1.2) with $\lambda > \lambda_1$ has a positive solution v, and let u be a positive eigenfunction corresponding to λ_1. There exists a constant $t > 0$ large enough such that $tv \geq u$. Clearly, $\widetilde{v} := tv$ is also an eigenfunction of (1.2). Define Φ and Ψ on X by

$$\Phi(w) = \frac{1}{2}\|w\|_X^2, \quad \Psi(w) = \frac{1}{2}\Big(\int_\Omega w^q dx\Big)^{2/q}.$$

Then the energy functional corresponding to (1.2) is

$$J(w) = (\Phi - \lambda\Psi)(w) = \frac{1}{2}\|w\|_X^2 - \frac{\lambda}{2}\Big(\int_\Omega w^q dx\Big)^{2/q}.$$

For all $\varphi \in C_c^\infty(\Omega)$, let

$$\Phi'(w)\varphi = \int_\Omega \nabla w \nabla \varphi,$$
$$\Psi'(w)\varphi = \Big(\int_\Omega w^q dx\Big)^{\frac{2}{q}-1} \int_\Omega w^{q-1}\varphi.$$

Then w is a weak solution of (1.2) if and only if $\Phi'(w) = \lambda\Psi'(w)$. For $\varphi \geq 0$, we have $\langle \Psi'(u), \varphi\rangle \leq \langle \Psi'(\widetilde{v}), \varphi\rangle$. Then

$$\langle \Phi'(u), \varphi\rangle = \langle \lambda_1 \Psi'(u), \varphi\rangle \leq \langle \lambda_1 \Psi'(\widetilde{v}), \varphi\rangle = \langle \lambda \Psi'(\eta\widetilde{v}), \varphi\rangle = \langle \Phi'(\eta\widetilde{v}), \varphi\rangle,$$

where $\eta = \lambda_1/\lambda < 1$. Taking $\varphi = (u - \eta\widetilde{v})^+$ as a test function in $\langle \Phi'(u), \varphi\rangle \leq \langle \Phi'(\eta\widetilde{v}), \varphi\rangle$, it follows that $\nabla(u - \eta\widetilde{v})^+ = 0$, this implies $(u - \eta\widetilde{v})^+ = 0$ and so $u \leq \eta\widetilde{v}$ in Ω. Repeating this argument n times, we obtain that $0 \leq u \leq \eta^n\widetilde{v}$. Letting $n \to +\infty$, we obtain $u \equiv 0$. This is a contradiction. So v must change sign. $\square$

Lemma 2.6. λ_1 *is isolated.*

Proof. Let v be any eigenfunction associated to an eigenvalue $\lambda > \lambda_1$ and $\mathcal{N}$ be its any nodal domain. Then we have $v|_{\mathcal{N}} \in W_0^{1,2}(\mathcal{N})$. Define

$$w = \begin{cases} v & \text{for } x \in \mathcal{N} \\ 0 & \text{for } x \in \Omega\backslash\mathcal{N} \end{cases}$$

It is easy to see that $w \in X$ and we claim $|\mathcal{N}| \geq C(\lambda)$, where $C(\lambda)$ is a constant and is only related to λ. We only consider the case $n \geq 3$, and $n = 2$ is simple. We have

$$\int_{\mathcal{N}} |\nabla w|^2 dx = \lambda\Big(\int_{\mathcal{N}} |v|^q dx\Big)^{\frac{2}{q}-1} \int_{\mathcal{N}} w^q dx.$$

By the Hölder inequality and the Sobolev embeddings we have

$$c\|w\|_{2^*}^2 \leq \int_{\mathcal{N}} |\nabla w|^2 dx \leq \lambda\Big(\int_{\mathcal{N}} |w|^q dx\Big)^{2/q} \leq \lambda \|w\|_{2^*}^2 |\mathcal{N}|^{(1-\frac{q}{2^*})\frac{2}{q}}.$$

where $c > 0$ is related to the best embedding constant. Based on the aforementioned fact, we substantiate the assertion:

$$|\mathcal{N}| \geq \Big(\frac{c}{\lambda}\Big)^{\frac{2^*}{2^*-q}\frac{q}{2}} \equiv C(\lambda).$$

Now we prove Lemma 2.6 by contradiction. Assume that there exists a sequence of eigenvalues $\lambda_n \in (\lambda_1, \delta)$ for some constant $\delta > \lambda_1$ which converges to λ_1. Let u_n be the corresponding eigenfunctions. Lemma 2.5 implies that u_n changes sign. Integration by parts gives

$$\int_\Omega |\nabla u_n|^2 dx = \lambda_n \Big(\int_\Omega |u_n|^q dx\Big)^{2/q}.$$

We define

$$v_n := \frac{u_n}{\|u_n\|_q}.$$

Obviously, v_n is bounded in X so there exists a subsequence, denoted again by v_n, and $v \in X$ such that $v_n \rightharpoonup v$ in X and $v_n \to v$ in $L^q(\Omega)$. Norm $\|\cdot\|_X$ is sequentially weakly lower semi-continuous, so we have that

$$\int_\Omega |\nabla v|^2 dx \le \liminf_{n\to+\infty} \int_\Omega |\nabla v_n|^2 dx = \liminf_{n\to+\infty} \lambda_n = \lambda_1.$$

On the other hand, $\|v_n\|_q = 1$ and $v_n \to v$ in $L^q(\Omega)$ imply that $\|v\|_q = 1$. It follows that

$$\int_\Omega |\nabla v|^2 dx \le \lambda_1 \Big(\int_\Omega v^q dx \Big)^{2/q}.$$

The above inequality and the variational characterization of λ_1 imply that

$$\int_\Omega |\nabla v|^2 dx = \lambda_1 \Big(\int_\Omega v^q dx \Big)^{2/q}.$$

Without loss of generality, we may assume that $v > 0$ in Ω. Since $v_n \rightharpoonup v$ in X, passing if necessary to a subsequence, we can assume that $v_n \to v$ a.e. in Ω. Therefore, we arrive at the conclusion that $|B_n^-| \to 0$, where B_n^- denotes the negative set of u_n. This presents a contradiction to the aforementioned assertion. □

Because of Lemma 2.6, we have shown that λ_1 is an isolated eigenvalue of (1.2); i.e. if we let

$$\lambda_2 = \inf\{\lambda > \lambda_1 : \lambda \text{ is an eigenvalue of problem (1.2)}\},$$

then $\lambda_1 < \lambda_2$. Moreover, we have the following conclusion.

Proposition 2.7. *There exists $\delta > 0$ such that for all $q \in [1,2]$, there is no eigenvalue of problem* (1.2) *in $(\lambda_1, \lambda_1 + \delta]$.*

2.2. **Equivalence of norms near $(\lambda_1, 0)$ and (λ_1, ∞).** The following lemma is a useful consequence of hypothesis (H2) or (H3).

Lemma 2.8. *Let $u \in L^\infty(\Omega)$ and $u \not\equiv 0$ in Ω.*

(i) *If* (H2) *is satisfied, then*

$$\frac{h(x,u,\lambda)}{\|u\|_{L^\infty(\Omega)}} \to 0 \quad \text{as } \|u\|_{L^\infty(\Omega)} \to 0$$

holds for a.e. $x \in \Omega$ and uniformly for every $\lambda \in \mathbb{R}$.

(ii) *If* (H3) *is satisfied, then*

$$\frac{h(x,u,\lambda)}{\|u\|_{L^\infty(\Omega)}} \to 0 \quad \text{as } \|u\|_{L^\infty(\Omega)} \to \infty$$

holds for a.e. $x \in \Omega$ and uniformly for every $\lambda \in \mathbb{R}$.

Proof. We first notice that $h(x,u,\lambda) = 0$ if $u(x) = 0$, and estimate

$$\frac{|h(x,u_n,\lambda)|}{\|u_n\|_{L^\infty(\Omega)}} = \frac{|h(x,u_n,\lambda)|}{|u_n(x)|}\frac{|u_n(x)|}{\|u_n\|_{L^\infty(\Omega)}} \le \frac{|h(x,u_n,\lambda)|}{|u_n(x)|}$$

if $u_n(x) \ne 0$. From $\|u_n\|_{L^\infty(\Omega)} \to 0$ we obtain $u_n(x) \to 0$ uniformly for a.e. $x \in \Omega$. This gives (i).

