

EXISTENCE AND REGULARITY OF SOLUTIONS FOR ELLIPTIC SYSTEMS WITH MIXED BOUNDARY CONDITIONS

IMANE BOUSSETOUAN, CHÉRIF AMROUCHE

Communicated by Jesus Ildefonso Diaz

ABSTRACT. In this article we study elliptic systems involving nonstandard mixed boundary conditions in a bounded domain, possibly multiply connected. We prove the existence and uniqueness of weak and strong solutions of such problems in the Hilbert setting. Then, we generalize our results to the L^p -theory using some inf-sup conditions.

1. INTRODUCTION

Elliptic problems arise in various fields as in mechanics, chemistry, biology, and other fields. There has been a considerable amount of works done in the analysis of general and particular elliptic systems (see for instance [12, 13, 14]). The aim of this work is to study the well-posedness of linear elliptic systems involving two different kinds of mixed boundary conditions in the Hilbert framework and then in L^p -theory.

Unless stated otherwise, we assume that Ω is a $\mathcal{C}^{1,1}$ domain in \mathbb{R}^3 , possibly multiply connected and its boundary is decomposed as $\Gamma_D \cup \Gamma_N$ such that $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$. In what follows, the unit outer normal to the boundary is denoted by \mathbf{n} .

Laplace's equation has been mostly studied with Dirichlet or Neumann boundary conditions. In the vector case, we can have the same boundary conditions but they do not lead to an elliptic system. To recover the ellipticity, various types of physical boundary conditions can be imposed. Among them, the conditions we assume in this paper.

We consider the Poisson equation

$$-\Delta \mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \tag{1.1}$$

with the boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \text{curl } \mathbf{u} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_D, \tag{1.2}$$

$$\mathbf{u} \times \mathbf{n} = \mathbf{0}, \quad \text{div } \mathbf{u} = 0 \quad \text{on } \Gamma_N. \tag{1.3}$$

The condition (1.2) is called “Navier-type boundary condition” [6] or “Hodge boundary condition” [15] or even “perfect wall” condition [1]. It is equivalent to the Navier boundary condition introduced in [16] in the case of flat boundary and where the friction coefficient vanishes. It appears in the study of climate modeling, in electromagnetism and in magnetohydrodynamics. It can be used for ferrofluid flow models, such as those by Rosensweig and Shliomis [17], where the condition concerning the divergence of internal and external magnetic fields may not be constrained to zero. In [6], the authors treated the Stokes problem with (1.2) on the boundary and in particular the system (1.1)-(1.2) with a free-divergence condition, which required more assumptions on the data. Recently in [3], the authors investigated the L^p -theory of an elliptic system involving the condition (1.2) on the boundary without a divergence constraint.

2020 *Mathematics Subject Classification*. 35J05, 35J57, 35D35.

Key words and phrases. Elliptic systems; mixed boundary conditions; Navier-type boundary condition; weak and strong solutions; L^p -theory.

©2025. This work is licensed under a CC BY 4.0 license.

Submitted September 10, 2024. Published November 3, 2025.

For a better understanding of the conditions (1.2) and (1.3), let us consider any point P on Γ and choose any neighborhood W of P in Γ , small enough to allow the existence of \mathcal{C}^2 curves on W . The lengths s_1, s_2 along each family of curves are a possible set of coordinates in W . The unit tangent vectors to each family of curves are denoted by $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$. With this notation we have $\mathbf{v} = \mathbf{v}_\tau + (\mathbf{v} \cdot \mathbf{n})\mathbf{n}$ and $\mathbf{v}_\tau = \sum_{k=1}^2 v_k \boldsymbol{\tau}_k$, where $v_k = \mathbf{v} \cdot \boldsymbol{\tau}_k$. Consequently, we have the following formula valuable for any $\mathbf{u} \in \mathcal{D}(\bar{\Omega})$ (or more generally for any $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ with $\Delta \mathbf{u} \in \mathbf{L}^{r(p)}(\Omega)$, see below the definition of the exponent $r(p)$)

$$\operatorname{div} \mathbf{u} = \operatorname{div}_\Gamma \mathbf{u}_\tau - 2K \mathbf{u} \cdot \mathbf{n} + \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \mathbf{n} \quad \text{on } \Gamma, \quad (1.4)$$

where $K = -\frac{1}{2} \sum_{j=1}^2 \frac{\partial \mathbf{n}}{\partial \mathbf{s}_j} \cdot \boldsymbol{\tau}_j$ is the mean curvature of Γ and $\operatorname{div}_\Gamma$ is the surface divergence. Moreover, we have the following relations for any $\mathbf{u} \in \mathcal{D}(\bar{\Omega})$

$$\operatorname{curl} \mathbf{u} \times \mathbf{n} = -\nabla_\tau (\mathbf{u} \cdot \mathbf{n}) + \Lambda \mathbf{u} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{n}} \right)_\tau \quad \text{on } \Gamma, \quad (1.5)$$

where Λ is an operator of order zero, defined by

$$\Lambda \mathbf{w} = \sum_{j=1}^2 \left(\frac{\partial \mathbf{n}}{\partial \mathbf{s}_j} \cdot \mathbf{w}_\tau \right) \boldsymbol{\tau}_j.$$

Taking account of (1.4), the condition (1.3) is equivalent to Fourier-Robin boundary condition given by

$$K \mathbf{u} \cdot \mathbf{n} + \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \mathbf{n} = 0.$$

For the well-posedness of the problem, a quantity that reflects the topological structure of the domain should be added (see the last relations in Problem (\mathcal{P}_D) Section 2).

The second elliptic problem we are interested in is given by the equation (1.1) with mixed normal and tangential components on the boundary, i.e.

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_D, \quad (1.6)$$

$$\mathbf{u} \times \mathbf{n} = \mathbf{0}, \quad \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N. \quad (1.7)$$

This kind of conditions occur in the nonlinear model of a nematic liquid crystal studied by Dias in [11], who proved the existence and uniqueness of a weak solution in the non mixed case. In the work of Shoenuer [18], the author studied the problem, with (1.6) or (1.7) on the whole boundary. Taking $\mathbf{f} \in \mathbf{L}^p(\Omega)$, he proved the existence and uniqueness of strong solutions in $W^{2,p}(\Omega)$. The idea of the proof is based on the regularity near the boundary for $p = 2$. By a bootstrap argument, the case $p > 2$ is then deduced. Finally, the case $1 < p < 2$ is obtained using a priori estimates. Brezis et al [8] treated the regularity of the solution in the non mixed case where the source term \mathbf{f} belongs to $\mathbf{L}^1(\Omega)$. Using an additional condition on $\operatorname{div} \mathbf{f}$, they proved that the solution belongs to $\mathbf{W}^{1,n/n-1}(\Omega)$ and not only to the Sobolev space $\mathbf{W}_\infty^{1,n/n-1}(\Omega)$. The functions belonging to the latter space have gradients in the Lorentz space $\mathbf{L}_\infty^{n/n-1}(\Omega)$. Up to our knowledge, the problem of mixed boundary conditions (1.6)-(1.7) has not been treated in the literature. We assume that $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$ where

$$r(p) = \begin{cases} \frac{3p}{p+3} & \text{if } p > \frac{3}{2} \\ 1 + \varepsilon & \text{if } p = \frac{3}{2} \\ 1 & \text{if } 1 \leq p < \frac{3}{2}, \end{cases}$$

where $\varepsilon > 0$ is arbitrary. Note that $\mathbf{W}^{1,r(p)}(\Omega) \hookrightarrow \mathbf{L}^p(\Omega)$ and $\mathbf{W}^{1,p'}(\Omega) \hookrightarrow \mathbf{L}^{[r(p)]'}(\Omega)$.

These problems are challenging due to the unusual mixtures of physical boundary conditions we are considering. Also, the complexity of the geometry of the domain brings more difficulties to the problems. In fact, the domain is possibly multiply connected and we do not assume that Γ_D and Γ_N are connected. We suppose that there exist J connected open surfaces Σ_j , $1 \leq j \leq J$,

called “cuts”, contained in Γ , such that each surface Σ_j is an open subset of a smooth manifold and $\bar{\Sigma}_i \cap \bar{\Sigma}_j = \emptyset$ for $i \neq j$. Also, $\partial\Sigma$ stands for the union of the boundaries of Σ_j .

Our investigations are separated into three categories: existence, uniqueness and regularity of the solutions, in the Hilbert framework and then in L^p -theory. We start with the existence and uniqueness of weak solutions in the Hilbert setting of the first system (1.1)-(1.3). This system is more general than what has been studied in the literature (see for instance [4, 6, 7]) where the authors assume that $\operatorname{div} \mathbf{u} = 0$ on Ω and add some assumptions on the source term \mathbf{f} . The fact that we propose here a more general setting allows this kind of problems to be applied in some different areas like in magnetohydrodynamics. Furthermore, the solution of the system without a divergence constraint requires less conditions on the data. In Proposition 4.12, we prove that we can recover the condition $\operatorname{div} \mathbf{u} = 0$ on Ω . Unlike the case of Dirichlet condition, where the Poincaré’s inequality plays a fundamental role, we need here some inequalities for vector fields involving the operators grad , div and curl . To prove the strong regularity results, we use a partition of unity combined with some regularity properties for the operators div and curl . Let us emphasize that we will deal separately with the cases where $\partial\Sigma$ is included in Γ_D or in Γ_N . The main purpose of this work is to give a complete L^p -theory for the proposed systems not only for $p \geq 2$ but also for any $1 < p < 2$. To do so, we establish an appropriate inf-sup condition for each problem leading to generalized solutions in $\mathbf{W}^{1,p}(\Omega)$ for any $1 < p < +\infty$. We also give some results related to the case of inhomogeneous boundary conditions. For the second problem (1.1),(1.6) and (1.7), we prove a particular Poincaré condition in the case of mixed normal and tangential components. The existence of a strong solution is deduced by using the same arguments as the first system. For the L^p -theory, we proceed using a strategy based on the first inf-sup condition leading to a solution in $\mathbf{W}^{1,p}(\Omega)$ for any $1 < p < +\infty$ but requiring some assumptions on the geometry of the domain. We also provide a $\mathbf{W}^{2,p}(\Omega)$ regularity result.

Let us now briefly outline the structure of this paper. In Section 2, we state some main results. In Section 3, we introduce the mathematical framework and review some preliminary results. In Section 4, we focus our attention on the existence, uniqueness and regularity of the solution of the first elliptic problems dealing with the conditions (1.2) and (1.3) on the boundary. Section 5 is devoted to the study of the second elliptic system with (1.6) and (1.7) in the Hilbert case and then in L^p -theory.

2. MAIN RESULTS

The first main result of this work deals with the existence and uniqueness of the solution of the following problem where we assume that $\partial\Sigma \subset \Gamma_D$,

$$\begin{aligned} -\Delta \mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} &= \mathbf{a} \times \mathbf{n}, \quad \operatorname{div} \mathbf{u} = g \quad \text{on } \Gamma_N, \\ \mathbf{u} \cdot \mathbf{n} &= b, \quad \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma_D, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} &= 0, \quad 1 \leq \ell \leq L_N, \quad \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J, \end{aligned} \tag{2.1}$$

where \mathbf{f} , \mathbf{a} , b , g and \mathbf{h} are given data. The last relations in (2.1) are zero-fluxes conditions across the connected components Γ_N^ℓ and the cuts Σ_j respectively and are necessary for the uniqueness of the solution.

Theorem 2.1. *(i) Assume that $\partial\Sigma \subset \Gamma_D$. Let $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$ with $3/2 \leq p < +\infty$, $\mathbf{a} \times \mathbf{n} \in \mathbf{W}^{1-1/p,p}(\Gamma_N)$, $g \in W^{-1/p,p}(\Gamma_N)$, $b \in W^{-1-1/p,p}(\Gamma_D)$ and $\mathbf{h} \times \mathbf{n} \in \mathbf{W}^{-1/p,p}(\Gamma_D)$ satisfying the compatibility condition*

$$\int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi} \, dx - \langle \mathbf{h} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\Gamma_D} + \langle g, \boldsymbol{\varphi} \cdot \mathbf{n} \rangle_{\Gamma_N} = 0.$$

Then Problem (2.1) has a unique solution $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ for any $1 < p < +\infty$ and it satisfies

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} &\leq C \left(\|\mathbf{f}\|_{\mathbf{L}^{r(p)}(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma_N)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma_D)} \right. \\ &\quad \left. + \|\mathbf{a} \times \mathbf{n}\|_{\mathbf{W}^{1-1/p,p}(\Gamma_N)} + \|b\|_{W^{-1-1/p,p}(\Gamma_D)} \right). \end{aligned}$$

(ii) If $\mathbf{f} \in \mathbf{L}^1(\Omega)$, then there exists a unique $\mathbf{u} \in \mathbf{W}^{1,s}(\Omega)$ for any $s < \frac{3}{2}$ solution of Problem (2.1) and satisfying

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{W}^{1,s}(\Omega)} \leq C & \left(\|\mathbf{f}\|_{\mathbf{L}^1(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma_N)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma_D)} \right. \\ & \left. + \|\mathbf{a} \times \mathbf{n}\|_{\mathbf{W}^{1-1/p,p}(\Gamma_N)} + \|b\|_{W^{1-1/p,p}(\Gamma_D)} \right). \end{aligned}$$

(iii) Furthermore, if Ω is of class $\mathcal{C}^{2,1}$, $\mathbf{f} \in \mathbf{L}^p(\Omega)$, $\mathbf{a} \times \mathbf{n} \in \mathbf{W}^{2-1/p,p}(\Gamma_N)$, $g \in W^{1-1/p,p}(\Gamma_N)$, $b \in W^{2-1/p,p}(\Gamma_D)$, and $\mathbf{h} \times \mathbf{n} \in \mathbf{W}^{1-1/p,p}(\Gamma_D)$, then \mathbf{u} belongs to $\mathbf{W}^{2,p}(\Omega)$ and

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{W}^{2,p}(\Omega)} \leq C & \left(\|\mathbf{f}\|_{\mathbf{L}^p(\Omega)} + \|g\|_{W^{1-1/p,p}(\Gamma_N)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{1-1/p,p}(\Gamma_D)} \right. \\ & \left. + \|\mathbf{a} \times \mathbf{n}\|_{\mathbf{W}^{2-1/p,p}(\Gamma_N)} + \|b\|_{W^{2-1/p,p}(\Gamma_D)} \right). \end{aligned}$$

The proof of this theorem is given in Section 4 using an inf-sup condition established in Lemma 4.7. We also discuss the case where $\partial\Sigma \subset \Gamma_N$ in the same section.

