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STOCHASTIC BURGERS EQUATIONS WITH FRACTIONAL DERIVATIVE DRIVEN BY FRACTIONAL NOISE

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Abstract. In this article, we study fractional stochastic Burgers equations perturbed by fractional noise. Existence and uniqueness of a mild solution is given by a fixed point argument. Then, we explore Hölder regularity of the mild solution in $C([0,T_*];L^p(\Omega;\dot{H}^\gamma))$ for some stopping time T_*.

1. Introduction

In this article, we study the fractional stochastic Burgers equation with fractional noise

$$\begin{gathered} D_t^\beta u(t,x) - u(t,x)\frac{\partial u(t,x)}{\partial x} + (-\Delta)^{\alpha/2}u(t,x) = \dot{W}^H(t), \\ u(0,x) = u_0(x), \\ u(t,x)\big|_{\partial D} = 0, \end{gathered} \tag{1.1}$$

where $(t,x)\in[0,T]\times D$, D is a bounded interval in $\mathbb{R}$, and $u_0\in L^2(D)$. The operator $(-\Delta)^{\alpha/2}$ is the fractional power of $-\Delta$, with $\alpha\in(1,2)$, defined as

$$(-\Delta)^{\alpha/2}e_n := \lambda_n^{\alpha/2}e_n, \quad n=1,2,\dots, \tag{1.2}$$

where λ_n is the eigenvalue of $-\Delta$, with the corresponding eigenvector e_n. The fractional derivative D_t^β is the Caputo derivative of order $\beta\in(0,1]$ in the time variable, which is defined as follows (see [15])

$$D_t^\beta u(t,x) = \begin{cases} \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{\partial u(s,x)}{\partial s}\frac{ds}{(t-s)^\beta}, & 0<\beta<1, \\ \frac{\partial u(t,x)}{\partial t}, & \beta=1, \end{cases} \tag{1.3}$$

in which the gamma function is defined as $\Gamma(\beta)=\int_0^\infty t^{\beta-1}e^{-t}dt$. The process $\{W^H(t), t\in[0,T]\}$ is a cylindrical fractional Brownian motion on a real and separable Hilbert space, with Hurst parameter $H\in(1/2,1)$.

The classical stochastic Burgers equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \frac{1}{2}u\frac{\partial u}{\partial x} + \sigma(u)\frac{\partial^2 W}{\partial t\partial x}.$$

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models a turbulent flow and is solved by the Hopf-Cole transformation [5, 6, 7]. In the past few years, stochastic Burgers equations perturbed by different random noises have been studied intensively. This equation plays an important role in nonlinear acoustics, cosmology, and statistical physics [1, 2, 3, 21]. The author in [16] considers one dimensional stochastic Burgers equation driven by white noise term, and obtains existence of a weak solution by proving tightness for a sequence of polygonal approximations and solving a martingale problem for the weak limit. In [9] the authors explore the existence and uniqueness of the global solution of a stochastic Burgers equation perturbed by white noise, and the existence of an invariant measure the corresponding transition semigroup.

Stochastic Burgers equations with fractional Laplacian in spatial variable have been also explored. For instance, the researchers in [25] study a model involving the Lipschitz continuity of the inhomogeneous term, and a diffusion coefficient with space-time white noise in local subspace. In [4] the authors explore existence and uniqueness of invariant measures for the stochastic Burgers equation driven by fractional Laplacian and space-time white noise. They show that the transition measures of the solution converge to the invariant measure in the norm of total variation.

Many researchers have developed interests in the time-fractional diffusion equations [24, 27, 30] which are also applied for describing the memory effect of the wall friction through the boundary layer [13]. The authors in [8] studied the nonlinear stochastic equation of fractional derivative both in space and time variables with space-time white noise. They obtained the existence and uniqueness of solution with the moment bounds of solutions under Dalang's condition. In [31], it is proved that there is a unique mild solution of the stochastic Burgers equation with time- and space-fractional derivative driven by white noise, by a Picard iteration method. Different from the white noise in the model in [31], here we consider the fractional noise in time variable.

The fractional Brownian motion was first introduced with a Hilbert space framework by Kolmogorov in [17]. In recent years, fractional Brownian motion has been attracted attention because of their useful feature of preserving long term memory, and a large number of interesting results from scaling invariance to the description of their laws as random fields have been established by various authors. The study of these Gaussian processes has its historical motivation from their applications in hydrology and telecommunication, and has been applied to the mathematical finance, biotechnology and biophysics, see for example [11, 18, 23] and their references. In [14], the researchers explore that the existence, uniqueness, and moment estimate for the solution of the stochastic Burgers equation driven by multi-parameter fractional noise. The authors in [26] show the local and global existence and uniqueness results for the stochastic Burgers equation driven by fractional Brownian motion with $H > \frac{1}{4}$. In our work, we consider a U-valued Q-cylindrical fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$.

The above research work motivates us to obtain the existence and uniqueness of the mild solution to the problem (1.1) with boundary and initial conditions, and explore Hölder regularity of the mild solution.

Definition 1.1. The domain of the fractional Laplace operator $(-\Delta)^{\alpha/2}$ in (1.2) is defined as

$$\dot{H}^\alpha := \big\{v \in L^2(D) : \sum_{n=1}^{\infty} \lambda_n^\alpha \langle v, e_n\rangle^2 < \infty\big\},$$

with the inner product $\langle\cdot,\cdot\rangle$ in $L^2(D)$. Thus, we define the norm as

$$\|v\|^2_{\dot{H}^\alpha} := \|A_\alpha v\|^2_{L^2(D)} = \sum_{n=1}^{\infty} \lambda_n^\alpha \langle v, e_n\rangle^2. \tag{1.4}$$

In this article we use the following notation.

- $A_\alpha := (-\Delta)^{\alpha/2}$.
- The eigenvalues of $-\Delta$ are $\lambda_1 \le \lambda_2 \le \dots \le \Lambda$, where Λ denote the maximum of the eigenvalues in D.
- $\|\cdot\| := \|\cdot\|_{L^2(D)}$.
- $B(u,v) := u\frac{\partial v}{\partial x}$, $B(u) := B(u,u)$.
- The domain of the operator B is $\mathscr{D}(B) := H_0^1(D) \times H_0^1(D)$.

Thus, problem (1.1) can be rewritten as

$$\begin{gathered} D_t^\beta u(t) = -A_\alpha u(t) + B(u(t)) + \dot{W}^H(t), \\ u(0) = u_0. \end{gathered} \tag{1.5}$$

Now, we introduce the Bochner spaces $L^p(\Omega; \mathbf{G}) = L^p((\Omega, \mathscr{F}, \mathbb{P}); \mathbf{G})$ as

$$L^p(\Omega;\mathbf{G}) = \big\{f : \mathbb{E}\|f\|^p_{\mathbf{G}} = \int_\Omega \|f(\omega)\|^p_{\mathbf{G}} d\mathbb{P}(\omega) < \infty, \omega \in \Omega\big\},$$

with the norm $\|f\|_{L^p(\Omega;\mathbf{G})} = (\mathbb{E}\|f\|^p_{\mathbf{G}})^{1/p}$, where $\mathbf{G}$ is a Banach space.

Next, we define mild solutions of problem (1.5), which is inspired by the definition of mild solution to the fractional stochastic Burgers equations driven by multiplicative white noise [31].

Definition 1.2. Let $\{u(t), t \in [0,T]\}$ be a random field that is continuous with respect to t. A function $u \in C([0,T]; L^p(\Omega; \dot{H}^\gamma))$ is a mild solution of (1.5) if

$$\begin{aligned} u(t) = L_\beta^\alpha(t)u_0 &+ \int_0^t (t-s)^{\beta-1} L^\alpha_{\beta,\beta}(t-s) B(u(s))ds \\ &+ \int_0^t (t-s)^{\beta-1} L^\alpha_{\beta,\beta}(t-s)\, dW^H(s), \end{aligned} \tag{1.6}$$

where $L_\beta^\alpha(t)$ and $L^\alpha_{\beta,\beta}(t)$ are the generalized Mittag-Leffler operators defined by (2.5) and (2.6).

A derivation of the mild solution is shown in the Appendix, which applies the Laplace transform method and the properties of the semigroup generated from the fractional Laplace operator. In the following, we give some assumptions about the operator B and the initial condition u_0.

Assumption 1.3. The bounded bilinear operator B satisfies

$$\begin{gathered} \|B(u)\| \leqslant M\|u\|^2, \\ \|B(u) - B(v)\| \leqslant M(\|u\| + \|v\|)\|u - v\|, \end{gathered}$$

for all $u, v \in L^2(D)$, where M is a positive constant.

Assumption 1.4. Let the initial value $u_0 : \Omega \to \dot{H}^\gamma$ be $\mathscr{F}_0$-measurable random variable, satisfying

$$\|u_0\|_{L^p(\Omega;\dot{H}^\gamma)} < \infty$$

for all $0 < \gamma < \alpha < 2$.