To prove (ii), we split the domain as $\Omega = M_n \cup N_n$ where

$$M_n = \big\{x \in \Omega : |u_n(x)| \le \|u_n\|_{L^\infty(\Omega)}^{\frac{1}{2(p-1)}}\big\},$$
$$N_n = \big\{x \in \Omega : |u_n(x)| > \|u_n\|_{L^\infty(\Omega)}^{\frac{1}{2(p-1)}}\big\}.$$

For $x \in M_n$ we infer that

$$\frac{|h(u_n)|}{\|u_n\|_{L^\infty(\Omega)}} = \frac{|h(u_n)|}{|u_n|^{p-1}}\frac{|u_n|^{p-1}}{\|u_n\|_{L^\infty(\Omega)}} \le C\frac{\|u_n\|^{1/2}_{L^\infty(\Omega)}}{\|u_n\|_{L^\infty(\Omega)}}.$$

For $x \in N_n$ we infer that

$$\frac{|h(u_n)|}{\|u_n\|_{L^\infty(\Omega)}} = \frac{|h(u_n)|}{|u_n|}\frac{|u_n|}{\|u_n\|_{L^\infty(\Omega)}} \le \frac{|h(u_n)|}{|u_n|}.$$

Now let χ_{M_n} and χ_{N_n} denote the characteristic functions of the sets M_n and N_n, respectively. By combining the previous inequalities, we obtain

$$\frac{|h(u_n)|}{\|u_n\|_{L^\infty(\Omega)}} \le C\frac{\|u_n\|^{1/2}_{L^\infty(\Omega)}}{\|u_n\|_{L^\infty(\Omega)}}\chi_{M_n}(x) + \frac{|h(u_n)|}{|u_n|}\chi_{N_n}(x).$$

It is easy to see that

$$\frac{h(u_n)}{\|u_n\|_{L^\infty(\Omega)}} \to 0 \quad \text{as } \|u_n\|_{L^\infty(\Omega)} \to +\infty$$

holds for a.e. $x \in \Omega$. □

Corollary 2.9. *Let $1 \le r < \infty$, and (H2) or (H3) is satisfied. Then*

$$\|\frac{h(x,u,\lambda)}{\|u\|_{L^\infty(\Omega)}}\|_r = \Big(\int_\Omega |\frac{h(x,u,\lambda)}{\|u\|_{L^\infty(\Omega)}}|^r\Big)^{1/r} \to 0$$

as $\|u\|_{L^\infty(\Omega)} \to 0$ or $\|u\|_{L^\infty(\Omega)} \to \infty$ uniformly for every $\lambda \in \mathbb{R}$.

Let us consider a sequence of nontrivial solutions $\{(\lambda_n, u_n)\}_{n=1}^\infty$ of problem (1.1), i.e., for each $n = 1, 2, \dots$, the integral identity

$$\int_\Omega \nabla u_n \nabla\phi \,\mathrm{d}x = \lambda_n\Big(\int_\Omega |u_n|^q dx\Big)^{\frac{2}{q}-1}\int_\Omega |u_n|^{q-2}u_n\phi\,\mathrm{d}x + \int_\Omega h(x,u_n;\lambda)\phi\,\mathrm{d}x$$

holds for all $\phi \in W_0^{1,2}(\Omega)$. We assume that

$$0 < \lambda_n \le \lambda_2 - \delta, \quad n = 1, 2, \dots,$$

where $\delta \in (\lambda_2 - \lambda_1)$ is a constant. Because of [2, Theorem 1.16], $u_n \in C^2(\overline{\Omega})$. More over we have the following result.

Theorem 2.10. *Let $\{(\lambda_n, u_n)\}_{n=1}^\infty$ be as above. Then the following three statements are equivalent, as $n \to \infty$,*

(i) $\|u_n\|_{W_0^{1,2}(\Omega)} \to 0$,
(ii) $\|u_n\|_{L^\infty(\Omega)} \to 0$,
(iii) $\|u_n\|_{C^{1,\beta}(\overline{\Omega})} \to 0$.

Proof. Clearly, (iii) implies (i) and (ii).

(i) ⇒ (ii): We denote $w_n = u_n/\|u_n\|_X$ satisfying $\|w_n\|_X = 1$ and

$$-\Delta w_n = \lambda_n\Big(\int_\Omega |w_n|^q dx\Big)^{\frac{2}{q}-1}|w_n|^{q-2}w_n + \frac{h(x, w_n\|u_n\|_X, \lambda_n)}{\|u_n\|_X}.$$

Based on the hypotheses $1 \le q \le 2$ and (H1), the right-hand side, represented as $f(x, w_n)$, satisfies

$$\begin{aligned} f(x,w_n) &= \lambda_n\|w_n\|_q^{2-q}|w_n|^{q-2}w_n + \frac{h\left(x, w_n\|u_n\|_X, \lambda_n\right)}{\|u_n\|_X} \\ &\le C_1|w_n|^{q-1} + C_2\|u_n\|_X^{p-2}|w_n|^{p-1} \end{aligned} \tag{2.2}$$

where C_1 and C_2 are constants that are independent of $\|u_n\|_X$. Our aim is to obtain a priori estimates and determine the L^∞-norm of w_n, which is controlled by the $W_0^{1,2}$-norm of w_n. However, it is important to recognize that the coefficient of $|w_n|^{p-1}$ is associated with the value of $\|u_n\|_X^{p-2}$, which is variable! Consequently, a comprehensive analysis utilizing a bootstrap argument is required. Now, we proceed as in the proof of [2, Theorem 1.16].

Step 1. By the Sobolev embedding theorem it follows that $w_n \in L^{2^*}$ and satisfies

$$\|w_n\|_{2^*} \le C\|w_n\|_X = C \tag{2.3}$$

where C is a constant that is independent of $\|u_n\|_X$.

Step 2. Based on the information provided in (2.2) and the condition $p > q$, it follows that $f(x, w_n) \in L^r$ with $r = \frac{2^*}{p-1}$ and

$$\begin{aligned}\|f(x,w_n)\|_r^r &\le \int_\Omega \big|C_1|w_n|^{q-1} + C_2\|u_n\|_X^{p-2}|w_n|^{p-1}\big|^r dx \\ &\le C_1\int_\Omega |w_n|^{(q-1)r}dx + C_2\|u_n\|_X^{(p-2)r}\int_\Omega |w_n|^{(p-1)r}dx \\ &= C_1\|w_n\|_{(q-1)r}^{(q-1)r} + C_2\|u_n\|_X^{(p-2)r}\|w_n\|_{(p-1)r}^{(p-1)r} \\ &\le C_1\|w_n\|_{2^*}^{(q-1)r} + C_2\|u_n\|_X^{(p-2)r}\|w_n\|_{2^*}^{(p-1)r}.\end{aligned}$$

Hence, we derive that

$$\|f(x,w_n)\|_r \le C_1\|w_n\|_{2^*}^{q-1} + C_2\|u_n\|_X^{p-2}\|w_n\|_{2^*}^{p-1} \tag{2.4}$$

where C_1 and C_2 are constants that are independent of $\|u_n\|_X$.

Step 3. Since $-\Delta w_n = f(x, w_n)$, L^p-estimates yields $u \in W^{2,r}$ and

$$\|w_n\|_{W^{2,r}(\Omega)} \le C\|f(x,w_n)\|_r \tag{2.5}$$

where C is a constant that is independent of $\|u_n\|_X$.

If $2r > n$ then $u \in C^{0,\gamma}(\overline{\Omega})$. Otherwise, we can repeat Steps 1–3. After a finite number of times, one finally finds a number r^* such that $u \in W^{2,r^*}$ with $2r^* > n$. Then the Sobolev embedding theorem yields again $W^{2,r^*}(\Omega) \hookrightarrow C^{0,\gamma}(\overline{\Omega})$ with $\gamma \le 1$ and

$$\|w_n\|_{C^{0,\gamma}(\overline{\Omega})} \le C\|w_n\|_{W^{2,r}(\Omega)} \tag{2.6}$$

where C is a constant that is independent of $\|u_n\|_X$.