The second main result concerns the elliptic system (1.1)-(1.7)-(1.6) in which we assume a more general right-hand side and the corresponding boundary conditions. The proof of this main result is detailed in Section 5.

$$\begin{aligned} -\Delta \mathbf{u} &= \mathbf{f} + \operatorname{div} \mathbb{F} \quad \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} &= \mathbf{0}, \quad [(\nabla \mathbf{u} + \mathbb{F})\mathbf{n}] \cdot \mathbf{n} = g \quad \text{on } \Gamma_N, \\ \mathbf{u} \cdot \mathbf{n} &= 0, \quad [(\nabla \mathbf{u} + \mathbb{F})\mathbf{n}] \times \mathbf{n} = \mathbf{h} \quad \text{on } \Gamma_D. \end{aligned} \tag{2.2}$$

where \mathbf{f} , g and \mathbf{h} are given data.

Theorem 2.2. (i) Assume that $\partial\Sigma \subset \Gamma_D$ or $\partial\Sigma \subset \Gamma_N$. Let $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$, $\mathbb{F} \in \mathbb{L}^p(\Omega)$, $g \in W^{-1/p,p}(\Gamma_N)$ and $\mathbf{h} \in \mathbf{W}^{-1/p,p}(\Gamma_D)$ then Problem (2.2) has a unique solution $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ and it satisfies

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \left(\|\mathbf{f}\|_{\mathbf{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma_N)} + \|\mathbf{h}\|_{\mathbf{W}^{-1/p,p}(\Gamma_D)} \right).$$

(ii) Furthermore, if Ω is of class $\mathcal{C}^{2,1}$, $\mathbf{f} \in \mathbf{L}^p(\Omega)$, $g \in W^{1-1/p,p}(\Gamma_N)$, and $\mathbf{h} \in \mathbf{W}^{1-1/p,p}(\Gamma_D)$ with $\mathbb{F} = 0$, then \mathbf{u} belongs to $\mathbf{W}^{2,p}(\Omega)$ and

$$\|\mathbf{u}\|_{\mathbf{W}^{2,p}(\Omega)} \leq C \left(\|\mathbf{f}\|_{\mathbf{L}^p(\Omega)} + \|g\|_{W^{1-1/p,p}(\Gamma_N)} + \|\mathbf{h}\|_{\mathbf{W}^{1-1/p,p}(\Gamma_D)} \right).$$

Note that the assumptions $\partial\Sigma \subset \Gamma_D$ or $\partial\Sigma \subset \Gamma_N$ are related to the chosen method.

3. PRELIMINARIES

In this section, we set some basic notation, we introduce the functional settings and we recall some tools necessary for our analysis. We follow the convention that C is a constant that may vary from expression to expression. We denote by X' the dual space of the space X and by $\langle \cdot, \cdot \rangle_{X',X}$ the duality product between X and X' . Vector fields are designated by bold letters and their corresponding spaces by bold capital characters.

We consider a bounded Lipschitz domain denoted ω . We assume that ω contains a number of obstacles denoted $\Omega_1, \Omega_2, \dots, \Omega_I$, some of them may be non-simply connected. We set

$$\Omega = \omega \setminus \left(\bigcup_{i=1}^I \bar{\Omega}_i \right).$$

The boundary of Ω is decomposed into two parts Γ_D and Γ_N , i.e.

$$\Gamma = \Gamma_D \cup \Gamma_N, \quad \Gamma_D \cap \Gamma_N = \emptyset.$$

More precisely,

$$\Gamma = \left(\bigcup_{\ell=0}^{L_D} \Gamma_D^\ell \right) \cup \left(\bigcup_{\ell=0}^{L_N} \Gamma_N^\ell \right),$$

where Γ_D^ℓ , $0 \leq \ell \leq L_D$ are the connected components of Γ_D and similarly Γ_N^ℓ , $0 \leq \ell \leq L_N$ are the connected components of Γ_N . We assume that there exists J connected open surfaces Σ_j , $1 \leq j \leq J$, called ‘‘cuts’’ contained in Ω such that

- (i) each surface Σ_j is an open part of a smooth manifold M_j
- (ii) the boundary of Σ_j is contained in $\partial\Omega$
- (iii) the intersection $\Sigma_i \cap \Sigma_j$ is empty for $i \neq j$
- (iv) the open set $\Omega^o = \Omega \setminus \cup_{j=1}^J \Sigma_j$ is simply-connected.

We denote by Γ_Σ^j the boundary of each connected open surface Σ_j and we have

$$\partial\Sigma = \cup_{j=1}^J \Gamma_\Sigma^j$$

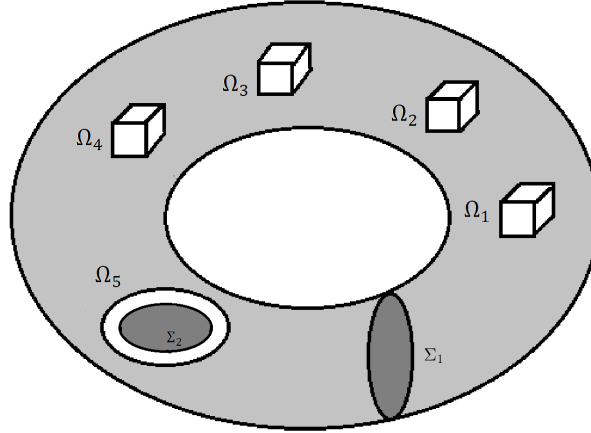


FIGURE 1. Lipschitz domain Ω

Note that we consider only the cases where $\partial\Sigma \subset \Gamma_N$ or $\partial\Sigma \subset \Gamma_D$. The case where $\partial\Sigma$ is included in Γ_D and in Γ_N in the same time is not handled in this article.

Referring Figure 1, when $\partial\Sigma \subset \Gamma_D$ (resp Γ_N), this means that $\partial\omega \cup \partial\Omega_5$ is included in Γ_D (resp Γ_N).

We denote by $[\cdot]_j$ the jump of a function over Σ_j , i.e. the differences of the traces for any $1 \leq j \leq J$. For any function $q \in \mathbf{W}^{1,p}(\Omega^o)$, ∇q is the gradient of q in the sense of distributions in $\mathcal{D}'(\Omega^o)$ which belongs to $\mathbf{L}^p(\Omega^o)$ and it can be extended to $\mathbf{L}^p(\Omega)$. Therefore, to distinguish this extension from the gradient of q , we denote it by $\widetilde{\text{grad}}q$.

We introduce the functional framework:

$$\begin{aligned} \mathbf{H}^p(\text{curl}, \Omega) &= \{\mathbf{v} \in \mathbf{L}^p(\Omega) : \text{curl } \mathbf{v} \in \mathbf{L}^p(\Omega)\}, \\ \mathbf{H}^p(\text{div}, \Omega) &= \{\mathbf{v} \in \mathbf{L}^p(\Omega) : \text{div } \mathbf{v} \in \mathbf{L}^p(\Omega)\} \end{aligned}$$

and we denote by $\mathbf{X}^p(\Omega)$ the space

$$\mathbf{X}^p(\Omega) = \mathbf{H}^p(\text{curl}, \Omega) \cap \mathbf{H}^p(\text{div}, \Omega)$$

provided with the norm for $1 < p < \infty$

$$\|\mathbf{v}\|_{\mathbf{X}^p(\Omega)} = \left(\|\mathbf{v}\|_{\mathbf{L}^p(\Omega)}^p + \|\text{curl } \mathbf{v}\|_{\mathbf{L}^p(\Omega)}^p + \|\text{div } \mathbf{v}\|_{\mathbf{L}^p(\Omega)}^p \right)^{1/p}.$$

The space $\mathcal{D}(\overline{\Omega})$ is dense in both spaces $\mathbf{H}^p(\text{curl}, \Omega)$ and $\mathbf{H}^p(\text{div}, \Omega)$ and the closure of $\mathcal{D}(\Omega)$ in $\mathbf{H}^p(\text{curl}, \Omega)$ and $\mathbf{H}^p(\text{div}, \Omega)$ is respectively denoted by $\mathbf{H}_0^p(\text{curl}, \Omega)$ and $\mathbf{H}_0^p(\text{div}, \Omega)$. We also define the subspaces

$$\begin{aligned} \mathbf{X}_T^p(\Omega) &= \{\mathbf{v} \in \mathbf{X}^p(\Omega) : \mathbf{v} \cdot \mathbf{n} = \mathbf{0} \text{ on } \Gamma\}, & \mathbf{X}_N^p(\Omega) &= \{\mathbf{v} \in \mathbf{X}^p(\Omega) : \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma\}, \\ \mathbf{X}_0^p(\Omega) &= \{\mathbf{v} \in \mathbf{X}^p(\Omega) : \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_D, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_N\}, \\ \widetilde{\mathbf{X}}_0^p(\Omega) &= \{\mathbf{v} \in \mathbf{X}^p(\Omega) : \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_N, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_D\}. \end{aligned}$$

Note that as long as \mathbf{u} belongs to $\mathbf{H}^p(\text{curl}, \Omega)$, the tangential boundary value of \mathbf{u} is defined in $\mathbf{W}^{-\frac{1}{p}, p}(\Gamma)$ and in the case where \mathbf{u} belongs to $\mathbf{H}^p(\text{div}, \Omega)$, the normal boundary value is also defined in $W^{-\frac{1}{p}, p}(\Gamma)$. Moreover, we have the Green's formulas

$$\forall \varphi \in \mathbf{W}^{1, p'}(\Omega), \langle \mathbf{u} \times \mathbf{n}, \varphi \rangle_{\Gamma} = \int_{\Omega} \mathbf{u} \cdot \text{curl } \varphi \, dx - \int_{\Omega} \text{curl } \mathbf{u} \cdot \varphi \, dx,$$

where $\langle \cdot, \cdot \rangle_{\Gamma}$ denotes the duality product between $\mathbf{W}^{-\frac{1}{p}, p}(\Gamma)$ and $\mathbf{W}^{\frac{1}{p}, p'}(\Gamma)$ and

$$\forall \varphi \in W^{1, p'}(\Omega), \langle \mathbf{u} \cdot \mathbf{n}, \varphi \rangle_{\Gamma} = \int_{\Omega} \mathbf{u} \cdot \nabla \varphi \, dx + \int_{\Omega} (\text{div } \mathbf{u}) \varphi \, dx,$$

where $\langle \cdot, \cdot \rangle_{\Gamma}$ denotes the duality product between $W^{-\frac{1}{p}, p}(\Gamma)$ and $W^{\frac{1}{p}, p'}(\Gamma)$. It is also necessary to remind the following regularity lemma given in [7].