We set a subspace of $C([0,T];L^p(\Omega;\dot{H}^\gamma))$ for a stopping time T' as follows

$$S^{T'} := \{u \in C([0,T];L^p(\Omega;\dot{H}^\gamma)) : \sup_{t\in[0,T']} \mathbb{E}\|u(t)\|^p_{\dot{H}^\gamma} \leqslant K\}, \tag{1.7}$$

where $\gamma > 0$, $0 < T' \leqslant T$. The main results in this article reads as follows.

Theorem 1.5. *Let Assumption 1.3 and 1.4 be fulfilled with $p > 2$, $1/2 < \beta < 1$, $1 < \alpha < 2$, and $0 < \gamma < \frac{\alpha(2\beta-1)}{2\beta}$. Then there exists a unique mild solution of (1.1) in the space S^{T_*}, with some stopping time $T_* \in [0,T]$.*

Theorem 1.6. *Let Assumption 1.3 and 1.4 be fulfilled with $p > 2$, $1/2 < \beta < 1$, $1 < \alpha < 2$, and $0 < \gamma < \frac{\alpha(2\beta-1)}{2\beta}$. Let u be a solution of (1.1) in S^{T_*}, with T_* satisfying the conditions in Theorem 1.5. Then for any $0 \leqslant t_1 < t_2 \leqslant T_*$, the solution $u(t)$ is Hölder continuous with respect to the norm $\|\cdot\|_{L^p(\Omega;\dot{H}^\gamma)}$ and satisfies*

$$\|u(t_2) - u(t_1)\|^p_{L^p(\Omega;\dot{H}^\gamma)} \leqslant C(t_2 - t_1)^\tau,$$

with

$$\tau = \begin{cases} \min\left\{p\left(\beta - \frac{\beta\gamma}{\alpha} - \frac{1}{2}\right), \frac{p\beta\gamma}{\alpha}\right\}, & 0 < t_2 - t_1 < 1, \\ p(\beta - \frac{\beta\gamma}{\alpha}), & t_2 - t_1 \geqslant 1. \end{cases}$$

Now we highlight the contribution of this article in the field of the fractional stochastic Burgers equations. Firstly, our model with time- and space-fractional stochastic Burgers equation driven by fractional noise is new, compared to the problems studied in [4, 9, 14, 31]. Secondly, the U-valued Q-cylindrical fractional Brownian motion makes some difficulties in the analysis, we apply the embedding theorem to solve these difficulties. Finally, we compose the fractional Laplacian and the generalized Mittag-Leffler operators to estimate the norm of the mild solution of problem (1.1).

This article is organized as follows. In Section 2, we present some notation and introduce fractional Brownian motion, the generalized Mittag-Leffler operators. Then we give properties of fractional Laplacian and the generalized Mittag-Leffler operators. In Section 3, we prove Theorem 1.5 to obtain the existence and uniqueness of mild solution by the Banach Fixed Point Theorem for some stopping time. In Section 4, we prove Theorem 1.6, to obtain the Hölder continuity of the mild solution finally.

2. Preliminaries

2.1. **Fractional Brownian motion.** We provide an overview and systematization of stochastic calculus with respect to fractional Brownian motion. First, we introduce the one-dimensional fractional Brownian motion briefly; see [22] for details. A one-dimensional fractional Brownian motion with Hurst parameter $H \in (0,1)$ is a centered Gaussian process $B^H := \{B^H(t),\ t \geq 0\}$ with the covariance function

$$R_H(t,s) = \mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}).$$

Note that $B^{1/2}(t)$ is standard Brownian motion. We denote by $\mathcal{E}$ the set of step functions on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}} = R_H(t,s).$$

The mapping $1_{[0,t]} \to B^H(t)$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space associated with B^H. When $H > \frac{1}{2}$, it has been proved the covariance of fractional Brownian motion can be written as

$$R_H(t,s) = H(2H-1)\int_0^t \int_0^s |r-u|^{2H-2}\,du\,dr.$$

Consider the square integrable kernel

$$\begin{gathered} K_H(t,s) := c_H s^{H-\frac{1}{2}} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\,du, \\ c_H = \Big(\frac{H(2H-1)}{\mathbf{B}(2-2H, H-\frac{1}{2})}\Big)^{1/2} \end{gathered} \tag{2.1}$$

where $t > s > 0$, $\mathbf{B}(\cdot,\cdot)$ is the beta function. We deduce that this kernel satisfies

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,u)K_H(s,u)du.$$

From the definition of K_H, we obtain that

$$\frac{\partial K_H}{\partial t}(t,s) = c_H\Big(\frac{t}{s}\Big)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}}\,.$$

We consider the linear operator K_H^* from $\mathcal{E}$ to $L^2([0,T])$ defined by

$$(K_H^*\varphi)(s) = \int_s^T \varphi(t)\frac{\partial K_H}{\partial t}(t,s)dt.$$

Notice that

$$\left(K_H^* 1_{[0,t]}\right)(s) = K_H(t,s)1_{[0,t]}(s).$$

The operator K_H^* is an isometry between $\mathcal{E}$ and $L^2([0,T])$ that can be extended to the Hilbert space $\mathcal{H}$. In fact, for any $s,t \in [0,T]$ we have

$$\begin{aligned} \langle K_H^*1_{[0,t]}, K_H^*1_{[0,s]}\rangle_{L^2([0,T])} &= \langle K_H(t,\cdot)1_{[0,t]}, K_H(s,\cdot)1_{[0,s]}\rangle_{L^2([0,T])} \\ &= \int_0^{t\wedge s} K_H(t,u)K_H(s,u)du \\ &= R_H(t,s) = \langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}}. \end{aligned}$$

Since the operator K_H^* provides an isometry between the Hilbert space $\mathcal{H}$ and $L^2([0,T])$, it follows that for any $t \in [0,T]$ there exists a Brownian motion

$$B_t = B^H((K_H^*)^{-1}(1_{[0,t]}))$$

such that

$$B_t^H = \int_0^t K_H(t,s)dB_s.$$

Moreover, for any $\varphi \in \mathcal{H}$, we have

$$\int_0^T \varphi(t)dB_t^H = \int_0^T (K_H^*\varphi)(t)dB_t.$$

Next we introduce the fractional Brownian motion with values in a Hilbert space and give the definition of the corresponding stochastic integral. Let U, V be separable Hilbert spaces, and $L(U,V)$ denote the space of all bounded linear operators from U to V. Let $Q \in L(U,U)$ be a nonnegative self-adjoint operator, and let $\{\sigma_n\}_{n\in\mathbb{N}}$ be a bounded sequence of nonnegative real numbers such that $Q\varsigma_n = \sigma_n\varsigma_n$ with $\sum_{n=1}^{\infty}\sigma_n < \infty$, where $\{\varsigma_n\}_{n\in\mathbb{N}}$ is a complete orthonormal basis in U.

We denote by $L_Q(U,V)$ the space of all $\varphi \in L(U,V)$ such that $\varphi Q^{1/2}$ is a Hilbert-Schmidt operator with the norm

$$\|\varphi\|^2_{L_Q(U,V)} = \sum_{n=1}^{\infty} \|\sqrt{\sigma_n}\varphi\varsigma_n\|^2_V. \tag{2.2}$$

Then φ is called a Q-Hilbert-Schmidt operator from U to V.

Let $\{B_n^H(t)\}_{n\in\mathbb{N}}$ be a sequence of two-sided one-dimensional standard fractional Brownian motions mutually independent on $(\Omega,\mathscr{F},\mathbb{P})$. When one considers the series

$$\sum_{i=1}^{\infty} B_n^H(t)\varsigma_n, \quad t \geqslant 0,$$

which not necessarily converges in the space U. Then we consider the U-valued stochastic process

$$W^H(t) = \sum_{i=1}^{\infty} B_n^H(t) Q^{1/2}\varsigma_n, \quad t \geqslant 0.$$

Since Q is a nonnegative self-adjoint operator, the above series converges in the space U, that is, it holds that $W^H(t) \in L^2(\Omega,U)$. Then, we say that $W^H(t)$ is a well defined U-valued Q-cylindrical fractional Brownian motion with covariance operator Q such that

$$W^H(t) = \sum_{n=1}^{\infty} B_n^H(t) Q^{1/2}\varsigma_n = \sum_{n=1}^{\infty} \sqrt{\sigma_n} B_n^H(t)\varsigma_n, \quad t \geqslant 0.$$

Definition 2.1. Let $\varphi : [0,T] \to L_Q(U,V)$ satisfy

$$\sum_{n=1}^{\infty} \|K_H^*(\varphi Q^{1/2}\varsigma_n)\|_{L^2([0,T];V)} < \infty. \tag{2.3}$$

Its stochastic integral with respect to the U-valued Q-cylindrical fractional Brownian motion W^H is defined, for $t \geq 0$, as

$$\int_0^t \varphi(s)\, dW^H(s) := \sum_{n=1}^{\infty} \int_0^t \varphi(s) Q^{1/2}\varsigma_n dB_n^H(s) = \sum_{n=1}^{\infty} \int_0^t K_H^*(\varphi(s)Q^{1/2}\varsigma_n) dB_s.$$

The following lemma estimates the stochastic integrals, see [29] for details.