From (2.3)-(2.6), we obtain

$$\|w_n\|_{C^{0,\gamma}(\overline{\Omega})} \le C_1 + C_2\|u_n\|_X^{p-2} \tag{2.7}$$

where C_1 and C_2 are constants that are independent of $\|u_n\|_X$. Furthermore, based on Schauder estimates, we obtain $w_n \in C^{2,\gamma}(\overline{\Omega})$ and

$$\|w_n\|_{C^{2,\gamma}(\overline{\Omega})} \le C_1 + C_2\|u_n\|_X^{p-2} \tag{2.8}$$

where C_1 and C_2 are constants that are independent of $\|u_n\|_X$. (2.8) implies that

$$\|u_n\|_\infty \le C\|u_n\|_{C^{2,\gamma}(\overline{\Omega})} \le \Big(C_1 + C_2\|u_n\|_X^{p-2}\Big)\|u_n\|_X. \tag{2.9}$$

However, because $p - 2 > 0$, (i) $\Rightarrow$ (ii) is proved.

(ii) $\Rightarrow$ (iii). proceeding as above, we obtain

$$\|u_n\|_{C^{1,\beta}(\overline{\Omega})} \le C\|u_n\|_{C^{2,\gamma}(\overline{\Omega})} \le \big(C_1 + C_2\|u_n\|_\infty^{p-2}\big)\|u_n\|_\infty \tag{2.10}$$

where C_1 and C_2 are constants that are independent of $\|u_n\|_\infty$. □

Corollary 2.11. *Let $\{(\lambda_n, u_n)\}_{n=1}^\infty$ be as above with $\|u_n\|_\infty \to 0$. Then there exists subsequence, which we denote again by λ_n and u_n, such that as $n \to \infty$ for some $\alpha \in (0, 1)$,*

$$\lambda_n \to \lambda_1, \quad \frac{u_n}{\|u_n\|_\infty} \to \pm\frac{\varphi_1}{\|\varphi_1\|_\infty} \quad in\ C^{1,\alpha}(\overline{\Omega}) \tag{2.11}$$

where λ_1 is the first eigenvalue of (1.2) and φ_1 is an eigenfunction corresponding λ_1.

Proof. We denote $w_n = u_n/\|u_n\|_\infty$ which satisfies

$$\int_\Omega \nabla w_n\nabla\phi = \lambda_n\Big(\int_\Omega |w_n|^q\Big)^{\frac{2}{q}-1}\int_\Omega |w_n|^{q-2}w_n\phi + \int_\Omega \frac{h(x, w_n\|u_n\|_\infty, \lambda_n)}{\|u_n\|_\infty}\phi \tag{2.12}$$

for all $\phi \in X$. Using the compact embedding $C^{1,\beta} \hookrightarrow C^{1,\alpha}$ with $0 < \alpha < \beta < 1$, the sequence w_n contains a subsequence that converges in $C^{1,\alpha}$ to some w; we denote it again by $w_n \to w$. We let $n \to \infty$ in (2.12) and use Corollary 2.9, to conclude that $w \in C^{1,\alpha}$ must satisfy

$$\int_\Omega \nabla w\nabla\phi\,dx = \lambda^*\Big(\int_\Omega |w|^q\,dx\Big)^{\frac{2}{q}-1}\int_\Omega |w|^{q-2}w\phi\,dx \tag{2.13}$$

where $\lambda^* = \lim_{n\to\infty} \lambda_n$. Since $0 \le \lambda^* \le \lambda_2 - \delta$ and λ_1 is the only eigenvalue of (1.2) in the open interval $(0, \lambda_2)$, we must have $\lambda^* = \lambda_1$. In addition, λ_1 being a simple eigenvalue, we have $w = k\varphi_1$ in Ω where k satisfies

$$|k| \cdot \|\varphi_1\|_\infty = \|w\|_\infty = \|w_n\|_\infty = 1.$$

This gives (2.11). □

Comparable results can be achieved concerning Theorem 2.10 and Corollary 2.11 in the context of (λ_1, ∞).

Theorem 2.12. *Let $\{(\lambda_n, u_n)\}_{n=1}^\infty$ be as above. Then the following three statements are equivalent, as $n \to \infty$,*

- (i) $\|u_n\|_{W_0^{1,2}(\Omega)} \to \infty$,
- (ii) $\|u_n\|_{L^\infty(\Omega)} \to \infty$,
- (iii) $\|u_n\|_{C^{1,\beta}(\overline{\Omega})} \to \infty$.

Corollary 2.13. *Let $\{(\lambda_n, u_n)\}_{n=1}^\infty$ be as above with $\|u_n\|_\infty \to \infty$. Then there exists subsequence, which we denote again by λ_n and u_n, such that as $n \to \infty$ for some $\alpha \in (0,1)$*

$$\lambda_n \to \lambda_1, \quad \frac{u_n}{\|u_n\|_\infty} \to \pm\frac{\varphi_1}{\|\varphi_1\|_\infty} \quad \text{in } C^{1,\alpha}(\overline{\Omega})$$

where λ_1 is the first eigenvalue of (1.2) and φ_1 is an eigenfunction corresponding λ_1.

3. Global bifurcation

It is known that $u \in W_0^{1,2}(\Omega)$ is a solution of problem (1.1) (in the weak sense) if and only if a pair $(\lambda, u) \in \mathbb{R} \times W_0^{1,2}(\Omega)$ that satisfies the integral identity

$$\int_\Omega \nabla u \nabla \phi \,\mathrm{d}x = \lambda\Big(\int_\Omega |u|^q dx\Big)^{\frac{2}{q}-1} \int_\Omega |u|^{q-2} u\phi \,\mathrm{d}x + \int_\Omega h(x,u;\lambda)\phi \,\mathrm{d}x$$

for all $\phi \in W_0^{1,2}(\Omega)$. The last equation is equivalent to the operator equation

$$\Phi(u) = \lambda\Psi(u) + H(\lambda, u)$$

with all terms valued in the dual space $X' = W^{-1,2'}(\Omega)$ of X and the operators $\Phi, \Psi, H(\lambda,\cdot) : X \to X'$ defined as follows, for all $u, \phi \in X$ and $\lambda \in \mathbb{R}$:

$$\langle \Phi(u), \phi\rangle_X = \int_\Omega \nabla u \nabla \phi \,\mathrm{d}x,$$
$$\langle \Psi(u), \phi\rangle_X = \Big(\int_\Omega |u|^q dx\Big)^{\frac{2}{q}-1} \int_\Omega |u|^{q-2} u\phi \,\mathrm{d}x,$$
$$\langle H(\lambda, u), \phi\rangle_X = \int_\Omega h(x,u,\lambda)\phi(x) \,\mathrm{d}x.$$

It is easy to see that the operator $\Phi : X \to X'$ is continuous, coercive, strictly monotone and satisfies condition (S_+). The operator $\Psi : X \to X'$ can be extended to a continuous operator $\widetilde{\Psi} : L^q(\Omega) \to (L^q(\Omega))' = L^{q'}(\Omega)$ in a unique way. Consequently, Ψ decomposed as

$$\Psi : X \hookrightarrow L^q(\Omega) \xrightarrow{\tilde{\Psi}} L^{q'}(\Omega) \hookrightarrow X'$$

is compact by Rellich's theorem. Finally, given $\lambda \in \mathbb{R}$, operator $H(\lambda, \cdot\,) : X \to X'$ is also compact by Rellich's theorem. Thus

$$G_\lambda(u) = G(\lambda, u) \equiv \Phi(u) - \Psi(u) - H(\lambda, u)$$

can be viewed as a compact perturbation of (S_+)-type operator Ψ and is also (S_+)-type. So its Browder-Petryshyn degree can be defined.

Now, we give a result that shows discontinuity for Browder-Petryshyn degree at the first eigenvalue λ_1.