Lemma 3.1. *Let $m \in \mathbb{N}^*$ and Ω of class $\mathcal{C}^{m,1}$. Then the spaces*

$$X^{m, p}(\Omega) = \{ \mathbf{v} \in \mathbf{L}^p(\Omega) : \text{div } \mathbf{v} \in W^{m-1, p}(\Omega), \text{curl } \mathbf{v} \in \mathbf{W}^{m-1, p}(\Omega), \mathbf{v} \cdot \mathbf{n} \in W^{m-\frac{1}{p}, p}(\Gamma) \}$$

and

$$Y^{m, p}(\Omega) = \{ \mathbf{v} \in \mathbf{L}^p(\Omega) : \text{div } \mathbf{v} \in W^{m-1, p}(\Omega), \text{curl } \mathbf{v} \in \mathbf{W}^{m-1, p}(\Omega), \mathbf{v} \times \mathbf{n} \in \mathbf{W}^{m-\frac{1}{p}, p}(\Gamma) \}$$

are continuously embedded in $\mathbf{W}^{m, p}(\Omega)$. For each \mathbf{v} in $W^{m, p}(\Omega)$, we have the following estimates

$$\|\mathbf{v}\|_{W^{m, p}(\Omega)} \leq C(\|\mathbf{v}\|_{\mathbf{L}^p(\Omega)} + \|\text{curl } \mathbf{v}\|_{\mathbf{W}^{m-1, p}(\Omega)} + \|\text{div } \mathbf{v}\|_{W^{m-1, p}(\Omega)} + \|\mathbf{v} \cdot \mathbf{n}\|_{W^{m-\frac{1}{p}, p}(\Gamma)}), \quad (3.1)$$

$$\|\mathbf{v}\|_{W^{m, p}(\Omega)} \leq C(\|\mathbf{v}\|_{\mathbf{L}^p(\Omega)} + \|\text{curl } \mathbf{v}\|_{\mathbf{W}^{m-1, p}(\Omega)} + \|\text{div } \mathbf{v}\|_{W^{m-1, p}(\Omega)} + \|\mathbf{v} \times \mathbf{n}\|_{\mathbf{W}^{m-\frac{1}{p}, p}(\Gamma)}). \quad (3.2)$$

We define the kernels

$$\begin{aligned} \mathbf{K}_T^p(\Omega) &= \{ \mathbf{v} \in \mathbf{X}_T^p(\Omega), \text{div } \mathbf{v} = 0, \text{curl } \mathbf{v} = \mathbf{0} \text{ in } \Omega \} \\ \mathbf{K}_N^p(\Omega) &= \{ \mathbf{v} \in \mathbf{X}_N^p(\Omega), \text{div } \mathbf{v} = 0, \text{curl } \mathbf{v} = \mathbf{0} \text{ in } \Omega \}, \end{aligned}$$

and the kernels when mixing normal and tangential components,

$$\begin{aligned} \mathbf{K}_0^p(\Omega) &= \{ \mathbf{v} \in \mathbf{X}_0^p(\Omega), \text{div } \mathbf{v} = 0, \text{curl } \mathbf{v} = \mathbf{0} \text{ in } \Omega \}, \\ \widetilde{\mathbf{K}}_0^p(\Omega) &= \{ \mathbf{v} \in \widetilde{\mathbf{X}}_0^p(\Omega), \text{div } \mathbf{v} = 0, \text{curl } \mathbf{v} = \mathbf{0} \text{ in } \Omega \}. \end{aligned}$$

When $\partial\Omega \subset \Gamma_D$, the kernel $\widetilde{\mathbf{K}}_0^p(\Omega)$ is spanned by the functions $\widetilde{\text{grad}} s_j^\ell$ with $1 \leq j \leq J$ and $1 \leq \ell \leq L_N$, where each s_j^ℓ is the unique solution in $H^1(\Omega^o)$ of the following problem (see [4, Proposition 3.13])

$$\begin{aligned} -\Delta s_j^\ell &= 0 \quad \text{in } \Omega^o, \\ \frac{\partial s_j^\ell}{\partial \mathbf{n}} &= 0 \quad \text{on } \Gamma_D, \\ s_j^\ell|_{\Gamma_N^o} &= 0 \quad \text{and} \quad s_j^\ell|_{\Gamma_N^m} = \text{const}, \quad 1 \leq m \leq L_N, \\ [s_j^\ell]_k &= \text{const} \quad \text{and} \quad \left[\frac{\partial s_j^\ell}{\partial \mathbf{n}} \right]_k = 0, \quad 1 \leq k \leq J, \\ \left\langle \frac{\partial s_j^\ell}{\partial \mathbf{n}}, 1 \right\rangle_{\Sigma_k} &= \delta_{jk}, \quad 1 \leq k \leq J, \\ \left\langle \frac{\partial s_j^\ell}{\partial \mathbf{n}}, 1 \right\rangle_{\Gamma_N^o} &= -1 \quad \text{and} \quad \left\langle \frac{\partial s_j^\ell}{\partial \mathbf{n}}, 1 \right\rangle_{\Gamma_N^m} = \delta_{\ell m}, \quad 1 \leq m \leq L_N. \end{aligned} \quad (3.3)$$

When $\partial\Sigma \subset \Gamma_N$, the space $\widetilde{\mathbf{K}}_0^p(\Omega)$ is spanned by the functions ∇s_ℓ with $1 \leq \ell \leq L_N$ where each s_ℓ is the unique solution in $H^1(\Omega)$, of the following problem (see [4])

$$\begin{aligned} -\Delta s_\ell &= 0 \quad \text{in } \Omega, \\ \frac{\partial s_\ell}{\partial \mathbf{n}} &= 0 \quad \text{on } \Gamma_D, \\ s_\ell|_{\Gamma_N^0} &= 0 \quad \text{and} \quad s_\ell|_{\Gamma_N^m} = \text{const}, \quad 1 \leq m \leq L_N, \\ \left\langle \frac{\partial s_\ell}{\partial \mathbf{n}}, 1 \right\rangle_{\Gamma_N^0} &= -1 \quad \text{and} \quad \left\langle \frac{\partial s_\ell}{\partial \mathbf{n}}, 1 \right\rangle_{\Gamma_N^m} = \delta_{\ell m}, \quad 1 \leq m \leq L_N. \end{aligned} \tag{3.4}$$

The kernel $\mathbf{K}_0^p(\Omega)$ is spanned by the functions $\widetilde{\text{grad}} q_j^\ell$, $1 \leq j \leq J$, $1 \leq \ell \leq L_D$ when $\partial\Sigma \subset \Gamma_N$ and by ∇q_ℓ , $1 \leq \ell \leq L_D$ when $\partial\Sigma \subset \Gamma_D$ satisfying the same properties as $\widetilde{\text{grad}} s_j^\ell$ and ∇s_ℓ when Γ_D and Γ_N are swapped. We underline that the kernels $\mathbf{K}_0^p(\Omega)$ and $\widetilde{\mathbf{K}}_0^p(\Omega)$ do not depend on p in a $\mathcal{C}^{1,1}$ domain.

We recall the inf-sup conditions obtained in [4] in the case where $\partial\Sigma$ is included either in Γ_N or in Γ_D .

Proposition 3.2. (i) *If $\partial\Sigma \subset \Gamma_N$, there exists a constant $\beta_1 > 0$ depending only on Ω and p , such that the following inf-sup condition holds*

$$\inf_{\varphi \in \widetilde{\mathbf{V}}_{0,\sigma}^{p'}(\Omega), \varphi \neq 0} \left[\sup_{\xi \in \widetilde{\mathbf{V}}_{0,\sigma}^p(\Omega), \xi \neq 0} \frac{\left| \int_\Omega \text{curl } \xi \cdot \text{curl } \varphi \right|}{\|\xi\|_{\mathbf{W}^{1,p}(\Omega)} \|\varphi\|_{\mathbf{W}^{1,p'}(\Omega)}} \right] \geq \beta_1. \tag{3.5}$$

(ii) *If $\partial\Sigma \subset \Gamma_D$, there exists a constant $\beta_2 > 0$ depending only on Ω and p , such that the following inf-sup condition holds*

$$\inf_{\varphi \in \widetilde{\mathbf{W}}_{\Sigma,\sigma}^{p'}(\Omega), \varphi \neq 0} \left[\sup_{\xi \in \widetilde{\mathbf{W}}_{\Sigma,\sigma}^p(\Omega), \xi \neq 0} \frac{\left| \int_\Omega \text{curl } \xi \cdot \text{curl } \varphi \right|}{\|\xi\|_{\mathbf{W}^{1,p}(\Omega)} \|\varphi\|_{\mathbf{W}^{1,p'}(\Omega)}} \right] \geq \beta_2, \tag{3.6}$$

where

$$\widetilde{\mathbf{V}}_{0,\sigma}^p(\Omega) = \{ \mathbf{v} \in \widetilde{\mathbf{X}}_0^p(\Omega), \text{div } \mathbf{v} = 0 \text{ in } \Omega, \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} = 0, 1 \leq \ell \leq L_N \},$$

$$\widetilde{\mathbf{W}}_{\Sigma,\sigma}^p(\Omega) = \{ \mathbf{v} \in \widetilde{\mathbf{X}}_0^p(\Omega), \text{div } \mathbf{v} = 0 \text{ in } \Omega, \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, 1 \leq j \leq J, \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} = 0, 0 \leq \ell \leq L_N \}.$$

We review some important estimates established in [4]. Note that the space $\widetilde{\mathbf{X}}_0^p(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$.

Theorem 3.3. *Assume that Ω is Lipschitz (resp. $\mathcal{C}^{1,1}$).*

(i) *If $\partial\Sigma \subset \Gamma_N$, then on the space $\widetilde{\mathbf{X}}_0^p(\Omega)$, the semi-norm*

$$\mathbf{u} \mapsto \|\text{div } \mathbf{u}\|_{L^p(\Omega)} + \|\text{curl } \mathbf{u}\|_{\mathbf{L}^p(\Omega)} + \sum_{\ell=1}^{L_N} |\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell}| \tag{3.7}$$

is a norm equivalent to the norm $\|\cdot\|_{\mathbf{X}^p(\Omega)}$ (resp. $\|\cdot\|_{\mathbf{W}^{1,p}(\Omega)}$) on $\widetilde{\mathbf{X}}_0^p(\Omega)$.

(ii) *If $\partial\Sigma \subset \Gamma_D$, the semi-norm*

$$\mathbf{u} \mapsto \|\text{div } \mathbf{u}\|_{L^p(\Omega)} + \|\text{curl } \mathbf{u}\|_{\mathbf{L}^p(\Omega)} + \sum_{\ell=1}^{L_N} |\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell}| + \sum_{j=1}^J |\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j}|, \tag{3.8}$$

is a norm equivalent to the norm $\|\cdot\|_{\mathbf{X}^p(\Omega)}$ (resp. $\|\cdot\|_{\mathbf{W}^{1,p}(\Omega)}$) on $\widetilde{\mathbf{X}}_0^p(\Omega)$.

The following lemma has been proved for non mixed boundary conditions where Ω is of class $\mathcal{C}^{1,1}$ (see [2]) and the result remains valid when mixing Dirichlet and Neumann boundary conditions.

Lemma 3.4. *Let $\chi \in L^p(\Omega)$ such that $\Delta\chi = 0$ in Ω , $\chi = 0$ on Γ_N , $\frac{\partial\chi}{\partial\mathbf{n}} = 0$ on Γ_D . Then $\chi = 0$ in Ω .*

We also recall the following result [3, Lemma 3.2].

Lemma 3.5. *Suppose that Ω is Lipschitz and let $\mathbf{w} \in \mathbf{L}^p(\Omega)$ be such that $\text{curl } \mathbf{w} \in \mathbf{L}^p(\Omega)$ and $\mathbf{w} \times \mathbf{n} \in \mathbf{L}^p(\Gamma)$. Then $\text{div}_\Gamma(\mathbf{w} \times \mathbf{n}) \in W^{-1/p,p}(\Gamma)$ and*

$$\text{div}_\Gamma(\mathbf{w} \times \mathbf{n}) = \text{curl } \mathbf{w} \cdot \mathbf{n} \quad \text{on } \Gamma. \quad (3.9)$$

Let us introduce the Banach space

$$\mathbf{E}^p(\Omega; \Delta) = \{\mathbf{u} \in \mathbf{W}^{1,p}(\Omega) : \Delta \mathbf{u} \in \mathbf{L}^{r(p)}(\Omega)\},$$

equipped with the graph norm and the following mappings

$$\begin{aligned} \ell_1 : \mathcal{D}(\overline{\Omega}) &\rightarrow \mathbf{W}^{-1/p,p}(\Gamma) \quad \text{as } \ell_1(\mathbf{u}) = \text{curl } \mathbf{u} \times \mathbf{n}, \\ \ell_2 : \mathcal{D}(\overline{\Omega}) &\rightarrow \mathbf{W}^{-1/p,p}(\Gamma) \quad \text{as } \ell_2(\mathbf{u}) = \text{div } \mathbf{u}. \end{aligned}$$

A vector-valued Laplace operator of a vector field \mathbf{u} is equivalently defined as

$$\Delta \mathbf{u} = \nabla(\text{div } \mathbf{u}) - \text{curl } \text{curl } \mathbf{u}.$$

Lemma 3.6. (i) *The space $\mathcal{D}(\overline{\Omega})$ is dense in the space $\mathbf{E}^p(\Omega; \Delta)$.*

(ii) *The linear mappings ℓ_1 and ℓ_2 can be extended to linear and continuous mappings from $\mathbf{E}^p(\Omega; \Delta)$ to $\mathbf{W}^{-1/p,p}(\Gamma)$ and from $\mathbf{E}^p(\Omega; \Delta)$ to $W^{-1/p,p}(\Gamma)$ respectively.*

(iii) *Furthermore, the following Green's formula holds for any $\mathbf{u} \in \mathbf{E}^p(\Omega; \Delta)$ and $\varphi \in \mathbf{W}^{1,p'}(\Omega)$ such that $\varphi \times \mathbf{n} = \mathbf{0}$ on Γ_N and $\varphi \cdot \mathbf{n} = 0$ on Γ_D*

$$-\int_{\Omega} \Delta \mathbf{u} \cdot \varphi = \int_{\Omega} \text{curl } \mathbf{u} \cdot \text{curl } \varphi + \int_{\Omega} (\text{div } \mathbf{u})(\text{div } \varphi) + \langle \text{curl } \mathbf{u} \times \mathbf{n}, \varphi \rangle_{\Gamma_D} - \langle \text{div } \mathbf{u}, \varphi \cdot \mathbf{n} \rangle_{\Gamma_N}, \quad (3.10)$$

where $\langle \cdot, \cdot \rangle_{\Gamma_D}$ denotes the duality product between $\mathbf{W}^{-1/p,p}(\Gamma_D)$ and $\mathbf{W}^{1/p,p'}(\Gamma_D)$ and $\langle \cdot, \cdot \rangle_{\Gamma_N}$ the duality product between $W^{-1/p,p}(\Gamma_N)$ and $W^{1/p,p'}(\Gamma_N)$.