Lemma 2.2. *For each $\varphi : [0,T] \to L_Q(U,V)$ satisfying $\int_0^T \|\varphi(s)\|^2_{L_Q^2(U,V)} ds < \infty$, the integral $\int_0^t \varphi(s)\, dW^H(s)$ is well defined as an V-valued random variable, and for any $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ we have*

$$\mathbb{E}\Big\| \int_{t_1}^{t_2} \varphi(s)\, dW^H(s) \Big\|_V^p \leqslant C(H,p)(t_2 - t_1)^{\frac{p(2H-1)}{2}} \Big(\int_{t_1}^{t_2} \|\varphi(s)\|^2_{L_Q(U,V)} ds \Big)^{p/2},$$

where the constant $C(H,p)$ is positive.

2.2. **Fractional Laplace Operator.** For the operator A_α introduced in the previous section, we have the following property; see [28].

Lemma 2.3. *For any $\alpha > 0$, the operator $-A_\alpha$ generates an analytic semigroup $S_\alpha(t) = e^{-tA_\alpha}$, $t \geqslant 0$ on $L^2(D)$. And for each $\gamma \geqslant 0$, there exists a constant $C(\alpha, \gamma)$ such that*

$$\|A_\gamma S_\alpha(t)\|_{\mathcal{L}(L^2)} \leq C(\alpha, \gamma) t^{-\gamma/\alpha}, \quad t > 0.$$

Here $\mathcal{L}(L^2)$ denotes the Banach space of linear bounded operators from $L^2(D)$ to itself.

In this article, the constant C is different from line to line.

2.3. **Mittag-Leffler Operator.** In this subsection, we introduce the one-sided stable probability density function. For each $\beta \in (0, 1), \theta \in (0, +\infty)$, there exists

$$w_\beta(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-\beta n - 1} \frac{\Gamma(\beta n + 1)}{n!} \sin(n\pi\beta),$$

and the Mainardi's Wright-type function (see [19, 20]) given by

$$M_\beta(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^n}{n! \Gamma(1 - \beta(1 + n))} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \theta^{n-1}}{(n-1)!} \Gamma(n\beta) sin(n\pi\beta).$$

Thus, we can obtain the following properties

$$\int_0^\infty M_\beta(\theta)\, d\theta = 1, \quad M_\beta(\theta) = \frac{1}{\beta} \theta^{-\frac{1}{\beta} - 1} w_\beta(\theta^{-1/\beta}). \tag{2.4}$$

The above Mainardi function $M_\beta(\theta)$ acts as a bridge between the following generalized Mittag-Leffler operators and the fractional differential equation (1.5). The generalized Mittag-Leffler operators are defined as

$$L_\beta^\alpha(t) := \int_0^\infty M_\beta(\theta) S_\alpha(t^\beta \theta)\, d\theta, \tag{2.5}$$

$$L_{\beta,\beta}^\alpha(t) := \int_0^\infty \beta\theta M_\beta(\theta) S_\alpha(t^\beta \theta)\, d\theta. \tag{2.6}$$

Now we give some properties of these two operators; see [31].

Lemma 2.4. *For each $\beta \in (0, 1)$ and $-1 < \epsilon < \infty$, it holds*

$$M_\beta(\theta) \geqslant 0, \quad \int_0^\infty \theta^\epsilon M_\beta(\theta)\, d\theta = \frac{\Gamma(1 + \epsilon)}{\Gamma(1 + \beta\epsilon)}, \quad \textit{for all } \theta \geqslant 0.$$

Lemma 2.5. *For each $t > 0$, both $L_\beta^\alpha(t)$ and $L_{\beta,\beta}^\alpha(t)$ are linear and bounded operators. Moreover, for any η such that $0 \leq \eta < \alpha < 2$ and any $f \in L^2(D)$, it holds*

$$\|L_\beta^\alpha(t) f\|_{\dot{H}^\eta} \leq C(\alpha, \beta, \eta) t^{-\frac{\beta\eta}{\alpha}} \|f\|,$$

$$\|L_{\beta,\beta}^\alpha(t) f\|_{\dot{H}^\eta} \leq C(\alpha, \beta, \eta) t^{-\frac{\beta\eta}{\alpha}} \|f\|,$$

where the constant $C(\alpha, \beta, \eta)$ is positive.

Proof. For $t > 0$, $0 \leqslant \eta < \alpha < 2$, because of Lemmas 2.3 and 2.4, we have

$$\begin{aligned}\|L^\alpha_\beta(t)f\|_{\dot H^\eta} &= \|A_\eta L^\alpha_\beta(t)f\|\\ &\le \int_0^\infty M_\beta(\theta)\|A_\eta S_\alpha(t^\beta\theta)f\|\,d\theta\\ &\le \int_0^\infty C(\alpha,\eta)t^{-\frac{\beta\eta}{\alpha}}\theta^{-\frac{\eta}{\alpha}}M_\beta(\theta)\|f\|\,d\theta\\ &= C(\alpha,\eta)\frac{\Gamma(1-\frac{\eta}{\alpha})}{\Gamma(1-\frac{\beta\eta}{\alpha})}t^{-\frac{\beta\eta}{\alpha}}\|f\|\\ &= C(\alpha,\beta,\eta)t^{-\frac{\beta\eta}{\alpha}}\|f\|.\end{aligned}$$

Then

$$\begin{aligned}\|L^\alpha_{\beta,\beta}(t)f\|_{\dot H^\eta} &= \|A_\eta L^\alpha_{\beta,\beta}(t)f\|\\ &\le \int_0^\infty \beta\theta M_\beta(\theta)\|A_\eta S_\alpha(t^\beta\theta)f\|\,d\theta\\ &\le \int_0^\infty C(\alpha,\eta)\beta t^{-\frac{\beta\eta}{\alpha}}\theta^{1-\frac{\eta}{\alpha}}M_\beta(\theta)\|f\|\,d\theta\\ &= C(\alpha,\eta)\beta\frac{\Gamma(2-\frac{\eta}{\alpha})}{\Gamma(1+\beta(1-\frac{\eta}{\alpha}))}t^{-\frac{\beta\eta}{\alpha}}\|f\|\\ &= C(\alpha,\beta,\eta)t^{-\frac{\beta\eta}{\alpha}}\|f\|,\end{aligned}$$

Obviously, the linearity of $L^\alpha_\beta(t)$ and $L^\alpha_{\beta,\beta}(t)$ is the same as in the semigroup $S_\alpha(t)$. Thus, L^α_β and $L^\alpha_{\beta,\beta}$ are linear and bounded operators. □

Lemma 2.6. *For each $t > 0$, the operators $L^\alpha_\beta(t)$ and $L^\alpha_{\beta,\beta}(t)$ are strongly continuous with respect to t. Moreover, for $t_0 > 0$, η such that $0 < \eta < \alpha < 2$, it holds that for any $f \in L^2(D)$ and $t \in (t_0, T]$,*

$$\begin{gathered}\|(L^\alpha_\beta(t)-L^\alpha_\beta(t_0))f\|_{\dot H^\eta} \le C(\alpha,\beta,\eta)(t-t_0)^{\frac{\beta\eta}{\alpha}}\|f\|,\\ \|(L^\alpha_{\beta,\beta}(t)-L^\alpha_{\beta,\beta}(t_0))f\|_{\dot H^\eta} \le C(\alpha,\beta,\eta)(t-t_0)^{\frac{\beta\eta}{\alpha}}\|f\|,\end{gathered}$$

where $C(\alpha,\beta,\eta) > 0$.

Proof. We know from the properties of the semigroup $S_\alpha(t)$ and A_α that

$$\frac{d}{dt}S_\alpha(t)f = A_\alpha S_\alpha(t)f, \quad A_\eta A_s f = A_{\eta+s}f.$$

Since $0 < t_0 < t \le T$, we can deduce that for each $f \in L^2(D)$,

$$\begin{aligned}\|(L^\alpha_\beta(t)-L^\alpha_\beta(t_0))f\|_{\dot H^\eta} &\le \int_0^\infty M_\beta(\theta)\|A_\eta(S_\alpha(t^\beta\theta)-S_\alpha(t_1^\beta\theta))f\|\,d\theta\\ &= \int_0^\infty M_\beta(\theta)\Big\|A_\eta\int_{t_0}^t \frac{dS_\alpha(t^\beta\theta)}{dt}f\Big\|\,d\theta\\ &= \int_0^\infty M_\beta(\theta)\Big\|\int_{t_0}^t \beta s^{\beta-1}\theta A_\eta A_\alpha S_\alpha(s^\beta\theta)f ds\Big\|\,d\theta\\ &\le \int_0^\infty \beta\theta M_\beta(\theta)\int_{t_0}^t \big\|s^{\beta-1}A_{\eta+\alpha}S_\alpha(s^\beta\theta)f\big\|\,ds\,d\theta.\end{aligned}$$