Lemma 3.1. *For all $r > 0$ and all $0 < \delta < \lambda_2 - \lambda_1$ we have*

$$\operatorname{Deg}\big[\Phi - (\lambda_1 \pm \delta)\Psi; B_r(0), 0\big] = \mp 1.$$

Proof. Given $R > 0$ fixed, we define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$\psi(t) = \begin{cases} 0 & \text{for } 0 \le t \le R, \\ \frac{\delta}{R}(t-R)^2 & \text{for } R < t < 2R, \\ 2\delta(t-2R) + \delta R & \text{for } 2R \le t < \infty. \end{cases} \tag{3.1}$$

Clearly, ψ is continuously differentiable, monotone increasing, and convex on $\mathbb{R}^+$ satisfying

$$\psi'(t) = \begin{cases} 0 & \text{for } 0 \le t \le R, \\ \frac{2\delta}{R}(t-R) & \text{for } R < t < 2R, \\ 2\delta & \text{for } 2R \le t < \infty. \end{cases} \tag{3.2}$$

Now we consider the functional $T_\lambda : X \to \mathbb{R}$ defined as

$$T_\lambda(u) = \frac{1}{2}\big\langle \Phi(u), u\big\rangle_X - \frac{\lambda}{2}\big\langle \Psi(u), u\big\rangle_X + \psi\Big(\frac{1}{2}\big\langle \Psi(u), u\big\rangle_X\Big).$$

Every critical point $u_0 \in X$ of T_λ is a solution of the operator equation

$$T'_\lambda(u) = \Phi(u) - \Big[\lambda - \psi'\Big(\frac{1}{2}\langle \Psi(u), u\rangle_X\Big)\Big]\Psi(u) = 0 \quad \text{in} X'\,. \tag{3.3}$$

Next we investigate the above equation. Assuming $\lambda \le \lambda_1 + \delta < \lambda_2$ we have

$$\lambda - 2\delta \le \lambda - \psi'\Big(\frac{1}{2}\langle \Psi(u), u\rangle_X\Big) \le \lambda \le \lambda_1 + \delta < \lambda_2.$$

Therefore, if (3.3) is valid, we have following two cases: (i) $u_0 = 0$, and (ii) $\lambda - \psi'(\frac{1}{2}\langle \Psi(u_0), u_0\rangle_X) = \lambda_1$ and $u_0 = \alpha\varphi_1$ for some constant $\alpha \in \mathbb{R} \setminus \{0\}$.

Here φ_1 is an eigenfunction corresponding to λ_1 satisfies $\|\varphi_1\|_q^2 = 1$, and α, because of the homogeneity, satisfies

$$\lambda - \psi'\Big(\frac{|\alpha|^2}{2}\Big) = \lambda_1.$$

Because α depends on λ, we sometimes write $\alpha = \alpha_\lambda$. Next, we discuss the value of λ in three cases.

Case 1: $\lambda < \lambda_1$. We have

$$\psi'\big(|\alpha|^2/2\big) = \lambda - \lambda_1 < 0.$$

Combining with (3.2), the only zero of T_λ is the $u = 0 \in X$; it is the global minimizer for T_λ. We apply Theorem 5.8 to conclude that

$$\operatorname{Deg}[T_\lambda; B_r(0), 0] = 1 \quad \text{for all } r > 0. \tag{3.4}$$

Case 2: $\lambda = \lambda_1$. In this situation,

$$\psi'(|\alpha|^2/2) = \lambda_1 - \lambda_1 = 0.$$

Combining with (3.2), we have $0 \le \frac{|\alpha_{\lambda_1}|^2}{2} \le R$. Combining with (3.1) again, it is easy to see that

$$T_{\lambda_1}(\alpha_{\lambda_1}\varphi_1) = 0.$$

Case 3: $\lambda_1 < \lambda \le \lambda_1 + \delta$. In this situation,

$$\psi'\big(|\alpha|^2/2\big) \in (0, \delta].$$

Combining this with (3.2), we have $R < |\alpha_\lambda|^2/2 \le \frac{3R}{2}$. It can be known by direct calculation that $T_{\lambda_1+\delta}(\pm\alpha_{\lambda_1+\delta}\varphi_1) < 0 = T_0$. According to (3.3), 0 and $\pm\alpha_{\lambda_1+\delta}\varphi_1$ are the whole set of zeros of $T_{\lambda_1+\delta}$. Next, we proof T_λ is coercive on X. In fact, we fix $\frac{1}{2}\langle \Psi(u), u\rangle_X = 2R$. This is a C^1 manifold and contains origin $u = 0$. Then we have

$$\begin{aligned} T_\lambda(u) &= \frac{1}{2}\langle \Phi(u), u\rangle_X - \frac{\lambda}{2}\langle \Psi(u), u\rangle_X + 2\delta\Big(\frac{1}{2}\langle \Psi(u), u\rangle_X - 2R\Big) + \delta R \\ &= \frac{1}{2}\langle \Phi(u), u\rangle_X - \frac{\lambda - 2\delta}{2}\langle \Psi(u), u\rangle_X - 3\delta R \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{2}\langle \Phi(u), u\rangle_X - \frac{\lambda_1-\delta}{2}\langle \Phi(u), u\rangle_X - 3\delta R\\
&\geq \frac{1}{2}\Big(1-\frac{\lambda_1-\delta}{\lambda_1}\Big)\langle \Phi(u), u\rangle_X - 3\delta R\\
&= \delta\Big(\frac{1}{2\lambda_1}\langle \Phi(u), u\rangle_X - 3R\Big).
\end{aligned}$$

Let $\|u\|_X \to \infty$, we obtain $T_\lambda(u) \to \infty$. Thus, $\pm\alpha_{\lambda_1+\delta}\varphi_1$ are the global minimizers for $T_{\lambda_1+\delta}$. By Theorem 5.8 again, we obtain

$$\operatorname{Deg}\big[T'_{\lambda_1+\delta}; B_\sigma(\pm\alpha_{\lambda_1+\delta}\varphi_1), 0\big] = 1 \quad \text{for every } \sigma > 0 \text{ small enough.} \tag{3.5}$$

Without loss of generality, we assume $\sigma < \frac{1}{2}\alpha_{\lambda_1+\delta}\|\varphi_1\|_X$. Set $r_\delta = 2\alpha_{\lambda_1+\delta}\|\varphi_1\|_X$, then $r_\delta > \alpha_{\lambda_1+\delta}\|\varphi_1\|_X + \sigma$. Let

$$T_t(u) = tT'_{\lambda_1-\delta}(u) + (1-t)T'_{\lambda_1+\delta}(u) \quad t\in[0,1] \text{ and } \|u\|_X > r_\delta.$$

Noticing that $T_t(u) \neq 0$ holds for all $u \in \partial B_{r_\delta}(0)$ and $t \in [0,1]$ due to choice of σ and r_δ, by the homotopy invariance of degree, we obtain

$$\operatorname{Deg}\big[T'_{\lambda_1+\delta}; B_r(0), 0\big] = \operatorname{Deg}\big[T'_{\lambda_1-\delta}; B_r(0), 0\big] = 1 \quad \text{for every} \quad r > r_\delta. \tag{3.6}$$

On the other hand, 0 is also an isolated zero of T'_λ with $\lambda = \lambda_1 + \delta$ and

$$\operatorname{Deg}\big[T'_{\lambda_1+\delta}; B_{\sigma'}(0), 0\big]$$

is well defined for every $\sigma' > 0$ small enough. Without loss of generality, we still assume $\sigma' < \frac{1}{2}\alpha_{\lambda_1+\delta}\|\varphi_1\|_X$.

From (3.4), (3.6) and additivity property of the degree we deduce for all $r > r_\delta$,

$$\begin{aligned}
&\operatorname{Deg}\big[T'_{\lambda_1+\delta}; B_\sigma(\alpha_{\lambda_1+\delta}\varphi_1), 0\big] + \operatorname{Deg}\big[T'_{\lambda_1+\delta}; B_\sigma(-\alpha_{\lambda_1+\delta}\varphi_1), 0\big] + \operatorname{Deg}\big[T'_{\lambda_1+\delta}; B_{\sigma'}(0), 0\big]\\
&= \operatorname{Deg}\big[T'_{\lambda_1+\delta}; B_r(0), 0\big]\\
&= \operatorname{Deg}\big[T'_{\lambda_1-\delta}; B_r(0), 0\big] = 1.
\end{aligned}$$

From (3.5), we obtain

$$\operatorname{Deg}\big[T'_{\lambda_1+\delta}; B_{\sigma'}(0), 0\big] = -1.$$

To complete the proof, we only need to prove that there exists σ' small enough such that $\frac{1}{2}\langle \Psi(u), u\rangle_X < R$ for $\|u\|_X < \sigma'$. This is because, if that is correct, from (3.2) we have

$$\Phi(u) - (\lambda_1+\delta)\Psi(u) = \Phi(u) - \Big[(\lambda_1+\delta) + \psi'\Big(\frac{1}{2}\langle \Psi(u), u\rangle\Big)\Big]\Psi(u) = T'_{\lambda_1+\delta}(u).$$

Then

$$\operatorname{Deg}[\Phi - (\lambda_1+\delta)\Psi; B_{\sigma'}(0), 0] = \operatorname{Deg}\big[T'_{\lambda_1+\delta}; B_{\sigma'}(0), 0\big] = -1.$$