Proof. (i) The proof of the density of $\mathcal{D}(\overline{\Omega})$ into $\mathbf{E}^p(\Omega; \Delta)$ is similar to the one given by Grisvard in [12, Lemma 1.5.3.9, page 59]grisvard.

(ii) Let $\mathbf{u} \in \mathcal{D}(\overline{\Omega})$ and $\varphi \in \mathbf{W}^{1,p'}(\Omega)$, then

$$-\int_{\Omega} \Delta \mathbf{u} \cdot \varphi = \int_{\Omega} \text{curl } \mathbf{u} \cdot \text{curl } \varphi + \int_{\Omega} (\text{div } \mathbf{u})(\text{div } \varphi) + \langle \text{curl } \mathbf{u} \times \mathbf{n}, \varphi \rangle_{\Gamma} - \langle \text{div } \mathbf{u}, \varphi \cdot \mathbf{n} \rangle_{\Gamma}. \quad (3.11)$$

Let $\boldsymbol{\mu} \in \mathbf{W}^{1/p,p'}(\Gamma)$. We know that there exists $\boldsymbol{\psi} \in \mathbf{W}^{1,p'}(\Omega)$ such that

$$\text{div } \boldsymbol{\psi} = 0 \text{ in } \Omega, \quad \boldsymbol{\psi} = \boldsymbol{\mu}_\tau \text{ on } \Gamma,$$

satisfying the estimate

$$\|\boldsymbol{\psi}\|_{\mathbf{W}^{1,p'}(\Omega)} \leq C \|\boldsymbol{\mu}_\tau\|_{\mathbf{W}^{1/p,p'}(\Gamma)} \leq C \|\boldsymbol{\mu}\|_{\mathbf{W}^{1/p,p'}(\Gamma)}.$$

We infer that

$$\langle \text{curl } \mathbf{u} \times \mathbf{n}, \boldsymbol{\mu} \rangle_{\Gamma} = \langle \text{curl } \mathbf{u} \times \mathbf{n}, \boldsymbol{\mu}_\tau \rangle_{\Gamma} = \langle \text{curl } \mathbf{u} \times \mathbf{n}, \boldsymbol{\psi} \rangle_{\Gamma}.$$

By replacing in (3.11), we obtain

$$\langle \text{curl } \mathbf{u} \times \mathbf{n}, \boldsymbol{\mu} \rangle_{\Gamma} = -\int_{\Omega} \Delta \mathbf{u} \cdot \boldsymbol{\psi} - \int_{\Omega} \text{curl } \mathbf{u} \cdot \text{curl } \boldsymbol{\psi},$$

$$|\langle \text{curl } \mathbf{u} \times \mathbf{n}, \boldsymbol{\mu} \rangle_{\Gamma}| \leq C \|\mathbf{u}\|_{\mathbf{E}^p(\Omega; \Delta)} \|\boldsymbol{\psi}\|_{\mathbf{W}^{1,p'}(\Omega)} \leq C \|\mathbf{u}\|_{\mathbf{E}^p(\Omega; \Delta)} \|\boldsymbol{\mu}\|_{\mathbf{W}^{1/p,p'}(\Gamma)}.$$

Consequently,

$$\|\text{curl } \mathbf{u} \times \mathbf{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma)} \leq C \|\mathbf{u}\|_{\mathbf{E}^p(\Omega; \Delta)},$$

which means that the linear mapping ℓ_1 is continuous in $\mathcal{D}(\overline{\Omega})$ for the graph norm of $\mathbf{E}^p(\Omega; \Delta)$. Since the space $\mathcal{D}(\overline{\Omega})$ is dense in $\mathbf{E}^p(\Omega; \Delta)$, the mapping ℓ_1 can be extended to a linear and continuous one from $\mathbf{E}^p(\Omega; \Delta)$ to $\mathbf{W}^{-1/p,p}(\Gamma_D)$.

Now, let $\mu \in W^{1/p,p'}(\Gamma)$. We know that there exists $\psi \in W^{2,p'}(\Omega)$ such that

$$\psi = 0 \text{ on } \Gamma, \quad \frac{\partial \psi}{\partial \mathbf{n}} = \mu \text{ on } \Gamma,$$

satisfying

$$\|\psi\|_{W^{2,p'}(\Omega)} \leq C \|\mu\|_{W^{1/p,p'}(\Gamma)}.$$

By setting $\varphi = \nabla\psi$ in (3.11), we have

$$\langle \operatorname{div} \mathbf{u}, \mu \rangle_\Gamma = \int_\Omega \Delta \mathbf{u} \cdot \nabla \psi + \int_\Omega (\operatorname{div} \mathbf{u})(\Delta \psi),$$

which yields

$$|\langle \operatorname{div} \mathbf{u}, \mu \rangle_\Gamma| \leq C \|\mathbf{u}\|_{\mathbf{E}^p(\Omega; \Delta)} \|\psi\|_{W^{2,p'}(\Omega)} \leq C \|\mathbf{u}\|_{\mathbf{E}^p(\Omega; \Delta)} \|\mu\|_{W^{1/p,p'}(\Gamma)}.$$

As above, the mapping ℓ_2 can be extended to a linear and continuous one from $\mathbf{E}^p(\Omega; \Delta)$ to $W^{-1/p,p}(\Gamma)$.

(iii) From the density of $\mathcal{D}(\overline{\Omega})$ in $\mathbf{E}^p(\Omega; \Delta)$ and the continuity of ℓ_1 and ℓ_2 , we deduce that the Green's formula (3.11) is valid for any $\mathbf{u} \in \mathbf{E}^p(\Omega; \Delta)$ and $\varphi \in \mathbf{W}^{1,p'}(\Omega)$. Taking $\varphi \times \mathbf{n} = \mathbf{0}$ on Γ_N and $\varphi \cdot \mathbf{n} = 0$ on Γ_D in (3.11) leads readily to (3.10). \square

4. FIRST ELLIPTIC SYSTEM

The aim of this section is to study the elliptic system where we assume that $\partial\Sigma \subset \Gamma_D$,

$$\begin{aligned} -\Delta \mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} &= \mathbf{a} \times \mathbf{n}, \quad \operatorname{div} \mathbf{u} = g \quad \text{on } \Gamma_N, \\ \mathbf{u} \cdot \mathbf{n} &= b, \quad \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma_D, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} &= 0, \quad 1 \leq \ell \leq L_N, \quad \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J, \end{aligned} \tag{4.1}$$

where \mathbf{f} , \mathbf{a} , b , g and \mathbf{h} are given data.

Before solving the general problem (4.1), we start with the case of homogeneous boundary conditions given by

$$\begin{aligned} -\Delta \mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} &= \mathbf{0}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{on } \Gamma_N, \\ \mathbf{u} \cdot \mathbf{n} &= 0, \quad \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_D, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} &= 0, \quad 1 \leq \ell \leq L_N, \quad \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J. \end{aligned} \tag{4.2}$$

We consider the space

$$\widetilde{\mathbf{W}}_\Sigma^p(\Omega) = \{ \mathbf{v} \in \widetilde{\mathbf{X}}_0^p(\Omega) : \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, 1 \leq j \leq J, \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} = 0, 0 \leq \ell \leq L_N \}.$$

4.1. Hilbertian case. In the following theorem, we prove the existence and uniqueness of the weak solution of Problem (4.2) in the Hilbert space. We also discuss its strong solutions.

Theorem 4.1. (i) Assume that $\partial\Sigma \subset \Gamma_D$. Let $\mathbf{f} \in \mathbf{L}^{6/5}(\Omega)$ satisfying the following compatibility condition for any $\varphi \in \widetilde{\mathbf{K}}_0^2(\Omega)$:

$$\int_\Omega \mathbf{f} \cdot \varphi \, dx = 0. \tag{4.3}$$

Then, Problem (4.2) admits a unique solution $\mathbf{u} \in \mathbf{H}^1(\Omega)$ satisfying

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \leq C \|\mathbf{f}\|_{\mathbf{L}^{6/5}(\Omega)}. \tag{4.4}$$

(ii) Furthermore, if Ω is of class $\mathcal{C}^{2,1}$ and $\mathbf{f} \in \mathbf{L}^2(\Omega)$ then \mathbf{u} belongs to $\mathbf{H}^2(\Omega)$ and satisfies the estimate

$$\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \leq C \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}. \tag{4.5}$$

Proof. (i) Since it is rather long, we split the proof into four steps.

Step 1: Compatibility condition. The weak formulation of (4.2) is

$$\begin{aligned} \text{Find } \mathbf{u} \in \widetilde{\mathbf{W}}_\Sigma^2(\Omega) \text{ such that for any } \varphi \in \widetilde{\mathbf{W}}_\Sigma^2(\Omega), \\ \int_\Omega \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \varphi + \int_\Omega (\operatorname{div} \mathbf{u})(\operatorname{div} \varphi) = \int_\Omega \mathbf{f} \cdot \varphi. \end{aligned} \tag{4.6}$$

As it will be shown is Step 3 that Problem (4.2) and Problem (4.6) are equivalent, so the first compatibility condition (4.3) appears directly by taking $\varphi \in \widetilde{\mathbf{K}}_0^2(\Omega)$ in (4.6).

Step 2: Existence and uniqueness. We define the bilinear form $a(\cdot, \cdot)$ as

$$a(\mathbf{u}, \boldsymbol{\varphi}) = \int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \boldsymbol{\varphi} + \int_{\Omega} (\operatorname{div} \mathbf{u})(\operatorname{div} \boldsymbol{\varphi}),$$

which is coercive on $\widetilde{\mathbf{W}}_{\Sigma}^2(\Omega)$ because (3.8). Then, by using Lax Milgram's theorem we infer that Problem (4.6) admits a unique solution in $\widetilde{\mathbf{W}}_{\Sigma}^2(\Omega)$. Moreover,

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}^2 \leq C \|\mathbf{f}\|_{\mathbf{L}^{6/5}(\Omega)} \|\mathbf{u}\|_{\mathbf{L}^6(\Omega)} \leq C \|\mathbf{f}\|_{\mathbf{L}^{6/5}(\Omega)} \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)},$$

and the required estimate (4.4) is immediately checked.

Step 3: Equivalence. We show now that Problem (4.6) is equivalent to

$$\begin{aligned} &\text{Find } \mathbf{u} \in \widetilde{\mathbf{W}}_{\Sigma}^2(\Omega) \text{ such that for any } \boldsymbol{\varphi} \in \widetilde{\mathbf{X}}_0^2(\Omega), \\ &\int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \boldsymbol{\varphi} + \int_{\Omega} (\operatorname{div} \mathbf{u})(\operatorname{div} \boldsymbol{\varphi}) = \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi}. \end{aligned} \quad (4.7)$$

It is clear that any solution of (4.7) also solves (4.6). Conversely, we take $\boldsymbol{\varphi} \in \widetilde{\mathbf{X}}_0^2(\Omega)$ and we define $\widetilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \nabla \theta$ where θ is solution of the problem

$$\Delta \theta = \operatorname{div} \boldsymbol{\varphi} \text{ in } \Omega, \quad \theta = 0 \text{ on } \Gamma_N, \quad \frac{\partial \theta}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_D. \quad (4.8)$$

Since $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$, we can prove that θ belongs to $H^2(\Omega)$ by an argument of partition of unity leading to two problems with Dirichlet or Neumann boundary condition in the non mixed case.