From Lemmas 2.3 and 2.4, we have

$$\begin{aligned}
&\int_0^\infty \beta\theta M_\beta(\theta)\int_{t_0}^t \|s^{\beta-1}A_{\eta+\alpha}S_\alpha(s^\beta\theta)f\|ds\,d\theta\\
&\le C(\alpha,\eta)\int_0^\infty \beta\theta^{-\frac{\eta}{\alpha}}M_\beta(\theta)\|f\|\,d\theta\int_{t_0}^t s^{-1-\frac{\beta\eta}{\alpha}}ds\\
&= C(\alpha,\eta)\frac{\Gamma(1-\frac{\eta}{\alpha})}{\Gamma(1-\frac{\beta\eta}{\alpha})t_0^{\frac{2\beta\eta}{\alpha}}}(t_0^{-\frac{\beta\eta}{\alpha}}-t^{-\frac{\beta\eta}{\alpha}})\|f\|\\
&\le C(\alpha,\beta,\eta)(t-t_0)^{\frac{\beta\eta}{\alpha}}\|f\|.
\end{aligned}$$

Also, we use a similar method to obtain that

$$\begin{aligned}
\|(L^\alpha_{\beta,\beta}(t)-L^\alpha_{\beta,\beta}(t_0))f\|_{\dot H^\eta} &\le \int_0^\infty \beta\theta M_\beta(\theta)\|A_\eta(S_\alpha(t_2^\beta\theta)-S_\alpha(t_0^\beta\theta))f\|_{L^2(D)}\,d\theta\\
&\le \int_0^\infty \beta^2\theta^2 M_\beta(\theta)\int_{t_0}^t \|s^{\beta-1}A_{\eta+\alpha}S_\alpha(s^\beta\theta)f\|ds\,d\theta\\
&\le C(\alpha,\eta)\int_0^\infty \beta^2\theta^{1-\frac{\eta}{\alpha}}M_\beta(\theta)\|f\|\,d\theta\int_{t_0}^t s^{-1-\frac{\beta\eta}{\alpha}}ds\\
&= C(\alpha,\beta,\eta)\frac{\Gamma(2-\frac{\eta}{\alpha})}{\Gamma(1+\beta(1-\frac{\eta}{\alpha}))t_0^{\frac{2\beta\eta}{\alpha}}}(t_0^{-\frac{\beta\eta}{\alpha}}-t^{-\frac{\beta\eta}{\alpha}})\|f\|\\
&\le C(\alpha,\beta,\eta)(t-t_0)^{\frac{\beta\eta}{\alpha}}\|f\|.
\end{aligned}$$

Thus, $\|(L^\alpha_\beta(t)-L^\alpha_\beta(t_0))f\|_{\dot H^\eta}, \|(L^\alpha_{\beta,\beta}(t)-L^\alpha_{\beta,\beta}(t_0))f\|_{\dot H^\eta}\to 0$ as $t\to t_0$, and the operators $L^\alpha_\beta(t)$ and $L^\alpha_{\beta,\beta}(t)$ are strongly continuous. □

Corollary 2.7. *If we assume $\eta=0$ in Lemma 2.6, then for each $f\in L^2(D)$, $t\in(t_0,T]$, we have*

$$\begin{gathered}
\|(L^\alpha_\beta(t)-L^\alpha_\beta(t_0))f\|\le C(\alpha,\beta)(t-t_0)\|f\|,\\
\|(L^\alpha_{\beta,\beta}(t)-L^\alpha_{\beta,\beta}(t_0))f\|\le C(\alpha,\beta)(t-t_0)\|f\|.
\end{gathered}$$

Proof. Following as similar method as in Lemma 2.6, we have that

$$\begin{aligned}
\|(L^\alpha_\beta(t)-L^\alpha_\beta(t_0))f\| &= \Big\|\int_0^\infty M_\beta(\theta)(S_\alpha(t^\beta\theta)-S_\alpha(t_0^\beta\theta))f\,d\theta\Big\|\\
&\leqslant \int_0^\infty \beta\theta M_\beta(\theta)\int_{t_0}^t s^{\beta-1}\|A_\alpha S_\alpha(s^\beta\theta)f\|ds\,d\theta\\
&\leqslant \int_0^\infty C(\alpha)\beta M_\beta(\theta)\Big(\int_{t_0}^t s^{-1}ds\Big)\|f\|\,d\theta\\
&= C(\alpha)\beta(\ln t-\ln t_0)\|f\|\\
&\leqslant C(\alpha,\beta)(t-t_0)\|f\|,
\end{aligned}$$

and

$$\begin{aligned}
\|(L^\alpha_{\beta,\beta}(t)-L^\alpha_{\beta,\beta}(t_0))f\| &= \Big\|\int_0^\infty \beta\theta M_\beta(\theta)(S_\alpha(t^\beta\theta)-S_\alpha(t_0^\beta\theta))f\,d\theta\Big\|\\
&\leqslant \int_0^\infty \beta^2\theta^2 M_\beta(\theta)\int_{t_0}^t s^{\beta-1}\|A_\alpha S_\alpha(s^\beta\theta)f\|ds\,d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{C(\alpha)\beta^2\Gamma(2)}{\Gamma(1+\beta)}(\ln t - \ln t_0)\|f\| \\
&\leqslant C(\alpha,\beta)(t-t_0)\|f\|.
\end{aligned}
$$
□

3. Mild solution

In this section, we prove, Theorem 1.5, the existence and uniqueness of a mild solution of (1.1), by the Banach Fixed Point Theorem for some stopping time T_* in the space

$$
S^{T_*} := \{u \in C([0,T]; L^p(\Omega; \dot H^\gamma)) : \sup_{t\in[0,T_*]} \mathbb{E}\|u(t)\|^p_{\dot H^\gamma} \leqslant K\}.
$$

Proof of Theorem 1.5. We define a map $\mathcal{F} : S^T \to C([0,T]; L^p(\Omega; \dot H^\gamma))$ for $u \in S^T$ as follows

$$
\begin{aligned}
(\mathcal{F}u)(t) = L^\alpha_\beta(t)u_0 &+ \int_0^t (t-s)^{\beta-1} L^\alpha_{\beta,\beta}(t-s)B(u(s))ds \\
&+ \int_0^t (t-s)^{\beta-1} L^\alpha_{\beta,\beta}(t-s)\, dW^H(s).
\end{aligned} \tag{3.1}
$$

Firstly, we show that the map $\mathcal{F}$ is well defined. Indeed, for any $u \in S^T$, from $\|f\|_{\dot H^\gamma} = \|A_\gamma f\|$ in (1.4) and the definition of the operators L^α_β and $L^\alpha_{\beta,\beta}$ in (2.5) and (2.6), we have that

$$
\begin{aligned}
\mathbb{E}\|\mathcal{F}u(t)\|^p_{\dot H^\gamma} &= \mathbb{E}\Big\| L^\alpha_\beta(t)u_0 + \int_0^t (t-s)^{\beta-1} L^\alpha_{\beta,\beta}(t-s)B(u(s))ds \\
&\quad + \int_0^t (t-s)^{\beta-1} L^\alpha_{\beta,\beta}(t-s)\, dW^H(s)\Big\|^p_{\dot H^\gamma} \\
&\leq C\Big(\mathbb{E}\|\int_0^\infty M_\beta(\theta)A_\gamma S_\alpha(t^\beta\theta)u_0\, d\theta\|^p \\
&\quad + \mathbb{E}\|\int_0^t (t-s)^{\beta-1} L^\alpha_{\beta,\beta}(t-s)A_\gamma B(u(s))ds\|^p \\
&\quad + \mathbb{E}\|\int_0^t (t-s)^{\beta-1} A_\gamma L^\alpha_{\beta,\beta}(t-s)\, dW^H(s)\|^p\Big) \\
&=: C(I_1 + I_2 + I_3).
\end{aligned} \tag{3.2}
$$

From the properties of A_α in Lemma 2.3, and Assumption 1.4, we deduce that

$$
\begin{aligned}
I_1 &= \mathbb{E}\|\int_0^\infty M_\beta(\theta)A_\gamma S_\alpha(t^\beta\theta)u_0\, d\theta\|^p \\
&\leqslant \mathbb{E}\|\int_0^\infty M_\beta(\theta)(\|A_\gamma S_\alpha(t^\beta\theta)u_0\|^2)^{1/2}\, d\theta\|^p \\
&= \mathbb{E}\|\int_0^\infty M_\beta(\theta)\Big(\sum_{n=1}^\infty \langle A_\gamma e^{-t^\beta\theta A_\alpha}u_0, e_n\rangle^2\Big)^{1/2}\, d\theta\|^p \\
&= \mathbb{E}\|\int_0^\infty M_\beta(\theta)\Big(\sum_{n=1}^\infty \langle A_\gamma u_0, e^{-t^\beta\theta\lambda_n^{\alpha/2}} e_n\rangle^2\Big)^{1/2}\, d\theta\|^p \\
&\leqslant \mathbb{E}\|\int_0^\infty M_\beta(\theta)\, d\theta \|u_0\|_{\dot H^\gamma}\|^p
\end{aligned}
$$

$$= \mathbb{E}\|u_0\|^p_{\dot{H}^\gamma} < \infty. \tag{3.3}$$

From the properties of $L^\alpha_{\beta,\beta}(t)$ in Lemma 2.5, we have

$$\begin{aligned}
I_2 &= \mathbb{E}\Big\|\int_0^t (t-s)^{\beta-1} L^\alpha_{\beta,\beta}(t-s) A_\gamma B(u(s)) ds\Big\|^p \\
&\leqslant \mathbb{E}\Big(\int_0^t \|(t-s)^{\beta-1} A_\gamma L^\alpha_{\beta,\beta}(t-s) B(u(s))\| ds\Big)^p \\
&\leqslant C(\alpha,\beta)\mathbb{E}\Big(\int_0^t \|(t-s)^{\beta-1-\frac{\beta\gamma}{\alpha}} B(u(s))\| ds\Big)^p.
\end{aligned} \tag{3.4}$$