Indeed that is true. Because, from the definition of λ_1, we have

$$\lambda_1\langle \Psi(u), u\rangle_X \leq \langle \Phi(u), u\rangle = \|u\|_X.$$

Since $u = 0 \in X$ is the only solution to the operator equation $\Phi(u) = (\lambda_1+\delta)\Psi(u)$. We obtain

$$\operatorname{Deg}[\Phi - (\lambda_1+\delta)\Psi; B_r(0), 0] = -1 \quad \text{for every } r > 0.$$

By analogous arguments, we infer that

$$\operatorname{Deg}[\Phi - (\lambda_1-\delta)\Psi; B_r(0), 0] = 1 \quad \text{for every } r > 0.$$

□

Proof of Theorem 1.1. According to [6], if $(\lambda, 0)$ is a bifurcation point, then λ is an eigenvalue for the nonlinear eigenvalue problem $\Phi(u) - \lambda\Psi(u) = 0$. So for each $\lambda \in (0,\lambda_2)\backslash\{\lambda_1\}$, $u = 0 \in X$ is an isolated solution of $G_\lambda(u) = 0$. Thus one can find $R > 0$ small enough, such that $\operatorname{Deg}[G_{\lambda_1\pm\delta}; B_r(0), 0]$ remains constant with respect to $r \in (0, R)$.

Now, we assert that there exists $R' \in (0, R)$ such that

$$\Phi(u) - (\lambda_1\pm\delta)\Psi(u) - tH(\lambda_1\pm\delta, u) \neq 0 \tag{3.7}$$

for all $u \in \partial B_{R'}(0)$ and $t \in [0,1]$. If not, for all $R' \in (0,R)$ there exist $u \in \partial B_{R'}(0)$ and $t \in [0,1]$ such that

$$\Phi(u) - (\lambda_1 \pm \delta)\Psi(u) - tH(\lambda_1 \pm \delta, u) = 0.$$

Thus, we can find a sequence $\{r_n\}_{n=1}^{\infty} \subset (0,R)$, $r_n \to 0$, with $\{u_n\}_{n=1}^{\infty} \subset X$, $u_n \in \partial B_{r_n}(0)$, and $\{t_n\}_{n=1}^{\infty} \subset [0,1]$, such that above equation holds with u_n and t_n in place of u and t, respectively. So, for all $\phi \in C_c^{\infty}(\Omega)$, we have

$$\int_\Omega \nabla u_n \nabla \phi = (\lambda_1 \pm \delta)\Big(\int_\Omega |u_n|^q\Big)^{\frac{2}{q}-1} \int_\Omega |u_n|^{q-2} u_n \,\phi + t_n \int_\Omega h(x,u_n,\lambda)\phi. \tag{3.8}$$

Since $u_n \in \partial B_{r_n}(0)$, we have $\|u_n\|_X \to 0$. Dividing both sides of the equation by $\|u_n\|_X$ and denoting $u_n/\|u_n\|_X$ by w_n the sequence $\{w_n\}$ is bounded in X. This means that there exists a $w \in X$ and subsequence that we call again w_n, such that

- $w_n \rightharpoonup w$ in X;
- $w_n \to w$ in $L^p(\Omega)$, $\forall\, p \in [1, 2^*)$;
- $w_n(x) \to w(x)$ a.e. in Ω;
- there exists $v \in L^p(\Omega)$ such that $|w_n(x)| \le v(x)$ a.e. in Ω and for all n.

Then we obtain

$$\int_\Omega \nabla w_n \nabla \phi = (\lambda_1 \pm \delta)\Big(\int_\Omega |w_n|^q\Big)^{\frac{2}{q}-1} \int_\Omega |w_n|^{q-2} w_n \phi + t_n \int_\Omega \frac{h(x,u_n,\lambda)}{\|u_n\|_X}\phi. \tag{3.9}$$

Next, it is sufficient to show that

$$\lim_{n\to\infty} \int_\Omega \frac{h(x,u_n,\lambda)}{\|u_n\|_X}\phi \,\mathrm{d}x = 0. \tag{3.10}$$

Since combining with weak convergence, Hölder inequality and Lebesgue dominated convergence theorem, we can obtain that

$$\int_\Omega \nabla w_n \nabla \phi \,\mathrm{d}x \to \int_\Omega \nabla w \nabla \phi \,\mathrm{d}x,$$

$$\Big(\int_\Omega |w_n|^q dx\Big)^{\frac{2}{q}-1} \int_\Omega |w_n|^{q-2} w_n \phi \,\mathrm{d}x \to \Big(\int_\Omega |w|^q dx\Big)^{\frac{2}{q}-1} \int_\Omega |w|^{q-2} w \phi \,\mathrm{d}x.$$

This means that there exists $0 \ne w \in X$ satisfying

$$\int_\Omega \nabla w \nabla \phi \,\mathrm{d}x = (\lambda_1 \pm \delta)\Big(\int_\Omega |w|^q dx\Big)^{\frac{2}{q}-1} \int_\Omega |w|^{q-2} w \phi \,\mathrm{d}x.$$

This contradicts that $\lambda_1 \pm \delta$ are not eigenvalues. Now we prove (3.10).

From Lemma 2.9 and Theorem 2.10, we have

$$\begin{aligned}\int_\Omega \frac{h(x,u_n,\lambda)}{\|u_n\|_X}\phi \,\mathrm{d}x &= \int_\Omega \frac{h(x,u_n,\lambda)}{\|u_n\|_\infty}\frac{\|u_n\|_\infty}{\|u_n\|_X}\phi \,\mathrm{d}x \\ &\le \Big(C_1 + C_2\|u_n\|_X^{p-2}\Big)\int_\Omega \frac{h(x,u_n,\lambda)}{\|u_n\|_\infty}\phi \,\mathrm{d}x \to 0.\end{aligned}$$

Therefore, the degree of the homotopy operator

$$\Phi(u) - (\lambda_1 \pm \delta)\Psi(u) - tH(\lambda_1 \pm \delta, u), \quad t \in [0,1],$$

is well defined. This operator connects $G_{\lambda_1 \pm \delta}$ with $\Phi - (\lambda_1 \pm \delta)\Psi$. Consequently, by Lemma 3.1, we have

$$\mathrm{Deg}[G_{\lambda_1\pm\delta}; B_r(0), 0] = \mathrm{Deg}[G_{\lambda_1\pm\delta}; B_{R'}(0), 0] = \mathrm{Deg}[\Phi - (\lambda_1 \pm \delta)\Psi; B_{R'}(0), 0] = \mp 1.$$

Next, we can proceed step by step as in the original proof of Rabinowitz [9]. This concludes the proof. □ □

By Corollary 2.11 and Corollary 2.13, we have the following result.

Corollary 3.2. *Let $\mathscr{C}$ is a component of the set of nontrivial solutions of* (1.1) *in $\mathbb{R} \times X$ in Theorem 1.1 and $\mathscr{D}$ is a component of the set of nontrivial solutions of* (1.1) *in $\mathbb{R} \times X$ in Theorem 1.2.*

(i) *If* $(\mu_n, u_n) \in \mathscr{C}$ *and is near* $(\lambda_1, 0)$, *then* $u_n = s_n\varphi_1 + w_n$, *where* $w_n = o(|s_n|)$ *as* $|s_n| \to 0$ *and* $\int_\Omega w_n\varphi_1 dx = 0$.

(ii) *If* $(\mu_n, u_n) \in \mathscr{D}$ *and is near* (λ_1, ∞), *then* $u_n = s_n\varphi_1 + w_n$, *where* $w_n = o(|s_n|)$ *as* $|s_n| \to \infty$ *and* $\int_\Omega w_n\varphi_1 dx = 0$.