Now, we can define the function

$$\widetilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \sum_{\ell=1}^{L_N} \sum_{j=1}^J \left(\frac{1}{L_N} \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} + \frac{1}{J} \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^{\ell}} \right) \widetilde{\operatorname{grad}} s_j^{\ell}, \quad (4.9)$$

It is obvious that $\widetilde{\boldsymbol{\varphi}} \in \mathbf{L}^2(\Omega)$, $\operatorname{div} \widetilde{\boldsymbol{\varphi}} = 0$, $\operatorname{curl} \widetilde{\boldsymbol{\varphi}} = \operatorname{curl} \boldsymbol{\varphi} \in \mathbf{L}^2(\Omega)$. Also $\widetilde{\boldsymbol{\varphi}} \cdot \mathbf{n} = 0$ on Γ_D and $\widetilde{\boldsymbol{\varphi}} \times \mathbf{n} = \mathbf{0}$ on Γ_N , thus $\widetilde{\boldsymbol{\varphi}} \in \widetilde{\mathbf{X}}_0^2(\Omega)$. Moreover, since $\boldsymbol{\varphi} \cdot \mathbf{n} = 0$ on Γ_D and $\operatorname{div} \boldsymbol{\varphi} = 0$ in Ω , we have

$$\langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N} = \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_{\Gamma} = \int_{\Omega} \operatorname{div} \boldsymbol{\varphi} dx = 0. \quad (4.10)$$

Therefore using (3.3) and (4.9), we infer that for any $1 \leq m \leq L_N$

$$\langle \widetilde{\boldsymbol{\varphi}} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^m} = \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^m} - \sum_{j=1}^J \left(\frac{1}{L_N} \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} + \frac{1}{J} \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^{\ell}} \right).$$

So

$$\langle \widetilde{\boldsymbol{\varphi}} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^m} = -\frac{1}{L_N} \sum_{j=1}^J \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j}. \quad (4.11)$$

Clearly, using (4.10), from this relation, we obtain after summing

$$0 = \sum_{m=1}^{L_N} \langle \widetilde{\boldsymbol{\varphi}} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^m} = -\sum_{j=1}^J \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j}.$$

Using again (4.11), we obtain $\langle \widetilde{\boldsymbol{\varphi}} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^m} = 0$ for any $1 \leq m \leq L_N$. In the same way for any $1 \leq j \leq J$, we deduce from (4.10) and (4.9) that

$$\begin{aligned} \langle \widetilde{\boldsymbol{\varphi}} \cdot \mathbf{n}, 1 \rangle_{\Sigma_k} &= \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_k} - \sum_{\ell=1}^{L_N} \left(\frac{1}{L_N} \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_k} + \frac{1}{J} \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^{\ell}} \right) \\ &= \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_k} - \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_k} = 0. \end{aligned}$$

As a consequence $\widetilde{\boldsymbol{\varphi}}$ belongs to $\widetilde{\mathbf{W}}_{\Sigma}^2(\Omega)$, which means that any solution of (4.6) also solves (4.7).

Step 4: Interpretation. Now, we give the interpretation of Problem (4.7). More precisely, we will prove that Problem (4.6) is equivalent to find $\mathbf{u} \in \mathbf{H}^1(\Omega)$ solution of Problem (4.2). By choosing $\varphi \in \mathcal{D}(\Omega)$, we deduce that

$$-\Delta \mathbf{u} = \mathbf{f} \quad \text{in } \Omega. \tag{4.12}$$

Moreover, because $\mathbf{u} \in \widetilde{\mathbf{W}}_\Sigma^2(\Omega)$, it satisfies $\mathbf{u} \times \mathbf{n} = \mathbf{0}$ on Γ_N , $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ_D , $\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} = 0$ for any $1 \leq \ell \leq L_N$ and $\langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0$ for any $1 \leq j \leq J$. It remains to show that $\text{curl } \mathbf{u} \times \mathbf{n} = \mathbf{0}$ on Γ_D and $\text{div } \mathbf{u} = 0$ on Γ_N . Multiplying (4.12) by $\varphi \in [\widetilde{\mathbf{X}}_0^2(\Omega)]$ and using the Green's formula (3.10) leads to

$$\int_\Omega \mathbf{f} \cdot \varphi = \int_\Omega \text{curl } \mathbf{u} \cdot \text{curl } \varphi + \int_\Omega (\text{div } \mathbf{u})(\text{div } \varphi) + \langle \text{curl } \mathbf{u} \times \mathbf{n}, \varphi \rangle_{\Gamma_D} - \langle \text{div } \mathbf{u}, \varphi \cdot \mathbf{n} \rangle_{\Gamma_N}.$$

Using (4.7), we deduce that

$$\forall \varphi \in \widetilde{\mathbf{X}}_0^2(\Omega), \quad \langle \text{curl } \mathbf{u} \times \mathbf{n}, \varphi \rangle_{\Gamma_D} - \langle \text{div } \mathbf{u}, \varphi \cdot \mathbf{n} \rangle_{\Gamma_N} = 0$$

Let $\boldsymbol{\mu}$ any element of $\mathbf{H}^{1/2}(\Gamma_D)$. So, there exists φ of $\mathbf{H}^1(\Omega)$ such that $\varphi = \boldsymbol{\mu}_\tau$ on Γ_D and $\varphi = \mathbf{0}$ on Γ_N and

$$\langle \text{curl } \mathbf{u} \times \mathbf{n}, \boldsymbol{\mu} \rangle_{\Gamma_D} = \langle \text{curl } \mathbf{u} \times \mathbf{n}, \boldsymbol{\mu}_\tau \rangle_{\Gamma_D} = \langle \text{curl } \mathbf{u} \times \mathbf{n}, \varphi \rangle_{\Gamma_D} = 0.$$

This implies that $\text{curl } \mathbf{u} \times \mathbf{n} = \mathbf{0}$ on Γ_D . Now, let μ be any element of $H^{1/2}(\Gamma_N)$. So, there exists ψ in $H^2(\Omega)$ such that $\frac{\partial \psi}{\partial \mathbf{n}} = \mu$ on Γ_N and $\psi = 0$ on Γ_D . Thus

$$\langle \text{div } \mathbf{u}, \mu \rangle_{\Gamma_N} = \langle \text{div } \mathbf{u}, \frac{\partial \psi}{\partial \mathbf{n}} \rangle_{\Gamma_N} = \langle \text{div } \mathbf{u}, \varphi \cdot \mathbf{n} \rangle_{\Gamma_N} = 0,$$

which means that $\text{div } \mathbf{u} = 0$ on Γ_N which is the required property.

(ii) Regularity. Assume that $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and Ω is of class $\mathcal{C}^{2,1}$. Arguing with a partition of unity, let θ be a function defined in $\mathcal{C}_0^\infty(\mathbb{R}^3)$, $0 \leq \theta \leq 1$ and such that $\theta = 1$ at the neighborhood of Γ_N and $\theta = 0$ at the neighborhood of Γ_D . Setting $\eta = 1 - \theta$, $\mathbf{v} = \theta \mathbf{u}$, $\mathbf{w} = \eta \mathbf{u}$ and $\mathbf{F}_\theta = \theta \mathbf{f} - 2\nabla \theta \cdot \nabla \mathbf{u} - \mathbf{u} \Delta \theta$, we look for the regularity of $\mathbf{v} \in \mathbf{H}^1(\Omega)$ which satisfies

$$\begin{aligned} -\Delta \mathbf{v} &= \mathbf{F}_\theta \quad \text{in } \Omega, \\ \mathbf{v} \times \mathbf{n} &= \mathbf{0}, \quad \text{div } \mathbf{v} = 0 \quad \text{on } \Gamma, \end{aligned} \tag{4.13}$$

where $\mathbf{F}_\theta \in \mathbf{L}^2(\Omega)$. Observe that $\text{div } \mathbf{v} \in L^2(\Omega)$ and $\Delta \text{div } \mathbf{v} = \text{div } \mathbf{F}_\theta \in H^{-1}(\Omega)$ with $\text{div } \mathbf{v} = 0$ on Γ . Let $\chi \in H_0^1(\Omega)$ such that $\Delta \chi = \text{div } \mathbf{F}_\theta$ in Ω . The function $\lambda = \chi - \text{div } \mathbf{v}$ is harmonic and $\lambda = 0$ on Γ then λ is zero [2] and thus $\text{div } \mathbf{v}$ belongs to $H^1(\Omega)$. We know that $\text{curl } \text{curl } \mathbf{v} = -\Delta \mathbf{v} + \nabla(\text{div } \mathbf{v})$ which means that $\text{curl } \text{curl } \mathbf{v} \in \mathbf{L}^2(\Omega)$ and consequently $\text{curl } \mathbf{v} \in \mathbf{X}_T^2(\Omega) \hookrightarrow \mathbf{H}^1(\Omega)$. We infer that \mathbf{v} belongs to $\mathbf{H}^2(\Omega)$ because $\mathbf{v} \in \mathbf{L}^2(\Omega)$, $\text{curl } \mathbf{v} \in \mathbf{H}^1(\Omega)$, $\text{div } \mathbf{v} \in H^1(\Omega)$ and $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ on Γ according to (3.2).

Now, we look for the regularity of $\mathbf{w} \in \mathbf{H}^1(\Omega)$ satisfying

$$\begin{aligned} -\Delta \mathbf{w} &= \mathbf{F}_\eta \quad \text{in } \Omega, \\ \mathbf{w} \cdot \mathbf{n} &= 0, \quad \text{curl } \mathbf{w} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma, \end{aligned} \tag{4.14}$$

where $\mathbf{F}_\eta \in \mathbf{L}^2(\Omega)$. Setting $\mathbf{z} = \text{curl } \mathbf{w}$ so $\mathbf{z} \in \mathbf{L}^2(\Omega)$ and it satisfies

$$\begin{aligned} -\Delta \mathbf{z} &= \text{curl } \mathbf{F}_\eta \quad \text{and} \quad \text{div } \mathbf{z} = 0 \quad \text{in } \Omega, \\ \mathbf{z} \times \mathbf{n} &= \mathbf{0} \quad \text{on } \Gamma, \\ \langle \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} &= 0, \quad \forall 1 \leq i \leq I, \end{aligned} \tag{4.15}$$

where Γ_i , $1 \leq i \leq I$, are the connected components of $\Gamma = \Gamma_D \cup \Gamma_N$. There exists a unique $\boldsymbol{\psi} \in H^1(\Omega)$ such that $-\Delta \boldsymbol{\psi} = \text{curl } \mathbf{F}_\eta$ and $\text{div } \boldsymbol{\psi} = 0$ in Ω . Moreover, $\boldsymbol{\psi} \times \mathbf{n} = \mathbf{0}$ on Γ and $\langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = 0$, for $1 \leq i \leq I$ [7]. The function $\boldsymbol{\phi} = \boldsymbol{\psi} - \mathbf{z}$ satisfies $\Delta \boldsymbol{\phi} = \mathbf{0}$ in Ω , $\boldsymbol{\phi} \times \mathbf{n} = \mathbf{0}$ on Γ and $\langle \boldsymbol{\phi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = 0$ for any $1 \leq i \leq I$ thus $\boldsymbol{\phi} = \mathbf{0}$ and then $\mathbf{z} \in \mathbf{H}^1(\Omega)$. Moreover

$$\nabla(\text{div } \mathbf{w}) = \Delta \mathbf{w} + \text{curl } \text{curl } \mathbf{w} = -\mathbf{F}_\eta + \text{curl } \mathbf{z}$$

which belongs to $\mathbf{L}^2(\Omega)$ and then $\operatorname{div} \mathbf{w} \in H^1(\Omega)$. We deduce that \mathbf{w} belongs to $\mathbf{H}^2(\Omega)$ since $\mathbf{w} \in \mathbf{L}^2(\Omega)$, $\operatorname{div} \mathbf{w} \in H^1(\Omega)$, $\operatorname{curl} \mathbf{w} \in \mathbf{H}^1(\Omega)$ and $\mathbf{w} \cdot \mathbf{n} = 0$ on Γ thanks to (3.1). As a consequence, $\mathbf{u} = \mathbf{v} + \mathbf{w} \in \mathbf{H}^2(\Omega)$ and it satisfies (4.5). \square

Remark 4.2. If Ω is only Lipschitz, Problem (4.2) admits a unique solution in $\tilde{\mathbf{X}}_0^2(\Omega)$ that does not belong to $\mathbf{H}^1(\Omega)$ in general but to $\mathbf{H}^{1/2}(\Omega)$ due to the embedding of $\tilde{\mathbf{X}}_0^2(\Omega)$ into $\mathbf{H}^{1/2}(\Omega)$ (see [10] for the non mixed case).

Remark 4.3. If Ω is of class $\mathcal{C}^{2,1}$ and $\mathbf{f} \in \mathbf{L}^{6/5}(\Omega)$ then the solution of problem (4.2) belongs to $\mathbf{W}^{2,6/5}(\Omega)$. The proof follows the same lines of the one of Theorem 4.1 point (ii).

Remark 4.4. As in the step 3 of the proof of Theorem 4.1, we can check that if $\mathbf{a} \times \mathbf{n} = \mathbf{0}$ and $b = 0$ in (2.1), the problem: Find $\mathbf{u} \in \mathbf{H}^1(\Omega)$ solution of (2.1), is equivalent to the weak formulation (4.6), in which, we replace the linear form $\varphi \mapsto \int_{\Omega} \mathbf{f} \cdot \varphi$ by

$$\varphi \mapsto \int_{\Omega} \mathbf{f} \cdot \varphi + \langle \mathbf{h} \times \mathbf{n}, \varphi \rangle_{\Gamma_D} - \langle g, \varphi \cdot \mathbf{n} \rangle_{\Gamma_N}$$

In the next, we will see the equivalent version of Theorem 4.1 in the case where $\partial\Sigma \subset \Gamma_N$.