Since $u \in S^T$, by Assumption 1.3, $\|B(u)\| \leqslant M\|u\|^2$, and the Hölder inequality, (3.4) implies that

$$\begin{aligned}
&\mathbb{E}\Big(\int_0^t \|(t-s)^{\beta-1-\frac{\beta\gamma}{\alpha}} B(u(s))\| ds\Big)^p \\
&\leqslant \Big(\int_0^t (t-s)^{\frac{p(\beta-1-\frac{\beta\gamma}{\alpha})}{p-1}} ds\Big)^{p-1} \int_0^t \mathbb{E}\|B(u(s))\|^p ds \\
&\leqslant C(\alpha,\beta,\gamma,M,\lambda_1,\Lambda) t^{p\beta-\frac{p\beta\gamma}{\alpha}-1} \int_0^t \big(\mathbb{E}\|u(s)\|^p_{\dot{H}^\gamma}\big)^2 ds \\
&\leqslant C(\alpha,\beta,\gamma,M,\lambda_1,\Lambda) K^2 t^{p\beta-\frac{p\beta\gamma}{\alpha}}.
\end{aligned} \tag{3.5}$$

It follows that

$$I_2 \le C(\alpha,\beta,\gamma,M,\lambda_1,\Lambda) K^2 T^{p\beta-\frac{p\beta\gamma}{\alpha}} < \infty, \tag{3.6}$$

with $0 < \gamma < \frac{\alpha(p\beta-1)}{p\beta}$, where λ_1 and Λ are the minimum and maximum of the eigenvalues of the operator $-\Delta$ relatively in the notation in (1).

For I_3, since Q is a bounded operator, set $\sum_{n=1}^\infty \sigma_n < R_0$, for some constant $R_0 > 0$. From the norm (2.2) of $L_Q(U, L^2(D))$, Lemma 2.2 and $1/2 < \beta < 1$, $1 < \alpha < 2$, $0 < \gamma < \frac{\alpha(2\beta-1)}{2\beta}$, we have that

$$\begin{aligned}
I_3 &= \mathbb{E}\|\int_0^t (t-s)^{\beta-1} A_\gamma L^\alpha_{\beta,\beta}(t-s)\, dW^H(s)\|^p \\
&\leqslant C(H,p) t^{\frac{p(2H-1)}{2}} \Big(\int_0^t \|(t-s)^{\beta-1} A_\gamma L^\alpha_{\beta,\beta}(t-s)\|^2_{L_Q(U,L^2(D))} ds\Big)^{p/2} \\
&= C(H,p) t^{\frac{p(2H-1)}{2}} \Big(\int_0^t \sum_{n=1}^\infty \|\sqrt{\sigma_n}(t-s)^{\beta-1} A_\gamma L^\alpha_{\beta,\beta}(t-s)\varsigma_n\|^2 ds\Big)^{p/2} \\
&\leqslant C(H,p,\alpha,\beta,\gamma) t^{\frac{p(2H-1)}{2}} \Big(\int_0^t (t-s)^{2(\beta-1-\frac{\beta\gamma}{\alpha})} ds \Big(\sum_{n=1}^\infty \|\sqrt{\sigma_n}\varsigma_n\|^2_U\Big)\Big)^{p/2} \\
&\leqslant C(H,p,\alpha,\beta,\gamma,R_0) T^{p(H+\beta-\frac{\beta\gamma}{\alpha}-1)}.
\end{aligned} \tag{3.7}$$

where $C(H,p,\alpha,\beta,\gamma,R_0)$ is a positive constant depending on H, $p,\alpha,\beta,\gamma,R_0$, and the second last inequality holds because of Lemma 2.5.

From estimates (3.2)-(3.7), we have that

$$\sup_{t\in[0,T]} \mathbb{E}\|\mathcal{F}u(t)\|^p_{\dot{H}^\gamma} \le C\Big(T^{p\beta-\frac{p\beta\gamma}{\alpha}} + T^{p(H+\beta-\frac{\beta\gamma}{\alpha}-1)}\Big) < \infty.$$

where the constant C is positive and depends on $\alpha,\beta,\gamma,H,R_0,T,M,K,\lambda_1,\Lambda$. Thus, the map $\mathcal{F}$ is well defined.

Secondly, we want to find $T_0 \in (0, T]$ such that $\mathcal{F} : S^{T_0} \to S^{T_0}$. By the same arguments as the above analysis (3.3)-(3.7), we obtain that

$$\sup_{t\in[0,T]} \mathbb{E}\|(\mathcal{F}u)(t)\|^p_{\dot{H}^\gamma} \le C\left(\mathbb{E}\|u_0\|^p_{\dot{H}^\gamma} + K^2 T^{p\beta-\frac{p\beta\gamma}{\alpha}} + T^{p(H+\beta-\frac{\beta\gamma}{\alpha}-1)}\right),$$

where the constant C is positive and depends on $\alpha, \beta, \gamma, H, R_0, T, M, K, \lambda_1, \Lambda$. Then, we choose T_0 such that $\mathbb{E}\|\mathcal{F}u\|^p_{\dot{H}^v} \leqslant K$, for any $t \in [0, T_0]$,

$$C\Big(\mathbb{E}\|u_0\|^p_{\dot{H}^\gamma} + K^2 T_0^{p\beta-\frac{p\beta\gamma}{\alpha}} + T_0^{p(H+\beta-\frac{\beta\gamma}{\alpha}-1)}\Big) \le K, \tag{3.8}$$

where γ such that $0 < \gamma < \frac{\alpha(2\beta-1)}{2\beta} < \frac{\alpha(p\beta-1)}{p\beta}$.

Finally, we show $\mathcal{F}$ is a contraction mapping on S^{T_*} with suitable selected T_* such that $0 < T_* < T_0$. For any $u, h \in S^{T_0}$, taking similar method as in the estimate (3.4)-(3.6), and by Assumption 1.3 and Hölder inequality, we have that

$$\begin{aligned}
&\mathbb{E}\|(\mathcal{F}u)(t) - (\mathcal{F}h)(t)\|^p_{\dot{H}^\gamma} \\
&= \mathbb{E}\|\int_0^t (t-s)^{\beta-1} L^\alpha_{\beta,\beta}(t-s)(B(u(s)) - B(h(s)))ds\|^p_{\dot{H}^\gamma} \\
&\leqslant \mathbb{E}\Big(\int_0^t \|(t-s)^{\beta-1} A_\gamma L^\alpha_{\beta,\beta}(t-s)(B(u(s)) - B(h(s)))\|ds\Big)^p \\
&\leqslant C(\alpha,\beta,\gamma)\Big(\int_0^t (t-s)^{\frac{p(\beta-1-\frac{\beta\gamma}{\alpha})}{p-1}} ds\Big)^{p-1} \int_0^t \mathbb{E}\|B(u(s)) - B(h(s))\|^p ds \\
&\leqslant C(M,p,\alpha,\beta,\gamma) t^{p(\beta-\frac{\beta\gamma}{\alpha})-1} \int_0^t \mathbb{E}\left((\|u(s)\| + \|h(s)\|)\|u(s) - h(s)\|\right)^p ds \\
&\leqslant C(M,p,\alpha,\beta,\gamma,\Lambda) t^{p(\beta-\frac{\beta\gamma}{\alpha})-1} \int_0^t \mathbb{E}\Big((\|u(s)\|_{\dot{H}^\gamma} + \|h(s)\|_{\dot{H}^\gamma})\|u(s) - h(s)\|_{\dot{H}^\gamma}\Big)^p ds \\
&\leqslant C(M,K,p,\alpha,\beta,\gamma,\Lambda,\lambda_1) t^{p(\beta-\frac{\beta\gamma}{\alpha})-1} \int_0^t \mathbb{E}\|u(s) - h(s)\|^p_{\dot{H}^\gamma} ds,
\end{aligned}$$

where Λ, λ_1 are the maximum and minimum of the eigenvalues of $(-\Delta)$ relatively. Then, it further implies that with $0 < \gamma < \frac{\alpha(2\beta-1)}{2\beta} < \frac{\alpha(p\beta-1)}{p\beta}$

$$\begin{aligned}
&\sup_{t\in[0,T_0]} \mathbb{E}\|(\mathcal{F}u)(t) - (\mathcal{F}h)(t)\|^p_{\dot{H}^\gamma} \\
&\leqslant C(M,K,p,\alpha,\beta,\gamma,\Lambda,\lambda_1) T_0^{p\beta-\frac{p\beta\gamma}{\alpha}} \sup_{t\in[0,T_0]} \mathbb{E}\|u(t) - h(t)\|^p_{\dot{H}^\gamma}.
\end{aligned}$$

We take $T_* \in (0, T_0)$ such that

$$C(M,K,p,\alpha,\beta,\gamma,\Lambda,\lambda_1) T_*^{p\beta-\frac{p\beta\gamma}{\alpha}} < 1,$$

By the Banach Fixed Point Theorem, there exist a unique point $u \in S^{T_*}$, which is a unique mild solution to the problem (1.5). Then by the equivalency of the problem (1.5) and (1.1), the Theorem 1.5 is proved. □

4. Hölder continuity

In this section, we prove Theorem 1.6, and obtain the Hölder continuity of the mild solution in (1.1).