4. Positive solutions

In this Section, we consider the problem

$$\begin{gathered} -\Delta u = \lambda\Big(\int_\Omega |u|^q dx\Big)^{\frac{2}{q}-1} f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.1}$$

Using (H5) we define $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$f(s) = f_0|s|^{q-2}s + \xi(s)$$

with

$$\lim_{s\to 0^+} \frac{\xi(s)}{|s|^{q-2}s} = 0.$$

Then (4.1) is transformed into

$$\begin{gathered} -\Delta u = \lambda f_0\Big(\int_\Omega |u|^q dx\Big)^{\frac{2}{q}-1}|u|^{q-2}u + \lambda\Big(\int_\Omega |u|^q dx\Big)^{\frac{2}{q}-1}\xi(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{4.2}$$

as a bifurcation problem from the trivial solution axis. We intend to utilize Theorem 1.1 in the context of Problem (4.2). However, we observe that Equations (1.1) and (4.2) are not entirely identical; they share a similar structure only in certain aspects. To be precise, it is observed that the second term on the right-hand side of Equation (4.2) includes a nonlocal term, $(\int_\Omega |u|^q dx)^{\frac{2}{q}-1}$, whereas the second term on the right-hand side of Equation (1.1) does not incorporate this term. Consequently, it is necessary to formulate a theorem analogous to Theorem 1.1 for Equation (1.1).

To ensure comprehensiveness, we generalize (4.2) into the following equation.

$$\begin{gathered} -\Delta u = \lambda\Big(\int_\Omega |u|^q dx\Big)^{\frac{2}{q}-1}|u|^{q-2}u + \Big(\int_\Omega |u|^q dx\Big)^{\frac{2}{q}-1} g(x,u,\lambda) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{4.3}$$

where $g : \Omega \times \mathbb{R} \times \mathbb{R}$ satisfies the Carathéodory condition in the first two variable and is locally Hölder continuous about the second variable. There exists a constant $C \in (0, \infty)$ and $p \in (2, 2^*)$ such that

$$|g(x,u,\lambda)| \le C|u|^{p-1}$$

for a.e. $x \in \Omega$ and all $(u, \lambda) \in \mathbb{R} \times \mathbb{R}$. Furthermore, we assume that g satisfies the following hypotheses:

(H7) (for bifurcations from zero)

$$\lim_{s\to 0^+} \frac{g(x,s,\lambda)}{s^{q-1}} = 0$$

uniformly for almost every $x \in \Omega$ and λ on bounded sets.

(H8) (for bifurcations from infinity)

$$\lim_{s\to +\infty} \frac{g(x,s,\lambda)}{s^{q-1}} = 0$$

uniformly for almost every $x \in \Omega$ and λ on bounded sets.

We aim to provide a proof of the subsequent theorem.

Theorem 4.1. *Let* $q \in [1, 2]$ *and* g *satisfies* (H7). *Then the pair* $(\lambda_1, 0)$ *is a bifurcation point of* (4.3). *Moreover, there is a component* $\mathscr{C}$ *of the set of nontrivial solutions of* (4.3) *in* $\mathbb{R} \times X$ *whose closure contains* $(\lambda_1, 0)$ *and it is either unbounded or contains a pair* $(\overline{\lambda}, 0)$ *for some* $\overline{\lambda}$, *an eigenvalue of* (1.2) *with* $\overline{\lambda} \neq \lambda_1$.

Upon examining the proof of Theorem 1.1, it becomes evident that the critical component lies in the demonstration of (3.10). However the validity of (3.10) is contingent upon Lemma 2.8 and Theorem 2.10. Consequently, it is imperative to establish analogous conclusions for the function $\|u\|_q^{2-q}g(x,u,\lambda)$. We shall now continue with this verification. Initially, it is evident that the analogue of Lemma 2.8 can be established through a proof that is analogous to that of Lemma 2.8. This constitutes the subsequent lemma.

Lemma 4.2. *Let $u \in L^\infty(\Omega)$, $u \not\equiv 0$ in Ω.*

(i) *If hypothesis* (H7) *is satisfied, then*

$$\frac{g(x,u,\lambda)}{\|u\|_{L^\infty(\Omega)}^{q-1}} \to 0 \quad \text{as } \|u\|_{L^\infty(\Omega)} \to 0$$

holds for a.e. $x \in \Omega$ and uniformly for every $\lambda \in \mathbb{R}$.

(ii) *If hypothesis* (H8) *is satisfied, then*

$$\frac{g(x,u,\lambda)}{\|u\|_{L^\infty(\Omega)}^{q-1}} \to 0 \quad \text{as} \quad \|u\|_{L^\infty(\Omega)} \to \infty$$

holds for a.e. $x \in \Omega$ and uniformly for every $\lambda \in \mathbb{R}$.

Next, we investigate Theorem 2.10, the equivalence of norms. An analysis of the proof of Theorem 2.10 reveals that a pivotal aspect is the establishment of (2.2) and Step 2. When h is replaced by $\|u\|_q^{2-q}g(x,u,\lambda)$, the resulting expression is as follows

$$\begin{aligned} f(x,w_n) &= \lambda_n \|w_n\|_q^{2-q}|w_n|^{q-2}w_n + \frac{\|u_n\|_q^{2-q}g(x,w_n\|u_n\|_X,\lambda_n)}{\|u_n\|_X} \\ &\le C|w_n|^{q-1} + \frac{\|u_n\|_q^{2-q}}{\|u_n\|_X^{2-q}} \cdot \frac{g(x,w_n\|u_n\|_X,\lambda_n)}{\|u_n\|_X^{q-1}} \\ &\le C|w_n|^{q-1} + C^{2-q}\frac{C|w_n|^{p-1}\|u_n\|_X^{p-1}}{\|u_n\|_X^{q-1}} \\ &= C_1|w_n|^{q-1} + C_2\|u_n\|_X^{p-q}|w_n|^{p-1} \end{aligned}$$

where C_1 and C_2 are constants that are independent of $\|u_n\|_X$. Considering that $p-q>0$, it can be inferred that Theorem 2.10 remains valid when h is substituted with $\|u\|_q^{2-q}g(x,u,\lambda)$. This constitutes the subsequent theorem.

Theorem 4.3. *Let $\{(\lambda_n,u_n)\}_{n=1}^\infty$ be solutions of* (4.3)*. Then the following three stateents are equivalent, as $n \to \infty$:*

(i) $\|u_n\|_{W_0^{1,2}(\Omega)} \to 0$,
(ii) $\|u_n\|_{L^\infty(\Omega)} \to 0$,
(iii) $\|u_n\|_{C^{1,\beta}(\overline{\Omega})} \to 0$.

Similar to Theorem 1.1, we have Theorem 4.1. Moreover, similar to Theorem 1.2, we can draw conclusions regarding bifurcation from infinity.

Theorem 4.4. *Let $q \in [1,2]$ and g satisfy* (H8)*. Then the pair (λ_1,∞) is a bifurcation point of* (4.3)*. Moreover, there is a component $\mathscr{D}$ of the set of nontrivial solutions of* (4.3) *in $\mathbb{R}\times X$ which meets (λ_1,∞). If $\Lambda \subset \mathbb{R}$ is an interval such that $\Lambda \cap r(L) = \{\lambda_1\}$ where $r(L)$ is the set of real eigenvalue values of* (1.2) *and $\mathscr{M}$ is a neighborhood of (λ_1,∞) whose projection on $\mathbb{R}$ lies in Λ and whose projection on X is bounded away from 0, then either*

(i) *$\mathscr{D}-\mathscr{M}$ is bounded in $\mathbb{R}\times X$ in which case $\mathscr{D}-\mathscr{M}$ meets $\mathscr{R}=\{(\lambda,0) \mid \lambda\in\mathbb{R}\}$, or*
(ii) *$\mathscr{D}-\mathscr{M}$ is unbounded.*

If (ii) occurs and $\mathscr{D}-\mathscr{M}$ has a bounded projection on $\mathbb{R}$, then $\mathscr{D}-\mathscr{M}$ meets $(\widehat{\lambda},\infty)$ where $\lambda_1 \neq \widehat{\lambda} \in r(L)$.

Similar to Corollary 3.2, we have the following statement.

Corollary 4.5. *Let $\mathscr{C}$ be a component of the set of nontrivial solutions of* (4.3) *in $\mathbb{R}\times X$ in Theorem 4.1 and $\mathscr{D}$ is a component of the set of nontrivial solutions of* (4.3) *in $\mathbb{R}\times X$ in Theorem 4.4.*

(i) *If $(\mu_n, u_n)\in\mathscr{C}$ and is near $(\lambda_1, 0)$, then $u_n = s_n\varphi_1 + w_n$, where $w_n = o(|s_n|)$ as $|s_n|\to 0$ and $\int_\Omega w_n\varphi_1 dx = 0$.*

(ii) *If $(\mu_n, u_n)\in\mathscr{D}$ and is near (λ_1, ∞), then $u_n = s_n\varphi_1 + w_n$, where $w_n = o(|s_n|)$ as $|s_n|\to\infty$ and $\int_\Omega w_n\varphi_1 dx = 0$.*

We will now proceed with the examination of (4.1). Let

$$E = \{u\in C^1(\overline{\Omega}) : u = 0 \quad \text{on } \partial\Omega\}$$

with the usual norm

$$\|u\|_{C^1} = \max_{\overline{\Omega}}|u| + \max_{\overline{\Omega}}|\nabla u|.$$

Set

$$P^+ = \big\{u\in E : u > 0 \quad \text{in } \Omega \text{ and } \frac{\partial u}{\partial\omega} < 0 \text{ on } \partial\Omega\big\}$$

where ω is the outward pointing normal vector to $\partial\Omega$.