Theorem 4.5. (i) Assume that $\partial\Sigma \subset \Gamma_N$ and $\mathbf{f} \in \mathbf{L}^{6/5}(\Omega)$ satisfying (4.3). Then the problem

$$\begin{aligned} -\Delta \mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} &= \mathbf{0}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{on } \Gamma_N, \\ \mathbf{u} \cdot \mathbf{n} &= 0, \quad \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_D, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} &= 0, \quad 1 \leq \ell \leq L_N, \end{aligned} \tag{4.16}$$

admits a unique solution $\mathbf{u} \in \mathbf{H}^1(\Omega)$ satisfying estimate (4.4).

(ii) Moreover, if Ω is of class $\mathcal{C}^{2,1}$ and $\mathbf{f} \in \mathbf{L}^2(\Omega)$ then \mathbf{u} belongs to $\mathbf{H}^2(\Omega)$ and satisfies the estimate (4.5).

Sketch of the proof. The space we are working here is

$$\tilde{\mathbf{V}}_0^p(\Omega) = \{ \mathbf{v} \in \tilde{\mathbf{X}}_0^p(\Omega), \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} = 0, 0 \leq \ell \leq L_N \}.$$

The weak formulation of (\mathcal{P}_N^0) is

$$\begin{aligned} \text{Find } \mathbf{u} \in \tilde{\mathbf{V}}_0^2(\Omega) \text{ such that for any } \varphi \in \tilde{\mathbf{V}}_0^2(\Omega), \\ \int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \varphi + \int_{\Omega} (\operatorname{div} \mathbf{u})(\operatorname{div} \varphi) = \int_{\Omega} \mathbf{f} \cdot \varphi. \end{aligned} \tag{4.17}$$

To extend (4.17) to any test function $\varphi \in \tilde{\mathbf{X}}_0^2(\Omega)$, we set

$$\tilde{\varphi} = \varphi - \nabla \theta - \sum_{\ell=1}^{L_N} \langle (\varphi - \nabla \theta) \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} \nabla s_\ell,$$

where θ is defined in (4.8). It is clear that $\tilde{\varphi} \in \tilde{\mathbf{V}}_0^2(\Omega)$. The remainder of the questions of the existence, uniqueness and regularity can be solved with the same arguments as in Theorem 4.1. \square

Remark 4.6. If Ω is only Lipschitz, Problem (4.16) admits a unique solution \mathbf{u} in $\tilde{\mathbf{X}}_0^2(\Omega)$ that is not necessary in $\mathbf{H}^1(\Omega)$.

In the next section, we will analyze generalized and strong solutions in L^p -theory in the case where $\partial\Sigma \subset \Gamma_D$ and then $\partial\Sigma \subset \Gamma_N$. We will also treat the case of non-homogeneous boundary conditions.

4.2. L^p -theory. Unlike the inf-sup conditions of Proposition 3.2 which involve divergence-free vector fields, we extend in the following the condition (3.6) to a more general case where the divergence is not specified, this result is useful in the forthcoming analysis.

Lemma 4.7. *Assume that $\partial\Sigma \subset \Gamma_D$ and $1 < p < +\infty$. Then there exists a constant $\beta > 0$ depending only on Ω and p such that the following inf-sup condition holds*

$$\inf_{\varphi \in \widetilde{\mathbf{W}}_{\Sigma}^{p'}(\Omega), \varphi \neq 0} \sup_{\mathbf{u} \in \widetilde{\mathbf{W}}_{\Sigma}^p(\Omega), \mathbf{u} \neq 0} \frac{\left| \int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \varphi + (\operatorname{div} \mathbf{u})(\operatorname{div} \varphi) \right|}{\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \|\varphi\|_{\mathbf{W}^{1,p'}(\Omega)}} \geq \beta. \tag{4.18}$$

Proof. Let $\mathbf{g} \in \mathbf{L}^p(\Omega)$ and $\lambda \in L^p(\Omega)$. Then there exists a unique $\theta \in W^{1,p}(\Omega)$ such that $\theta = 0$ on Γ_D , $(\nabla\theta - \mathbf{g}) \cdot \mathbf{n} = 0$ on Γ_N satisfying $\Delta\theta = \operatorname{div} \mathbf{g}$ in Ω and

$$\|\theta\|_{W^{1,p}(\Omega)} \leq C \|\mathbf{g}\|_{\mathbf{L}^p(\Omega)}.$$

We set $\mathbf{z} = \mathbf{g} - \nabla\theta$ and

$$\tilde{\mathbf{z}} = \mathbf{z} - \sum_{\ell=1}^{L_D} \langle \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^{\ell}} \nabla q_{\ell}.$$

Thus $\tilde{\mathbf{z}} \in \mathbf{L}^p(\Omega)$, $\operatorname{div} \tilde{\mathbf{z}} = 0$ in Ω , $\tilde{\mathbf{z}} \cdot \mathbf{n} = 0$ on Γ_N and $\langle \tilde{\mathbf{z}} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^m} = 0$, for any $1 \leq m \leq L_D$.

From [4, Theorem 4.18] where Γ_D and Γ_N are swapped, there exists a vector potential $\psi \in \widetilde{\mathbf{W}}_{\Sigma,\sigma}^p(\Omega)$ such that $\tilde{\mathbf{z}} = \operatorname{curl} \psi$ and it satisfies the estimate

$$\|\psi\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \|\tilde{\mathbf{z}}\|_{\mathbf{L}^p(\Omega)} \leq C \|\mathbf{g}\|_{\mathbf{L}^p(\Omega)}.$$

Now, let $\chi \in W^{2,p}(\Omega)$ be the unique solution of

$$\Delta\chi = \lambda \text{ in } \Omega, \quad \frac{\partial\chi}{\partial\mathbf{n}} = 0 \text{ on } \Gamma_D, \quad \chi = 0 \text{ on } \Gamma_N$$

and satisfying

$$\|\chi\|_{W^{2,p}(\Omega)} \leq C \|\lambda\|_{L^p(\Omega)}.$$

Now, we set $\mathbf{v} = \psi + \nabla\chi$ and

$$\mathbf{u} = \mathbf{v} - \sum_{\ell=1}^{L_N} \sum_{j=1}^J \left(\frac{1}{L_N} \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} + \frac{1}{J} \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^{\ell}} \right) \widetilde{\operatorname{grad}} s_j^{\ell}.$$

Then $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ and

$$\begin{aligned} \operatorname{curl} \mathbf{u} &= \operatorname{curl} \psi = \tilde{\mathbf{z}}, \quad \operatorname{div} \mathbf{u} = \lambda \quad \text{in } \Omega \\ \mathbf{u} \cdot \mathbf{n} &= 0 \text{ on } \Gamma_D, \quad \mathbf{u} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_N \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^m} &= 0, \quad 1 \leq m \leq L_N, \quad \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J, \end{aligned}$$

with the estimate

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C(\|\tilde{\mathbf{g}}\|_{\mathbf{L}^p(\Omega)} + \|\lambda\|_{L^p(\Omega)}). \tag{4.19}$$

Let $\varphi \in \widetilde{\mathbf{W}}_{\Sigma}^{p'}(\Omega)$, so the following estimate holds

$$\begin{aligned} \|\varphi\|_{\mathbf{W}^{1,p'}(\Omega)} &\leq C \left(\|\operatorname{curl} \varphi\|_{\mathbf{L}^{p'}(\Omega)} + \|\operatorname{div} \varphi\|_{L^{p'}(\Omega)} \right) \\ &= C \sup_{\mathbf{g} \in \mathbf{L}^p(\Omega), \mathbf{g} \neq 0, \lambda \in L^p(\Omega), \lambda \neq 0} \frac{\left| \int_{\Omega} \operatorname{curl} \varphi \cdot \mathbf{g} + (\operatorname{div} \varphi) \lambda \, dx \right|}{\|\mathbf{g}\|_{\mathbf{L}^p(\Omega)} + \|\lambda\|_{L^p(\Omega)}}. \end{aligned}$$

But

$$\int_{\Omega} \operatorname{curl} \varphi \cdot \mathbf{g} \, dx = \int_{\Omega} \operatorname{curl} \varphi \cdot \left(\tilde{\mathbf{z}} + \nabla\theta + \sum_{\ell=1}^{L_D} \langle \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^{\ell}} \nabla q_{\ell} \right) dx.$$

Note that

$$\int_{\Omega} \operatorname{curl} \varphi \cdot \nabla\theta \, dx = \int_{\Omega} \operatorname{curl} \varphi \cdot \nabla q_{\ell} \, dx = 0.$$

So, from (4.19) we obtain

$$\begin{aligned} \frac{\left| \int_{\Omega} \operatorname{curl} \varphi \cdot \mathbf{g} + (\operatorname{div} \varphi) \lambda \, dx \right|}{\|\mathbf{g}\|_{\mathbf{L}^p(\Omega)} + \|\lambda\|_{L^p(\Omega)}} &= \frac{\left| \int_{\Omega} \operatorname{curl} \varphi \cdot \operatorname{curl} \mathbf{u} + (\operatorname{div} \varphi)(\operatorname{div} \mathbf{u}) \, dx \right|}{\|\mathbf{g}\|_{\mathbf{L}^p(\Omega)} + \|\lambda\|_{L^p(\Omega)}} \\ &\leq \frac{1}{C} \frac{\left| \int_{\Omega} \operatorname{curl} \varphi \cdot \operatorname{curl} \mathbf{u} + (\operatorname{div} \varphi)(\operatorname{div} \mathbf{u}) \, dx \right|}{\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)}}. \end{aligned}$$

Consequently, the required condition (4.18) is obtained. □

In a similar way as above, we can establish the following condition when $\partial\Sigma \subset \Gamma_N$.

Lemma 4.8. *Assume that $\partial\Sigma \subset \Gamma_N$ and $1 < p < +\infty$. Then there exists a constant $\beta > 0$ depending only on Ω and p such that the following inf-sup condition holds*

$$\inf_{\varphi \in \tilde{\mathbf{V}}_{\Sigma}^{p'}(\Omega), \varphi \neq 0} \sup_{\mathbf{u} \in \tilde{\mathbf{V}}_{\Sigma}^p(\Omega), \mathbf{u} \neq 0} \frac{\left| \int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \varphi + (\operatorname{div} \mathbf{u})(\operatorname{div} \varphi) \right|}{\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \|\varphi\|_{\mathbf{W}^{1,p'}(\Omega)}} \geq \beta. \tag{4.20}$$

In the next theorem, we investigate the L^p -theory of the solutions of Problem (4.2) where the role of the inf-sup condition (4.18) is illustrated. We recall the definition of the exponent

$$r(p) = \begin{cases} \frac{3p}{p+3} & \text{if } p > 3/2 \\ 1 + \varepsilon & \text{if } p = 3/2 \\ 1 & \text{if } 1 \leq p < 3/2, \end{cases} \tag{4.21}$$

where $\varepsilon > 0$ is arbitrary.

Theorem 4.9. *Assume that $\partial\Sigma \subset \Gamma_D$.*

(i) *Let $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$ with $3/2 \leq p < +\infty$ satisfying the compatibility condition (4.3). Then, there exists a unique solution $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ to Problem (4.2). Moreover, \mathbf{u} satisfies the following estimate*

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \|\mathbf{f}\|_{\mathbf{L}^{r(p)}(\Omega)}. \tag{4.22}$$

(ii) *If $\mathbf{f} \in \mathbf{L}^1(\Omega)$, then there exists unique $\mathbf{u} \in \mathbf{W}^{1,s}(\Omega)$ for any $s < \frac{3}{2}$, solution of Problem (4.2) and satisfying*

$$\|\mathbf{u}\|_{\mathbf{W}^{1,s}(\Omega)} \leq C \|\mathbf{f}\|_{\mathbf{L}^1(\Omega)}. \tag{4.23}$$

(iii) *Furthermore, if Ω is of class $\mathcal{C}^{2,1}$ and $\mathbf{f} \in \mathbf{L}^p(\Omega)$ with $1 < p < +\infty$ then \mathbf{u} belongs to $\mathbf{W}^{2,p}(\Omega)$ and satisfies the estimate*

$$\|\mathbf{u}\|_{\mathbf{W}^{2,p}(\Omega)} \leq C \|\mathbf{f}\|_{\mathbf{L}^p(\Omega)}. \tag{4.24}$$

Proof. Step 1: Uniqueness. Suppose that $\mathbf{f} = \mathbf{0}$ in Ω and let $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ solution of Problem (4.2). Applying the divergence operator, we obtain $\Delta(\operatorname{div} \mathbf{u}) = 0$ in Ω . But $\operatorname{div} \mathbf{u} = 0$ on Γ_N and $\operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{0}$ on Γ_D in the sense of $\mathbf{W}^{-1/p,p}(\Gamma_D)$ which implies that $\operatorname{div}_{\Gamma_D}(\operatorname{curl} \mathbf{u} \times \mathbf{n}) = 0$ in the sense of $W^{-1-1/p,p}(\Gamma_D)$. Moreover, $\operatorname{curl} \operatorname{curl} \mathbf{u} \cdot \mathbf{n} = \operatorname{div}_{\Gamma_D}(\operatorname{curl} \mathbf{u} \times \mathbf{n}) = 0$ on Γ_D . Since $\Delta \mathbf{u} = \mathbf{0}$ in Ω , we have

$$\operatorname{curl} \operatorname{curl} \mathbf{u} = \nabla(\operatorname{div} \mathbf{u}) \quad \text{in } \Omega.$$

So, $\frac{\partial(\operatorname{div} \mathbf{u})}{\partial \mathbf{n}} = 0$ on Γ_D this yields $\operatorname{div} \mathbf{u} = 0$ in Ω according to Lemma 3.4.