Proof of Theorem 1.6. For any $0 \leqslant t_1 < t_2 \leqslant T_*$, since u is a mild solution of (1.5), we have

$$
\begin{aligned}
&\mathbb{E}\|u(t_2) - u(t_1)\|^p_{\dot{H}^\gamma} \\
&= \mathbb{E}\Big\| L^\alpha_\beta(t_2)u_0 - L^\alpha_\beta(t_1)u_0 + \int_0^{t_2} (t_2 - s)^{\beta-1} L^\alpha_{\beta,\beta}(t_2 - s)B(u(s))ds \\
&\quad - \int_0^{t_1} (t_1 - s)^{\beta-1} L^\alpha_{\beta,\beta}(t_1 - s)B(u(s))ds \\
&\quad + \int_0^{t_2} (t_2 - s)^{\beta-1} L^\alpha_{\beta,\beta}(t_2 - s)\, dW^H(s) \\
&\quad - \int_0^{t_1} (t_1 - s)^{\beta-1} L^\alpha_{\beta,\beta}(t_1 - s)\, dW^H(s) \Big\|^p_{\dot{H}^\gamma} \\
&\leqslant C\mathbb{E}\|L^\alpha_\beta(t_2)u_0 - L^\alpha_\beta(t_1)u_0\|^p_{\dot{H}^\gamma} \\
&\quad + C\Big(\mathbb{E}\Big\| \int_0^{t_2} (t_2 - s)^{\beta-1} L^\alpha_{\beta,\beta}(t_2 - s)B(u(s))ds \\
&\quad - \int_0^{t_1} (t_1 - s)^{\beta-1} L^\alpha_{\beta,\beta}(t_1 - s)B(u(s))ds \Big\|^p_{\dot{H}^\gamma}\Big) \\
&\quad + C\Big(\mathbb{E}\Big\| \int_0^{t_2} (t_2 - s)^{\beta-1} L^\alpha_{\beta,\beta}(t_2 - s)\, dW^H(s) \\
&\quad - \int_0^{t_1} (t_1 - s)^{\beta-1} L^\alpha_{\beta,\beta}(t_1 - s)\, dW^H(s) \Big\|^p_{\dot{H}^\gamma}\Big) \\
&=: C(I_1 + I_2 + I_3).
\end{aligned} \tag{4.1}
$$

Firstly, we consider the term I_1. From Lemma 2.6 and Assumption 1.4, we deduce that

$$
\begin{aligned}
I_1 &= \mathbb{E}\|A_\gamma(L^\alpha_\beta(t_2) - L^\alpha_\beta(t_1))u_0\|^p_{\dot{H}^\gamma} \\
&\leqslant C(\alpha, \beta, \gamma, p)(t_2 - t_1)^{\frac{p\beta\gamma}{\alpha}} \mathbb{E}\|u_0\|^p_{\dot{H}^\gamma} \\
&\leqslant C(\alpha, \beta, \gamma, p)(t_2 - t_1)^{\frac{p\beta\gamma}{\alpha}}.
\end{aligned} \tag{4.2}
$$

Secondly, for the term I_2, we divide it into three parts as follows

$$
\begin{aligned}
I_2 &= \mathbb{E}\Big\| \int_0^{t_2} (t_2 - s)^{\beta-1} L^\alpha_{\beta,\beta}(t_2 - s)B(u(s))ds \\
&\quad - \int_0^{t_1} (t_1 - s)^{\beta-1} L^\alpha_{\beta,\beta}(t_1 - s)B(u(s))ds \Big\|^p_{\dot{H}^\gamma} \\
&\leqslant C\mathbb{E}\Big\| \int_0^{t_1} (t_1 - s)^{\beta-1} \big(L^\alpha_{\beta,\beta}(t_2 - s) - L^\alpha_{\beta,\beta}(t_1 - s)\big) B(u(s))ds\Big\|^p_{\dot{H}^\gamma} \\
&\quad + C\mathbb{E}\Big\| \int_0^{t_1} \big((t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}\big) L^\alpha_{\beta,\beta}(t_2 - s)B(u(s))ds\Big\|^p_{\dot{H}^\gamma} \\
&\quad + C\mathbb{E}\Big\| \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} L^\alpha_{\beta,\beta}(t_2 - s)B(u(s))ds\Big\|^p_{\dot{H}^\gamma} \\
&=: C(I_{21} + I_{22} + I_{23}).
\end{aligned} \tag{4.3}
$$

For I_{21}, by Assumption 1.3 and Lemma 2.6, we have

$$\begin{aligned} I_{21} &= \mathbb{E}\|\int_0^{t_1}(t_1-s)^{\beta-1}\left(L^{\alpha}_{\beta,\beta}(t_2-s)-L^{\alpha}_{\beta,\beta}(t_1-s)\right)B(u(s))ds\|^p_{\dot{H}^{\gamma}} \\ &= \mathbb{E}\|\int_0^{t_1}(t_1-s)^{\beta-1}A_{\gamma}\left(L^{\alpha}_{\beta,\beta}(t_2-s)-L^{\alpha}_{\beta,\beta}(t_1-s)\right)B(u(s))ds\|^p \\ &\leqslant \mathbb{E}\Big(\int_0^{t_1}(t_1-s)^{\beta-1}\|A_{\gamma}\left(L^{\alpha}_{\beta,\beta}(t_2-s)-L^{\alpha}_{\beta,\beta}(t_1-s)\right)B(u(s))\|ds\Big)^p \\ &\leqslant C(\alpha,\beta,\gamma)(t_2-t_1)^{\frac{p\beta\gamma}{\alpha}}\mathbb{E}\Big(\int_0^{t_1}(t_1-s)^{\beta-1}\|B(u(s))\|ds\Big)^p. \end{aligned} \tag{4.4}$$

Then, by using the Hölder inequality, we obtain

$$\begin{aligned} &(t_2-t_1)^{\frac{p\beta\gamma}{\alpha}}\mathbb{E}\Big(\int_0^{t_1}(t_1-s)^{\beta-1}\|B(u(s))\|ds\Big)^p \\ &\leqslant C(\lambda_1,\Lambda,M)(t_2-t_1)^{\frac{p\beta\gamma}{\alpha}}\Big(\int_0^{t_1}(t_1-s)^{\frac{p(\beta-1)}{p-1}}ds\Big)^{p-1}\int_0^{t_1}\mathbb{E}\|u(s)\|^p_{\dot{H}^{\gamma}}ds \\ &\leqslant C(\alpha,\beta,\gamma,\lambda_1,\Lambda,M,K,T_*,p)(t_2-t_1)^{\frac{p\beta\gamma}{\alpha}}, \end{aligned} \tag{4.5}$$

where the last inequality in (4.5) holds because $u \in S^{T_*}$ and $1/2 < \beta < 1$, $p > 2$. Thus, we have

$$I_{21} \leqslant C(\alpha,\beta,\gamma,\lambda_1,\Lambda,M,K,T_*,p)(t_2-t_1)^{\frac{p\beta\gamma}{\alpha}}. \tag{4.6}$$

Next we estimate I_{22} and I_{23} similarly as for (4.4) and (4.5). By applying Lemma 2.5, we can deduce that with $0<\gamma<\frac{\alpha(p\beta-1)}{p\beta}$ and $p>2$,

$$\begin{aligned} I_{22} &= \mathbb{E}\|\int_0^{t_1}\left((t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\right)L^{\alpha}_{\beta,\beta}(t_2-s)B(u(s))ds\|^p_{\dot{H}^{\gamma}} \\ &= \mathbb{E}\|\int_0^{t_1}\left((t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\right)A_{\gamma}L^{\alpha}_{\beta,\beta}(t_2-s)B(u(s))ds\|^p \\ &\leqslant C(\alpha,\beta,\gamma)\mathbb{E}\Big(\int_0^{t_1}\|((t_2-s)^{\beta-1}-(t_1-s)^{\beta-1})(t_2-s)^{-\frac{\beta\gamma}{\alpha}}\|\|B(u(s))\|ds\Big)^p \\ &\leqslant C(\alpha,\beta,\gamma,M,p)\Big(\int_0^{t_1}\Big((t_1-s)^{\beta-1}-(t_2-s)^{\beta-1}\Big)^{\frac{p}{p-1}}(t_2-s)^{-\frac{p\beta\gamma}{\alpha(p-1)}}ds\Big)^{p-1} \\ &\quad\times\int_0^{t_1}\Big(\mathbb{E}\|u\|^p_{\dot{H}^{\gamma}}\Big)^2ds \\ &\leqslant C(\alpha,\beta,\gamma,M,K,p,T_*)(t_2-t_1)^{\frac{p\beta(\alpha-\gamma)}{\alpha}-1}, \end{aligned}$$

and

$$\begin{aligned} I_{23} &= \mathbb{E}\|\int_{t_1}^{t_2}A_{\gamma}(t_2-s)^{\beta-1}L^{\alpha}_{\beta,\beta}(t_2-s)B(u(s))ds\|^p \\ &\leqslant C(\alpha,\beta,\gamma)\mathbb{E}\Big(\int_{t_1}^{t_2}(t_2-s)^{\beta-1-\frac{\beta\gamma}{\alpha}}\|B(u(s))\|\Big)^p \\ &\leqslant C(\alpha,\beta,\gamma,M)\Big(\int_{t_1}^{t_2}(t_2-s)^{\frac{p(\beta-1-\frac{\beta\gamma}{\alpha})}{p-1}}ds\Big)^{p-1}\int_{t_1}^{t_2}\left(\mathbb{E}\|u\|^p_{\dot{H}^{\gamma}}\right)^2ds \\ &\leqslant C(\alpha,\beta,\gamma,M,K,p)(t_2-t_1)^{\frac{p\beta(\alpha-\gamma)}{\alpha}}. \end{aligned} \tag{4.7}$$