Lemma 4.6. *Assume* (H4) *and* (H5) *hold. Then $(\lambda_1/f_0, 0)$ is a bifurcation point of* (4.1) *and the associated bifurcation branch $\mathscr{C}\subset\mathbb{R}\times E$ whose closure contains $(\lambda_1/f_0, 0)$ is either unbounded or contains a pair $(\overline{\lambda}/f_0, 0)$ where $\overline{\lambda}$ is an eigenvalue of* (1.2) *and $\overline{\lambda}\neq\lambda_1$.*

The above lemma is an application of Theorem 4.1.

Lemma 4.7. *Assume* (H4) *and* (H5) *hold. Then $\mathscr{C}\subseteq((\mathbb{R}\times P^+)\cup(\lambda_1/f_0, 0))$ and the last alternative in Lemma 4.6 is impossible.*

Proof. By the strong maximum principle [8], any nontrivial solution (λ, u) belongs to $\mathbb{R}\times P^+$. So we have $\mathscr{C}\subseteq((\mathbb{R}\times P^+)\cup(\mathbb{R}\times\{0\}))$. Suppose on the contrary that there exists $(\lambda_n, u_n)\to(\overline{\lambda}/f_0, 0)$ with $(\lambda_n, u_n)\in\mathscr{C}$, $u_n\neq 0$ and $\overline{\lambda}\neq\lambda_1$. Let $v_n = u_n/\|u_n\|_\infty$, then (λ_n, v_n) satisfies

$$\int_\Omega \nabla v_n\nabla\phi = \lambda_n f_0\Big(\int_\Omega |v_n|^q\Big)^{\frac{2}{q}-1}\int_\Omega |v_n|^{q-2}v_n\phi + \lambda_n\frac{\|u_n\|_q^{2-q}}{\|u_n\|_\infty^{2-q}}\int_\Omega\frac{\xi(u_n)}{\|u_n\|_\infty^{q-1}}\phi$$

for all $\phi\in X$. We obtain that there exists a subsequence $v_m\to v$ as $m\to+\infty$. Now v verifies the equation

$$\int_\Omega \nabla v\nabla\phi\,dx = \bar{\lambda}\Big(\int_\Omega |v|^q dx\Big)^{\frac{2}{q}-1}\int_\Omega |v|^{q-2}v\phi\,dx.$$

Hence v must change sign, and this is a contradiction. Furthermore, it follows that $\mathscr{C}\subseteq(\mathbb{R}\times P^+)\cup(\lambda_1/f_0, 0)$ and $\mathscr{C}$ is unbounded in $\mathbb{R}\times E$. $\square$

Using (H6), we define $\eta:\mathbb{R}^+\to\mathbb{R}^+$ as

$$f(s) = f_\infty|s|^{q-2}s + \eta(s)$$

with

$$\lim_{s\to+\infty}\frac{\eta(s)}{|s|^{q-2}s} = 0.$$

Then (4.1) is transformed into

$$\begin{gathered}-\Delta u = \lambda f_\infty\Big(\int_\Omega |u|^q dx\Big)^{\frac{2}{q}-1}|u|^{q-2}u + \lambda\Big(\int_\Omega |u|^q dx\Big)^{\frac{2}{q}-1}\eta(u) \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{4.4}$$

as a bifurcation problem from infinity. Similar to Lemma 4.6, 4.7, we have the following statement.

Lemma 4.8. *Assume* (H4) *and* (H6) *hold. Then $(\lambda_1/f_\infty, \infty)$ is a bifurcation point of* (4.1). *Moreover, there exists a continuum $\mathscr{D}\subset((\mathbb{R}\times P^+)\cup(\lambda_1/f_\infty, \infty))$ of solutions of problem* (4.1) *meeting $(\lambda_1/f_\infty, \infty)$ and satisfying at least one of the alternatives of Theorem 4.4.*

Proof of Theorem 1.3. Firstly, we define

$$\widetilde{f}(s)=\begin{cases} f(s) & \text{if } 0\leqslant s\leqslant \tau,\\ 0 & \text{otherwise}\end{cases}$$

and consider the problem

$$\begin{gathered} -\Delta u=\lambda\Big(\int_\Omega |u|^q dx\Big)^{\frac{2}{q}-1}\widetilde{f}(u) \quad \text{in } \Omega,\\ u=0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.5}$$

Utilizing Lemma 4.7, there exists a continuum $\mathscr{C}$ of nontrivial solutions of problem 4.5 emanating from $(\lambda_1/f_0,0)$ such that $\mathscr{C}\subset((\mathbb{R}\times P^+)\cup(\lambda_1/f_0,0))$, meets ∞ in $\mathbb{R}\times E$. We claim that $u\leqslant\tau$ for any $(\lambda,u)\in\mathscr{C}$. Suppose, by contradiction, that there exists $x\in\Omega$ so that $u(x)>\tau$. Since $u\in C^1(\overline{\Omega})$, we can find $\Omega_1\subset\Omega$ so that $u(x)>\tau$ in Ω_1 and $u(x)=\tau$ on $\partial\Omega_1$. This leads to the equation

$$\begin{gathered} -\Delta u=0 \quad \text{in } \Omega_1,\\ u=\tau \quad \text{on } \partial\Omega_1. \end{gathered}$$

This leads to a contradiction, as it follows that $u(x)=\tau$ in Ω_1 by maximum principle. This substantiates the assertion and consequently indicates that u is also a solution of (4.1) for any $(\lambda,u)\in\mathscr{C}$. Subsequently, we will demonstrate that the projection of $\mathscr{C}$ on $\mathbb{R}$ is unbounded. It is adequate to demonstrate that the set $\{(\lambda,u)\in\mathscr{C}:\lambda\in(0,d]\}$ is bounded for any fixed $d\in(0,+\infty)$. Arguing by contradiction, if there exists $(\lambda_n,u_n)\in\mathscr{C}$, such that $\lambda_n\to\lambda'\leqslant d, u_n\to+\infty$ as $n\to+\infty$. Let $w_n=u_n/\|u_n\|_{C^1}$. Then we have that

$$w_n=Q\Big(\lambda_n\frac{\|u_n\|_q^{2-q}}{\|u_n\|_{C^1}^{2-q}}\frac{\widetilde{f}(u_n)}{\|u_n\|_{C^1}^{q-1}}\Big).$$

where $Q=(-\Delta)^{-1}$. Clearly, we have that

$$\widetilde{f}(u_n)\leqslant\max_{[0,\tau]}|f(s)|.$$

It means that

$$\lambda_n\frac{\|u_n\|_q^{2-q}}{\|u_n\|_{C^1}^{2-q}}\frac{\widetilde{f}(u_n)}{\|u_n\|_{C^1}^{q-1}}\to 0 \quad \text{as } n\to+\infty.$$

By the compactness of Q, we obtain that for some subsequence $w_n\to 0$ as $n\to+\infty$. This result stands in contradiction to the assertion that $\|w_n\|_{C^1}=1$. This, in conjunction with the observation that $\mathscr{C}$ joins $(\lambda_1/f_0,0)$ to infinity, indicates that

$$(\lambda_1/f_0,+\infty)\subseteq\operatorname{Proj}(\mathscr{C})$$

where $\operatorname{Proj}(\mathscr{C})$ denotes the projection of $\mathscr{C}$ on $\mathbb{R}$.