Now, we set $\mathbf{z} = \operatorname{curl} \mathbf{u}$ in (4.2), then $\operatorname{curl} \mathbf{z} = \nabla(\operatorname{div} \mathbf{u}) = 0$ and $\operatorname{div} \mathbf{z} = \mathbf{0}$ in Ω , $\mathbf{z} \times \mathbf{n} = \mathbf{0}$ on Γ_D , $\mathbf{z} \cdot \mathbf{n} = 0$ on Γ_N , which means that $\mathbf{z} \in \mathbf{K}_0^p(\Omega)$ so we can write $\mathbf{z} = \sum_{\ell=1}^{L_D} \langle \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell} \nabla q_\ell$ but since $\langle \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{\Gamma_D^\ell} = 0$ for any $1 \leq \ell \leq L_D$, we infer that $\mathbf{z} = \mathbf{0}$. Finally, we have

$$\operatorname{curl} \mathbf{u} = \mathbf{0}, \quad \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega, \quad \mathbf{u} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_N, \quad \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma_D.$$

So, $\mathbf{u} \in \tilde{\mathbf{K}}_0^p(\Omega)$ and since the fluxes of \mathbf{u} over Σ_j and Γ_N^ℓ are equal to zero, we deduce that $\mathbf{u} = \mathbf{0}$ in Ω and this completes the uniqueness proof.

Step 2: Existence. We consider the problem

$$\begin{aligned} \text{Find } \mathbf{u} \in \widetilde{\mathbf{W}}_{\Sigma}^p(\Omega) \text{ such that for any } \varphi \in \widetilde{\mathbf{W}}_{\Sigma}^{p'}(\Omega), \\ \int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \varphi + \int_{\Omega} (\operatorname{div} \mathbf{u})(\operatorname{div} \varphi) = \int_{\Omega} \mathbf{f} \cdot \varphi, \end{aligned} \tag{4.25}$$

We recall that the inf-sup condition (4.18) holds for any $1 < p < +\infty$. Then, it suffices to show that the right-hand side of (4.25) is a linear and continuous form on $\widetilde{\mathbf{W}}_{\Sigma}^p(\Omega)$ for any $1 < p < +\infty$.

(i) When $p \geq \frac{3}{2}$, since $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$ so the right-hand side is a linear and continuous form on $\widetilde{\mathbf{W}}_{\Sigma}^p(\Omega)$. By the inf-sup condition (4.18), Problem (4.25) has a unique solution in $\widetilde{\mathbf{W}}_{\Sigma}^p(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$. As in the proof of Theorem 4.1, we show that (4.25) is equivalent to

$$\begin{aligned} \text{Find } \mathbf{u} \in \widetilde{\mathbf{W}}_{\Sigma}^p(\Omega) \text{ such that for any } \varphi \in \widetilde{X}_0^{p'}(\Omega), \\ \int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \varphi + \int_{\Omega} (\operatorname{div} \mathbf{u})(\operatorname{div} \varphi) = \int_{\Omega} \mathbf{f} \cdot \varphi, \end{aligned} \tag{4.26}$$

and that (4.26) is equivalent to: find $\mathbf{u} \in \widetilde{\mathbf{W}}_{\Sigma}^p(\Omega)$ solution of Problem (4.2) by using the same arguments. Observe that $\widetilde{\mathbf{W}}_{\Sigma}^p(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$.

(ii) When $1 \leq p < \frac{3}{2}$, from the definition of the exponent $r(p)$ given in (4.21), we have $\mathbf{f} \in \mathbf{L}^1(\Omega)$ and then the right-hand side of (4.25) is a linear and continuous form on $\widetilde{\mathbf{W}}_{\Sigma}^p(\Omega)$ since $\widetilde{\mathbf{W}}_{\Sigma}^{p'}(\Omega) \hookrightarrow \mathbf{W}^{1,p'}(\Omega) \hookrightarrow \mathbf{L}^{\infty}(\Omega)$. Using the same argument than above, we obtain a solution $\mathbf{u} \in \cap \mathbf{W}^{1,s}(\Omega)$ for any $s < \frac{3}{2}$, satisfying the estimate (4.23).

(iii) Regularity. Assume that Ω is of class $\mathcal{C}^{2,1}$ and $\mathbf{f} \in \mathbf{L}^p(\Omega)$, we prove that the solution of (4.2) belongs to $\mathbf{W}^{2,p}(\Omega)$ with the same approach used in the last step of the proof of Theorem 4.1. □

Remark 4.10. If Ω is of class $\mathcal{C}^{2,1}$ and $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$ then the solution of problem (4.2) belongs to $\mathbf{W}^{2,r(p)}(\Omega)$.

Remark 4.11. In some previous works [6, 7], similar problems have been studied in the case where $\Gamma_N = \emptyset$ with $\operatorname{div} \mathbf{u} = 0$ in Ω and the solution is obtained when $\operatorname{div} \mathbf{f} = 0$ in Ω and $\mathbf{f} \cdot \mathbf{n} = 0$ on Γ are imposed. Having in mind these results, the question we may ask is under which assumptions, the solution of (4.2) satisfies $\operatorname{div} \mathbf{u} = 0$ in Ω ?

Proposition 4.12. *Let $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$ satisfying the compatibility condition (4.3), if in addition $\operatorname{div} \mathbf{f} = 0$ in Ω and $\mathbf{f} \cdot \mathbf{n} = 0$ on Γ_D , then the unique solution \mathbf{u} given in Theorem 4.9 satisfies $\operatorname{div} \mathbf{u} = 0$ in Ω .*

Proof. Note that the solution \mathbf{u} satisfies $\Delta(\operatorname{div} \mathbf{u}) = 0$ in Ω because $\Delta(\operatorname{div} \mathbf{u}) = \operatorname{div} \mathbf{f}$ in Ω . Since $\operatorname{div} \mathbf{u} \in L^p(\Omega)$ thus $\frac{\partial}{\partial \mathbf{n}} \operatorname{div} \mathbf{u} \in W^{-1-1/p,p}(\Gamma)$.

We need to prove that $\frac{\partial}{\partial \mathbf{n}} \operatorname{div} \mathbf{u} = 0$ on Γ , in particular that $\frac{\partial}{\partial \mathbf{n}} \operatorname{div} \mathbf{u} = 0$ on Γ_D . Using the Green formula (3.10), we have for any $\chi \in W^{2,p'}(\Omega)$ with $\frac{\partial \chi}{\partial \mathbf{n}} = 0$ on Γ_D and $\chi = 0$ on Γ_N

$$-\int_{\Omega} \Delta \mathbf{u} \cdot \nabla \chi = \int_{\Omega} (\operatorname{div} \mathbf{u})(\Delta \chi) - \langle \operatorname{curl} \mathbf{u} \times \mathbf{n}, \nabla \chi \rangle_{\Gamma_D} + \langle \operatorname{div} \mathbf{u}, \frac{\partial \chi}{\partial \mathbf{n}} \rangle_{\Gamma_N}.$$

But $\operatorname{div} \mathbf{u} = 0$ on Γ_N and $\operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{0}$ on Γ_D , so we have

$$\langle \operatorname{curl} \mathbf{u} \times \mathbf{n}, \nabla \chi \rangle_{\Gamma_D} = \langle \operatorname{curl} \mathbf{u} \times \mathbf{n}, \nabla_{\tau} \chi \rangle_{\Gamma_D} = -\langle \operatorname{div}_{\Gamma}(\operatorname{curl} \mathbf{u} \times \mathbf{n}), \chi \rangle_{\Gamma_D}.$$

As $-\Delta \mathbf{u} \cdot \mathbf{n} = \mathbf{f} \cdot \mathbf{n} = 0$ on Γ_D and $\chi = 0$ on Γ_N , we have

$$-\int_{\Omega} \Delta \mathbf{u} \cdot \nabla \chi = \int_{\Omega} \chi(\operatorname{div} \Delta \mathbf{u}) - \langle \Delta \mathbf{u} \cdot \mathbf{n}, \chi \rangle_{\Gamma} = 0$$

Now, using the following Green formula gives

$$0 = \int_{\Omega} (\operatorname{div} \mathbf{u})(\Delta \chi) = \int_{\Omega} \chi \Delta(\operatorname{div} \mathbf{u}) + \langle \frac{\partial}{\partial \mathbf{n}} \operatorname{div} \mathbf{u}, \chi \rangle_{\Gamma_D} = \langle \frac{\partial}{\partial \mathbf{n}} \operatorname{div} \mathbf{u}, \chi \rangle_{\Gamma_D}.$$

We deduce that $\frac{\partial}{\partial \mathbf{n}} \operatorname{div} \mathbf{u} = 0$ on Γ_D and we already know that $\operatorname{div} \mathbf{u} = 0$ on Γ_N and $\Delta(\operatorname{div} \mathbf{u}) = 0$ then $\operatorname{div} \mathbf{u} = 0$ in Ω according to Lemma 3.4. □

In a similar way as in Theorem 4.9, we can establish the existence, uniqueness and regularity of the solution of Problem (\mathcal{P}_N^0) in L^p -theory.

Theorem 4.13. *Assume that $\partial\Sigma \subset \Gamma_N$.*

(i) *Let $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$ with $3/2 \leq p < +\infty$, satisfying the compatibility condition (4.3) for any $\varphi \in \widetilde{\mathbf{K}}_0^p(\Omega)$. Then, there exists a unique solution $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ of Problem (4.16). Moreover, \mathbf{u} satisfies*

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C\|\mathbf{f}\|_{\mathbf{L}^{r(p)}(\Omega)}.$$

(ii) *If $\mathbf{f} \in \mathbf{L}^1(\Omega)$. Then, there exists a unique solution $\mathbf{u} \in \mathbf{W}^{1,s}(\Omega)$ for any $s < 3/2$ of Problem (4.16). Moreover, \mathbf{u} satisfies*

$$\|\mathbf{u}\|_{\mathbf{W}^{1,s}(\Omega)} \leq C\|\mathbf{f}\|_{\mathbf{L}^1(\Omega)}.$$

(iii) *Furthermore, if Ω is of class $\mathcal{C}^{2,1}$ and $\mathbf{f} \in \mathbf{L}^p(\Omega)$ then \mathbf{u} belongs to $\mathbf{W}^{2,p}(\Omega)$ and satisfies*

$$\|\mathbf{u}\|_{\mathbf{W}^{2,p}(\Omega)} \leq C\|\mathbf{f}\|_{\mathbf{L}^p(\Omega)}.$$

In Theorem 4.9, we have studied the problem with homogeneous boundary conditions. We go back now to the original problem including non-homogeneous boundary conditions (2.1) in the case where $\partial\Sigma \subset \Gamma_D$.