From (4.3)-(4.7), we obtain

$$\begin{aligned} I_2 \leqslant C(\alpha,\beta,\gamma,\lambda_1,\Lambda,M,K,T_*,p)\big((t_2-t_1)^{\frac{p\beta\gamma}{\alpha}} \\ +(t_2-t_1)^{\frac{p\beta(\alpha-\gamma)}{\alpha}-1}+(t_2-t_1)^{\frac{p\beta(\alpha-\gamma)}{\alpha}}\big), \end{aligned} \tag{4.8}$$

where $C(\alpha,\beta,\gamma,\lambda_1,\Lambda,M,K,T_*,p)$ is a positive constant.

Finally, we estimate the term I_3 in (4.1). Here as in (4.3), the term I_3 is divided into three parts,

$$\begin{aligned} I_3 &= \mathbb{E}\Big\|\int_0^{t_2}(t_2-s)^{\beta-1}L^{\alpha}_{\beta,\beta}(t_2-s)\,dW^H(s) \\ &\quad -\int_0^{t_1}(t_1-s)^{\beta-1}L^{\alpha}_{\beta,\beta}(t_1-s)\,dW^H(s)\Big\|^p_{\dot H^\gamma} \\ &\leqslant C\Big(\mathbb{E}\|\int_0^{t_1}(t_1-s)^{\beta-1}\left(L^{\alpha}_{\beta,\beta}(t_2-s)-L^{\alpha}_{\beta,\beta}(t_1-s)\right)\,dW^H(s)\|^p_{\dot H^\gamma} \\ &\quad +\mathbb{E}\|\int_0^{t_1}\left((t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\right)L^{\alpha}_{\beta,\beta}(t_2-s)\,dW^H(s)\|^p_{\dot H^\gamma} \\ &\quad +\mathbb{E}\|\int_{t_1}^{t_2}(t_2-s)^{\beta-1}L^{\alpha}_{\beta,\beta}(t_2-s)\,dW^H(s)\|^p_{\dot H^\gamma}\Big) \\ &=: C(I_{31}+I_{32}+I_{33}). \end{aligned}$$

For I_{31}, since $\|\varphi\|^2_{L_Q(U,L^2(D))}=\sum_{n=1}^{\infty}\|\sqrt{\sigma_n}\varphi\varsigma_n\|^2$ in(2.2), by applying Lemmas 2.2 and 2.6, we obtain

$$\begin{aligned} I_{31} &= \mathbb{E}\Big\|\int_0^{t_1}(t_1-s)^{\beta-1}A_\gamma\Big(L^{\alpha}_{\beta,\beta}(t_2-s)-L^{\alpha}_{\beta,\beta}(t_1-s)\Big)\,dW^H(s)\Big\|^p \\ &\leqslant C(H)t_1^{\frac{p(2H-1)}{2}}\Big(\int_0^{t_1}\|(t_1-s)^{\beta-1}A_\gamma(L^{\alpha}_{\beta,\beta}(t_2-s) \\ &\quad -L^{\alpha}_{\beta,\beta}(t_1-s))\|^2_{L_Q(U,L^2(D))}ds\Big)^{p/2} \\ &= C(H)t_1^{\frac{p(2H-1)}{2}}\Big(\int_0^{t_1}(t_1-s)^{2(\beta-1)}\sum_{n=1}^{\infty}\|\sqrt{\sigma_n}A_\gamma\Big(L^{\alpha}_{\beta,\beta}(t_2-s) \\ &\quad -L^{\alpha}_{\beta,\beta}(t_1-s)\Big)\varsigma_n\Big\|^2ds\Big)^{p/2}. \end{aligned} \tag{4.9}$$

Since Q is a bounded operator and $\sum_{n=1}^{\infty}\sigma_n\leqslant R_0$, $1/2<\beta<1$ and $1<\alpha<2$, the above inequality implies

$$\begin{aligned} &\int_0^{t_1}(t_1-s)^{2(\beta-1)}\sum_{n=1}^{\infty}\big\|\sqrt{\sigma_n}A_\gamma\big(L^{\alpha}_{\beta,\beta}(t_2-s)-L^{\alpha}_{\beta,\beta}(t_1-s)\big)\varsigma_n\big\|^2ds \\ &\leqslant C(\alpha,\beta,\gamma)(t_2-t_1)^{\frac{2\beta\gamma}{\alpha}}\int_0^{t_1}(t_1-s)^{2(\beta-1)}ds\Big(\sum_{n=1}^{\infty}\|\sqrt{\sigma_n}\varsigma_n\|^2_U\Big) \\ &\leqslant C(\alpha,\beta,\gamma,H,R_0,T_*)(t_2-t_1)^{\frac{2\beta\gamma}{\alpha}}, \end{aligned} \tag{4.10}$$

Thus, from (4.9) it follows that

$$I_{31}\leqslant C(\alpha,\beta,\gamma,H,R_0,T_*)(t_2-t_1)^{\frac{p\beta\gamma}{\alpha}}. \tag{4.11}$$

Similarly, we consider I_{32} and I_{33}. Set $0<\gamma<\frac{\alpha(2\beta-1)}{2\beta}$, by Lemmas 2.2 and 2.5, it follows that

$$\begin{aligned}
I_{32} &= \mathbb{E}\|\int_0^{t_1} ((t_2-s)^{\beta-1}-(t_1-s)^{\beta-1})A_\gamma L^\alpha_{\beta,\beta}(t_2-s)\,dW^H(s)\|^p \\
&\leqslant C(H)t_1^{\frac{p(2H-1)}{2}}\Big(\int_0^{t_1}\|\big((t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\big) \\
&\quad\times A_\gamma L^\alpha_{\beta,\beta}(t_2-s)\|^2_{L_Q(U,L^2(D))}ds\Big)^{p/2} \\
&= C(H)t_1^{\frac{p(2H-1)}{2}}\Big(\int_0^{t_1}\sum_{n=1}^\infty\|\sqrt{\sigma_n}\big((t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\big) \\
&\quad\times A_\gamma L^\alpha_{\beta,\beta}(t_2-s)\varsigma_n\|^2\Big)^{p/2} \\
&\leqslant C(\alpha,\beta,\gamma,H,p,T_*)\Big(\int_0^{t_1}\big((t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\big)^2 \\
&\quad\times(t_2-s)^{-\frac{2\beta\gamma}{\alpha}}ds\Big)^{p/2}\Big(\sum_{n=1}^\infty\|\sqrt{\sigma_n}\varsigma_n\|^2_U\Big)^{p/2} \\
&\leqslant C(\alpha,\beta,\gamma,H,R_0,T_*,p)(t_2-t_1)^{\frac{p\beta\alpha-p\beta\gamma}{\alpha}-\frac{p}{2}},
\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}
I_{33} &= \mathbb{E}\|\int_{t_1}^{t_2}(t_2-s)^{\beta-1}A_\gamma L^\alpha_{\beta,\beta}(t_2-s)\,dW^H(s)\|^p \\
&\leqslant C(H)(t_2-t_1)^{\frac{p(2H-1)}{2}}\Big(\int_{t_1}^{t_2}(t_2-s)^{2(\beta-1)}\|A_\gamma L^\alpha_{\beta,\beta}(t_2-s)\|^2_{L_Q(U,L^2(D))}ds\Big)^{p/2} \\
&= C(H)(t_2-t_1)^{\frac{p(2H-1)}{2}}\Big(\int_{t_1}^{t_2}(t_2-s)^{2(\beta-1)}\|\sqrt{\sigma_n}A_\gamma L^\alpha_{\beta,\beta}(t_2-s)\varsigma_n\|^2ds\Big)^{p/2} \\
&\leqslant C(\alpha,\beta,\gamma,H)(t_2-t_1)^{\frac{p(2H-1)}{2}}\Big(\int_{t_1}^{t_2}(t_2-s)^{2(\beta-1-\frac{\beta\gamma}{\alpha})}ds\Big)^{p/2}\big(\|\sqrt{\sigma_n}\varsigma_n\|^2_U\big)^{p/2} \\
&\leqslant C(\alpha,\beta,\gamma,H,R_0,p)(t_2-t_1)^{p(H+\beta-\frac{\beta\gamma}{\alpha}-1)}.
\end{aligned}$$