Applying Lemma 4.8, there exists a continuum $\mathscr{D}\subset(\mathbb{R}\times P^+)\cup(\lambda_1/f_\infty,\infty)$ of solutions of problem (4.1) meeting $(\lambda_1/f_\infty,\infty)$ and satisfying at least one of the alternatives of Theorem 4.4. In addition, it is relatively straightforward to confirm that $(\lambda_1/f_\infty,\infty)$ is the unique bifurcation point of positive solutions of (4.1) from ∞. We will demonstrate that these two components, $\mathscr{C}$ and $\mathscr{D}$, are disjoint. Let

$$F=\{u\in C(\overline{\Omega}):u=0 \text{ on } \partial\Omega\}$$

with the usual norm

$$\|u\|_\infty=\max_{\overline{\Omega}}|u|.$$

It is sufficient to show that $\mathscr{C}$ and $\mathscr{D}$ are disjoint in $\mathbb{R}\times F$. We first claim that $\mathscr{D}$ is unbounded in the direction of F. And this only requires proving that $(\lambda_1/f_\infty,0)$ is a blow-up point of $\mathscr{D}$ in $\mathbb{R}\times F$. Otherwise, there exists $M>0$ such that $\|u_n\|_\infty\leqslant M$ for any $(\lambda_n,u_n)\in\mathscr{D}$ with $\lambda_n\to\lambda_1/f_\infty$ as $n\to+\infty$. Applying [8, Theorem 8.33 of], we obtain that $\|u_n\|_{C^1}\leqslant M'$ for some positive constant M', which contradicts the fact of $\mathscr{D}$ meeting $(\lambda_1/f_\infty,\infty)$. Let us assume, for the sake of contradiction, that $\mathscr{C}\cap\mathscr{D}\neq\emptyset$ in $\mathbb{R}\times F$. Since $\mathscr{D}$ is unbounded in the direction of F, there exists $(\lambda^*, u^*) \in (\mathscr{C} \cap \mathscr{D})$ such that $\max_{\overline{\Omega}} u^* = \tau$. Due to (H4), there exists $0 < \kappa < +\infty$ such that $f(s) \leqslant \kappa(\tau - s)$ for any $s \in [0, \tau]$. So we have

$$-\Delta(\tau - u^*) + \lambda^* \kappa \|u^*\|_q^{2-q}(\tau - u^*) \geq 0 \quad \text{in } \Omega,$$
$$\tau - u^* > 0 \quad \text{on } \partial\Omega.$$

The strong maximum principle of [8] implies that $\tau > u$ in Ω. This statement presents a paradox.

Thus $\mathscr{D} - \mathscr{M}$ is unbounded, where $\mathscr{M}$ is a neighborhood of $(\lambda_1/f_\infty, \infty)$ whose projection on $\mathbb{R}$ contains λ_1/f_∞ and whose projection on E is bounded away from 0. We claim that $\mathscr{D} - \mathscr{M}$ has an unbounded projection on $\mathbb{R}$. This only needs to examine that the case of $\mathscr{D} - \mathscr{M}$ meeting $(\lambda_j/f_\infty, \infty)$ for some $j > 1$ is not feasible, where λ_j denotes the eigenvalue of jth of eigenvalue problem (1.2). If not, we assume that $\mathscr{D} - \mathscr{M}$ meets $(\lambda_j/f_\infty, \infty)$ for some $j > 1$. So there exists a neighborhood $\widetilde{\mathscr{N}} \subset \widetilde{\mathscr{M}}$ of $(\lambda_j/f_\infty, \infty)$ such that u must change sign for any $(\lambda, u) \in ((\mathscr{D} - \mathscr{M}) \cap (\widetilde{\mathscr{N}} \setminus (\lambda_j/f_\infty, \infty)))$, where $\widetilde{\mathscr{M}}$ is a neighborhood of $(\lambda_j/f_\infty, \infty)$ which satisfies the assumptions of Lemma 4.8. This contradicts that $\mathscr{D} \subset \big((\mathbb{R} \times P^+) \cup (\lambda_1/f_\infty, +\infty)\big)$. The anticipated conclusions are now evident, see Figure 1. □

5. Appendix: Browder-Petryshyn degree

We need some theories on the Browder-Petryshyn degree. We list some of its definitions and properties. For detailed information, we refer readers to [11, 12]. Now let X be a Banach space and D is a subset of X. We consider an operator A, in general nonlinear, defined on a subset of X, with values in X'.

Definition 5.1 (condition $(S)_+$ or condition $\alpha(D)$)**.** Operator A belongs to the class $(S)_+$ if for any sequence $u_n \in D$, $u_n \rightharpoonup u_0$ and $\lim_{n\to\infty} \langle Au_n, u_n - u_0 \rangle \leqslant 0$ imply $u_n \to u_0$.

Definition 5.2. The operator A is said to be demicontinuous on D, if for any sequence $u_n \in D$ strongly converging to $u_0 \in D$, we have the equality

$$\lim_{n\to\infty} \langle Au_n, v \rangle = \langle Au_0, v \rangle \quad \text{for all } v \in X.$$

Definition 5.3. For $F \subset \overline{D}$ we denote by $A(D, F)$ the set of all bounded demicontinuous mappings $A : \overline{D} \mapsto X'$ satisfying condition condition $\alpha(F)$. When $F = \overline{D}$, we write $A(D)$ instead of $A(D, \overline{D})$.

We define $\operatorname{Deg}(A, \overline{D}, 0)$ –the degree of a mapping A on the set $\overline{D}$ with respect to the origin of the space X'– under the conditions:

(a) $A \in A(D)$,
(b) $Au \neq 0$ for any element $u \in \partial D$.

Let $\{v_i\}$, $i = 1, 2, \dots$, be any complete system of the space X and suppose that for every n the elements $v_1, \dots, v_n$ are linear independent. Denote by F_n the linear hull of the elements $v_1, \dots, v_n$. Define for every $n = 1, 2, \dots$ the finite-dimensional approximation A_n of the mapping A in the following way:

$$A_n u = \sum_{i=1}^{n} \langle Au, v_i \rangle v_i \text{ for } u \in \overline{D}_n, \quad D_n = D \cap F_n. \tag{5.1}$$

Lemma 5.4. *Let A be an operator satisfying conditions (a), (b). Then there exists N such that for $n \geqslant N$ the following assertions hold:*

1. *the equation $A_n u = 0$ has no solutions belonging to ∂D_n;*
2. *Leray-Schauder degree $\deg(A_n, \overline{D}_n, 0)$ of the mapping A_n on the set D_n with respect to $0 \in F_n$ is defined and independent of n.*

By Theorem 5.4, $\lim_{n\to\infty} \deg(A_n, \overline{D}_n, 0)$ exists and we denote it by $D\{v_i\}$.

Lemma 5.5. *Suppose that the conditions (a), (b) are satisfied. Then the limit*

$$D\{v_i\} = \lim_{n\to\infty} \deg(A_n, \overline{D}_n, 0)$$

does not depend on the the choice of the sequence $\{v_i\}$.

The 5.4 and 5.5 justify the following definition.

Definition 5.6 (Browder-Petryshyn degree)**.** For an operator A satisfying conditions (a) and (b), by its degree on the set $\overline{D}$ with respect to the point $0 \in X'$ we mean the number

$$\lim_{n\to\infty} \deg(A_n, \overline{D}_n, 0),$$

where A_n, D_n are determined in accordance with (5.1). This degree is denoted by $\operatorname{Deg}(A, \overline{D}, 0)$.

The degree of a mapping, introduced above, possesses all the natural properties of the degree of finite-dimensional mappings.

Definition 5.7 (index of isolated zero point)**.** The number

$$\lim_{r\to 0} \operatorname{Deg}(A, \overline{B}_r(u_0), 0)$$

is called the index of the mapping A at the isolated zero point u_0 and is denoted by $\operatorname{Ind}(A, u_0)$.

Lemma 5.8. *Suppose that a mapping A of class $A(D)$ has only isolated zero points in $\overline{D}$ and $Au \neq 0$ for $u \in \partial D$. Then there exists only a finite number of zero points and the equality*

$$\operatorname{Deg}(A, \overline{D}, 0) = \sum_{i=1}^{I} \operatorname{Ind}(A, u_i),$$

holds, where u_i, $i = 1, \dots, I$, are all zero points of the mapping A in D.

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Qingbo Liu

School of Mathematical Sciences, Dalian University of Technology, Dalian, 116024, China

Email address: liuqingbo@mail.dlut.edu.cn

Lan Zhao (corresponding author)

School of Mathematical Sciences, Dalian University of Technology, Dalian, 116024, China

Email address: zhaolan@mail.dlut.edu.cn