From (4.11), (4.12), and the above inequality, for $0<\gamma<\frac{\alpha(2\beta-1)}{2\beta}<\frac{\alpha(p\beta-1)}{p\beta}$ we have

$$\begin{aligned}
I_3 &\leqslant C(\alpha,\beta,\gamma,H,R_0,T_*,p) \\
&\quad\times\Big((t_2-t_1)^{\frac{p\beta\gamma}{\alpha}}+(t_2-t_1)^{p(\beta-\frac{\beta\gamma}{\alpha}-\frac12)}+(t_2-t_1)^{p(H+\beta-\frac{\beta\gamma}{\alpha}-1)}\Big).
\end{aligned} \tag{4.13}$$

Thus, by (4.1), (4.2), (4.8), and (4.13), we conclude that

$$\mathbb{E}\|u(t_2)-u(t_1)\|^2_{\dot H^\gamma}\leqslant C(t_2-t_1)^\tau,$$

where C depends on $\alpha,\beta,\gamma,\lambda_1,\Lambda,H,T_*,M,K,R_0,p$, and

$$\tau=\begin{cases}\min\{p(\beta-\frac{\beta\gamma}{\alpha}-\frac12),\frac{p\beta\gamma}{\alpha}\}, & 0<t_2-t_1<1,\\ p(\beta-\frac{\beta\gamma}{\alpha}), & t_2-t_1\geqslant 1.\end{cases}$$

Therefore, we have Hölder continuity of the mild solutions of problems (1.5) and (1.1). This completes the proof. □

5. Appendix

Here, we give the derivation of the mild solution of the abstract problem (1.6); for more details see [31].

Proof. Laplace transform of a function is denoted by $\widehat{f} = \mathscr{L}(f)$:

$$\widehat{u}(\lambda) = \int_0^\infty e^{-\lambda s} u(s)\, ds, \quad \widehat{B}(\lambda) = \int_0^\infty e^{-\lambda s} B(u(s))\, ds, \widehat{G}(\lambda) = \int_0^\infty e^{-\lambda s}\, dW^H(s).$$

Applying the fractional integral operator

$$I^\beta h(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h(s) ds$$

to the equation in (1.5), we obtain

$$u(t) = u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (A_\alpha u(s) + B(u(s))) ds + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\, dW^H(s).$$

Then, by the Laplace transform,

$$\lambda^\beta \widehat{u} - \lambda^{\beta-1} u_0 = -A_\alpha \widehat{u} + \widehat{B} + \widehat{G};$$

that is,

$$\begin{aligned} \widehat{u} &= \lambda^{\beta-1} (\lambda^\beta I + A_\alpha)^{-1} u_0 + (\lambda^\beta I + A_\alpha)^{-1} (\widehat{B}(\lambda) + \widehat{G}(\lambda)) \\ &= \lambda^{\beta-1} \int_0^\infty e^{-\lambda^\beta s} S_\alpha(s) u_0 ds + \int_0^\infty e^{-\lambda^\beta s} S_\alpha(s) (\widehat{B}(\lambda) + \widehat{G}(\lambda)) ds \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{5.1}$$

Considering I_1, we have

$$\begin{aligned} I_1 &= \lambda^{\beta-1} \int_0^\infty e^{-\lambda^\beta s} S_\alpha(s) u_0 ds = \int_0^\infty \lambda^{\beta-1} e^{-\lambda^\beta t^\beta} S_\alpha(t^\beta) u_0 d(t^\beta) \\ &= \int_0^\infty \lambda^{\beta-1} \beta t^{\beta-1} e^{-(\lambda t)^\beta} S_\alpha(t^\beta) u_0 dt = \int_0^\infty -\frac{1}{\lambda} \frac{d}{dt} e^{-(\lambda t)^\beta} S_\alpha(t^\beta) u_0 dt \\ &= \int_0^\infty \int_0^\infty \theta w_\beta(\theta) e^{-\lambda \theta t} S_\alpha(t^\beta) u_0\, d\theta dt \\ &= \int_0^\infty e^{-\lambda t} \Big(\int_0^\infty w_\beta(\theta) S_\alpha(\frac{t^\beta}{\theta^\beta}) u_0\, d\theta \Big) dt. \end{aligned} \tag{5.2}$$

Next, we estimate the terms I_2 and I_3 as follows

$$\begin{aligned} I_2 &= \int_0^\infty e^{-\lambda^\beta s} S_\alpha(s) \widehat{B}(\lambda) ds = \int_0^\infty e^{-\lambda^\beta t^\beta} S_\alpha(t^\beta) \widehat{B}(\lambda) d(t^\beta) \\ &= \int_0^\infty \int_0^\infty \beta t^{\beta-1} e^{-(\lambda t)^\beta} S_\alpha(t^\beta) e^{-\lambda s} B(u(s))\, ds\, dt \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \beta e^{-\lambda t \theta} w_\beta(\theta) S_\alpha(t^\beta) e^{-\lambda s} t^{\beta-1} B(u(s))\, d\theta\, ds\, dt \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \beta w_\beta(\theta) e^{-\lambda(s+t)} S_\alpha(\frac{t^\beta}{\theta^\beta}) \frac{t^{\beta-1}}{\theta^\beta} B(u(s))\, d\theta\, ds\, dt \\ &= \int_0^\infty e^{-\lambda t} \Big(\beta \int_0^t \int_0^\infty w_\beta(\theta) S_\alpha(\frac{(t-s)^\beta}{\theta^\beta}) \frac{(t-s)^{\beta-1}}{\theta^\beta} B(u(s))\, d\theta\, ds \Big) dt. \end{aligned} \tag{5.3}$$

and

$$
\begin{aligned}
I_3 &= \int_0^\infty e^{-\lambda^\beta s} S_\alpha(s)\widehat{G}(\lambda)ds = \int_0^\infty \beta t^{\beta-1} e^{-(\lambda t)^\beta} S_\alpha(t^\beta)\widehat{G}(\lambda)dt \\
&= \int_0^\infty \int_0^\infty \beta t^{\beta-1} e^{-(\lambda t)^\beta} S_\alpha(t^\beta) e^{-\lambda s}\, dW^H(s)dt \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \beta w_\beta(\theta) e^{-\lambda t\theta} S_\alpha(t^\beta) e^{-\lambda s} t^{\beta-1}\, d\theta\, dW^H(s)dt \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \beta w_\beta(\theta) e^{-\lambda(t+s)} S_\alpha(\frac{t^\beta}{\theta^\beta}) \frac{t^{\beta-1}}{\theta^\beta}\, d\theta\, dW^H(s)dt \\
&= \int_0^\infty e^{-\lambda t}\Big(\beta \int_0^t \int_0^\infty w_\beta(\theta) S_\alpha(\frac{(t-s)^\beta}{\theta^\beta}) \frac{(t-s)^{\beta-1}}{\theta^\beta}\, d\theta\, dW^H(s)\Big)dt.
\end{aligned}
\tag{5.4}
$$

Based on estimates (5.2)-(5.4) and using the inverse Laplace transform, we obtain that the mild solution satisfies

$$
\begin{aligned}
u(t) = &\int_0^\infty w_\beta(\theta) S_\alpha(\frac{t^\beta}{\theta^\beta}) u_0\, d\theta \\
&+ \beta \int_0^t \int_0^\infty w_\beta(\theta) S_\alpha\Big(\frac{(t-s)^\beta}{\theta^\beta}\Big) \frac{(t-s)^{\beta-1}}{\theta^\beta} B(u(s))\, d\theta\, ds \\
&+ \beta \int_0^t \int_0^\infty w_\beta(\theta) S_\alpha\Big(\frac{(t-s)^\beta}{\theta^\beta}\Big) \frac{(t-s)^{\beta-1}}{\theta^\beta}\, d\theta\, dW^H(s) \\
= &\int_0^\infty \frac{1}{\beta}\theta^{-\frac{1}{\beta}-1} w_\beta(\theta^{-1/\beta}) S_\alpha(t^\beta\theta) u_0\, d\theta \\
&+ \int_0^t \int_0^\infty \theta^{-1/\beta} w_\beta(\theta^{-1/\beta}) S_\alpha((t-s)^\beta\theta)(t-s)^{\beta-1} B(u(s))\, d\theta\, ds \\
&+ \int_0^t \int_0^\infty \theta^{-1/\beta} w_\beta(\theta^{-1/\beta}) S_\alpha((t-s)^\beta\theta)(t-s)^{\beta-1}\, d\theta\, dW^H(s).
\end{aligned}
$$

According to (2.4), (2.5), and (2.6), the mild solution can be written as (1.6), i.e.,

$$
\begin{aligned}
u(t) = L_\beta^\alpha(t)u_0 + &\int_0^t (t-s)^{\beta-1} L_{\beta,\beta}^\alpha(t-s) B(u(s))ds \\
+ &\int_0^t (t-s)^{\beta-1} L_{\beta,\beta}^\alpha(t-s)\, dW^H(s).
\end{aligned}
$$

□

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