Theorem 4.14. (i) *Assume that $\partial\Sigma \subset \Gamma_D$. Let $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$ with $3/2 \leq p < +\infty$, $\mathbf{a} \times \mathbf{n} \in \mathbf{W}^{1-1/p,p}(\Gamma_N)$, $g \in W^{-1/p,p}(\Gamma_N)$, $b \in W^{1-1/p,p}(\Gamma_D)$ and $\mathbf{h} \times \mathbf{n} \in \mathbf{W}^{-1/p,p}(\Gamma_D)$ satisfying the following compatibility condition*

$$\int_{\Omega} \mathbf{f} \cdot \varphi \, dx - \langle \mathbf{h} \times \mathbf{n}, \varphi \rangle_{\Gamma_D} + \langle g, \varphi \cdot \mathbf{n} \rangle_{\Gamma_N} = 0. \quad (4.27)$$

Then Problem (2.1) has a unique solution $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ for any $1 < p < +\infty$ and it satisfies

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C & \left(\|\mathbf{f}\|_{\mathbf{L}^{r(p)}(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma_N)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma_D)} \right. \\ & \left. + \|\mathbf{a} \times \mathbf{n}\|_{\mathbf{W}^{1-1/p,p}(\Gamma_N)} + \|b\|_{W^{1-1/p,p}(\Gamma_D)} \right). \end{aligned} \quad (4.28)$$

(ii) *If $\mathbf{f} \in \mathbf{L}^1(\Omega)$, then there exists a unique $\mathbf{u} \in \mathbf{W}^{1,s}(\Omega)$ for any $s < \frac{3}{2}$ solution of Problem (2.1) and satisfying*

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{W}^{1,s}(\Omega)} \leq C & \left(\|\mathbf{f}\|_{\mathbf{L}^1(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma_N)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma_D)} \right. \\ & \left. + \|\mathbf{a} \times \mathbf{n}\|_{\mathbf{W}^{1-1/p,p}(\Gamma_N)} + \|b\|_{W^{1-1/p,p}(\Gamma_D)} \right). \end{aligned}$$

(iii) *Furthermore, if Ω is of class $\mathcal{C}^{2,1}$, $\mathbf{f} \in \mathbf{L}^p(\Omega)$ with $1 < p < +\infty$, $\mathbf{a} \times \mathbf{n} \in \mathbf{W}^{2-1/p,p}(\Gamma_N)$, $g \in W^{1-1/p,p}(\Gamma_N)$, $b \in W^{2-1/p,p}(\Gamma_D)$ and $\mathbf{h} \times \mathbf{n} \in \mathbf{W}^{1-1/p,p}(\Gamma_D)$ then \mathbf{u} belongs to $\mathbf{W}^{2,p}(\Omega)$ and the following estimate holds*

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{W}^{2,p}(\Omega)} \leq C & \left(\|\mathbf{f}\|_{\mathbf{L}^p(\Omega)} + \|g\|_{W^{1-1/p,p}(\Gamma_N)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{1-1/p,p}(\Gamma_D)} \right. \\ & \left. + \|\mathbf{a} \times \mathbf{n}\|_{\mathbf{W}^{2-1/p,p}(\Gamma_N)} + \|b\|_{W^{2-1/p,p}(\Gamma_D)} \right). \end{aligned} \quad (4.29)$$

Proof. (i) **Step 1: Uniqueness.** By taking $\mathbf{f} = \mathbf{0}$, $\mathbf{h} = \mathbf{0}$, $\mathbf{a} = \mathbf{0}$, $b = 0$ and $g = 0$ in Problem (2.1), we proceed as in the proof of Theorem 4.9 to deduce that \mathbf{u} is unique.

Step 2: Compatibility condition. According to (3.11), every solution of (2.1) also solves

$$\begin{aligned} & \text{Find } \mathbf{u} \in \mathbf{W}^{1,p}(\Omega) \text{ such that for any } \varphi \in \widetilde{\mathbf{W}}_{\Sigma}^{p'}(\Omega), \\ & \int_{\Omega} \text{curl } \mathbf{u} \cdot \text{curl } \varphi + \int_{\Omega} (\text{div } \mathbf{u})(\text{div } \varphi) = \int_{\Omega} \mathbf{f} \cdot \varphi - \langle \mathbf{h} \times \mathbf{n}, \varphi \rangle_{\Gamma_D} + \langle g, \varphi \cdot \mathbf{n} \rangle_{\Gamma_N}. \end{aligned} \quad (4.30)$$

By taking $\varphi \in \widetilde{K}_0^{p'}(\Omega)$, the compatibility condition (4.27) is obtained.

Step 3: Equivalence. We prove that (4.30) is equivalent to

$$\begin{aligned} &\text{Find } \mathbf{u} \in \mathbf{W}^{1,p}(\Omega) \text{ such that for any } \varphi \in \widetilde{\mathbf{X}}_0^{p'}(\Omega), \\ &\int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \varphi + \int_{\Omega} (\operatorname{div} \mathbf{u})(\operatorname{div} \varphi) = \int_{\Omega} \mathbf{f} \cdot \varphi - \langle \mathbf{h} \times \mathbf{n}, \varphi \rangle_{\Gamma_D} + \langle g, \varphi \cdot \mathbf{n} \rangle_{\Gamma_N}, \end{aligned} \tag{4.31}$$

in the same way as in the proof of Theorem 4.1.

Step 4: Existence and estimate. Let $\chi \in W^{2,p}(\Omega)$ such that

$$\Delta \chi = 0 \text{ in } \Omega, \quad \frac{\partial \chi}{\partial \mathbf{n}} = b \text{ on } \Gamma_D, \quad \chi = 0 \text{ on } \Gamma_N,$$

with the estimate

$$\|\chi\|_{W^{2,p}(\Omega)} \leq C \|b\|_{W^{1-1/p,p}(\Gamma_D)}.$$

We set

$$\boldsymbol{\xi} = \mathbf{u} - \nabla \chi - \sum_{j=1}^J \sum_{\ell=1}^{L_N} \left(\frac{1}{J} \langle (\mathbf{u} - \nabla \chi) \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} + \frac{1}{L_N} \langle (\mathbf{u} - \nabla \chi) \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} \right) \widetilde{\operatorname{grad}} s_j^\ell, \tag{4.32}$$

where the properties of $\widetilde{\operatorname{grad}} s_j^\ell$ are given in (3.3). We are now reduced to solve the following problem

$$\begin{aligned} -\Delta \boldsymbol{\xi} &= \mathbf{f} \quad \text{in } \Omega, \\ \boldsymbol{\xi} \times \mathbf{n} &= \mathbf{a} \times \mathbf{n}, \quad \operatorname{div} \boldsymbol{\xi} = g \quad \text{on } \Gamma_N, \\ \boldsymbol{\xi} \cdot \mathbf{n} &= 0, \quad \operatorname{curl} \boldsymbol{\xi} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma_D, \\ \langle \boldsymbol{\xi} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} &= 0, \quad 1 \leq \ell \leq L_N, \quad \langle \boldsymbol{\xi} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J. \end{aligned}$$

Since $\mathbf{a}_\tau \in \mathbf{W}^{1-1/p,p}(\Gamma_N)$, there exists a unique $(\boldsymbol{\xi}_0, \pi_0) \in \mathbf{W}^{1,p}(\Omega) \times L^p(\Omega)$ solution of

$$\begin{aligned} \Delta \boldsymbol{\xi}_0 + \nabla \pi_0 &= \mathbf{0}, \quad \operatorname{div} \boldsymbol{\xi}_0 = 0, \quad \text{in } \Omega, \\ \boldsymbol{\xi}_0 &= \mathbf{a}_\tau \quad \text{on } \Gamma_N, \quad \boldsymbol{\xi}_0 = \mathbf{0} \quad \text{on } \Gamma_D. \end{aligned}$$

Note that since $\boldsymbol{\xi}_0 \in \mathbf{W}^{1,p}(\Omega)$ and $\Delta \boldsymbol{\xi}_0 \in [\mathbf{H}_0^{p'}(\operatorname{div}, \Omega)]'$, so $\operatorname{curl} \boldsymbol{\xi}_0 \times \mathbf{n}$ makes sense in $\mathbf{W}^{-1/p,p}(\Gamma)$ according to Corollary 4.2 [6]. Setting $\tilde{\mathbf{z}} = \boldsymbol{\xi} - \boldsymbol{\xi}_0$ so $\tilde{\mathbf{z}}$ satisfies

$$\begin{aligned} -\Delta \tilde{\mathbf{z}} &= \mathbf{f} + \Delta \boldsymbol{\xi}_0 \quad \text{in } \Omega, \\ \tilde{\mathbf{z}} \times \mathbf{n} &= \mathbf{0}, \quad \operatorname{div} \tilde{\mathbf{z}} = g \quad \text{on } \Gamma_N, \\ \tilde{\mathbf{z}} \cdot \mathbf{n} &= 0, \quad \operatorname{curl} \tilde{\mathbf{z}} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} - \operatorname{curl} \boldsymbol{\xi}_0 \times \mathbf{n} \quad \text{on } \Gamma_D, \\ \langle \tilde{\mathbf{z}} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} &= 0, \quad 1 \leq \ell \leq L_N, \quad \langle \tilde{\mathbf{z}} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = -\langle \boldsymbol{\xi}_0 \cdot \mathbf{n}, 1 \rangle_{\Sigma_j}, \quad 1 \leq j \leq J. \end{aligned}$$

We finally set

$$\mathbf{z} = \tilde{\mathbf{z}} - \sum_{j=1}^J \sum_{\ell=1}^{L_N} \left(\frac{1}{J} \langle \tilde{\mathbf{z}} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} + \frac{1}{L_N} \langle \tilde{\mathbf{z}} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} \right) \widetilde{\operatorname{grad}} s_j^\ell,$$

we obtain

$$\begin{aligned} -\Delta \mathbf{z} &= \mathbf{f} + \Delta \boldsymbol{\xi}_0 \quad \text{in } \Omega, \\ \mathbf{z} \times \mathbf{n} &= \mathbf{0}, \quad \operatorname{div} \mathbf{z} = g \quad \text{on } \Gamma_N, \\ \mathbf{z} \cdot \mathbf{n} &= 0, \quad \operatorname{curl}(\mathbf{z} + \boldsymbol{\xi}_0) \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma_D, \\ \langle \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} &= 0, \quad 1 \leq \ell \leq L_N, \quad \langle \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J. \end{aligned} \tag{4.33}$$

This problem is equivalent to

$$\begin{aligned} &\text{Find } \mathbf{z} \in \widetilde{\mathbf{W}}_\Sigma^p(\Omega) \text{ such that for any } \varphi \in \widetilde{\mathbf{W}}_\Sigma^{p'}(\Omega), \\ &\int_{\Omega} \operatorname{curl} \mathbf{z} \cdot \operatorname{curl} \varphi + \int_{\Omega} (\operatorname{div} \mathbf{z})(\operatorname{div} \varphi) = \int_{\Omega} \mathbf{f} \cdot \varphi - \int_{\Gamma_D} \operatorname{curl} \boldsymbol{\xi}_0 \cdot \operatorname{curl} \varphi - \langle \mathbf{h} \times \mathbf{n}, \varphi \rangle_{\Gamma_D} + \langle g, \varphi \cdot \mathbf{n} \rangle_{\Gamma_N}. \end{aligned} \tag{4.34}$$

Note that the right-hand side is a linear and continuous form on $\widetilde{\mathbf{W}}_\Sigma^p(\Omega)$ for any $p \geq \frac{3}{2}$, by use of the inf-sup condition (4.18), this problem admits a unique solution $\mathbf{z} \in \widetilde{\mathbf{W}}_\Sigma^p(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$ satisfying

$$\begin{aligned} \|\mathbf{z}\|_{\mathbf{W}^{1,p}(\Omega)} &\leq C \left(\|\mathbf{f}\|_{\mathbf{L}^{r(p)}(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma_N)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma_D)} + \|\operatorname{curl} \boldsymbol{\xi}_0\|_{\mathbf{L}^p(\Omega)} \right) \\ &\leq C \left(\|\mathbf{f}\|_{\mathbf{L}^{r(p)}(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma_N)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma_D)} + \|\mathbf{a} \times \mathbf{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma_N)} \right). \end{aligned}$$

From the construction of \mathbf{z} , we have

$$\|\boldsymbol{\xi}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \left(\|\mathbf{a} \times \mathbf{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma_N)} + \|\mathbf{z}\|_{\mathbf{W}^{1,p}(\Omega)} \right).$$

By (4.32), we have

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq \|\boldsymbol{\xi}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\nabla \chi\|_{\mathbf{W}^{1,p}(\Omega)} \leq \|\boldsymbol{\xi}\|_{\mathbf{W}^{1,p}(\Omega)} + \|b\|_{W^{1-1/p,p}(\Gamma_D)}.$$

The estimate (4.29) is then deduced.

Step 5: Interpretation. To give the interpretation of (4.31), otherwise, to prove that (4.31) is equivalent to find $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ solution of Problem (2.1), we use Remark 4.4 and we follow the same lines of the proof of Theorem 4.1.

(ii) When $1 \leq p < \frac{3}{2}$, we prove that there exists a unique solution $\mathbf{u} \in \cap \mathbf{W}^{1,s}(\Omega)$ for any $s < \frac{3}{2}$ using the same arguments than in the proof of Theorem 4.9 point ii).

(iii) Regularity. Assume that $\mathbf{f} \in \mathbf{L}^p(\Omega)$ and Ω is of class $\mathcal{C}^{2,1}$ as in Theorem 4.1. We can prove the $\mathbf{W}^{2,p}(\Omega)$ regularity by means of a partition of unity such that $\mathbf{v} = \theta \mathbf{z}$ and $\mathbf{w} = \eta \mathbf{z}$. Indeed, $\Delta(\operatorname{div} \mathbf{v}) = \operatorname{div} \mathbf{F}_\theta \in W^{-1,p}(\Omega)$, $\operatorname{div} \mathbf{v} = g$ on Γ where $g \in W^{1-1/p,p}(\Gamma)$ so $\operatorname{div} \mathbf{v} \in W^{1,p}(\Omega)$. Moreover, $\mathbf{v} \in \mathbf{W}^{1,p}(\Omega)$ thus $\operatorname{curl} \mathbf{v} \in \widetilde{\mathbf{X}}_T^p(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$. As, $\mathbf{v} \in \mathbf{L}^p(\Omega)$ and $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ on Γ hence, $\mathbf{v} \in \mathbf{W}^{2,p}(\Omega)$ according to (3.2).

Setting $\boldsymbol{\lambda} = \operatorname{curl} \mathbf{w}$, so $-\Delta \boldsymbol{\lambda} = \operatorname{curl} \mathbf{F}_\eta$, $\operatorname{div} \boldsymbol{\lambda} = 0$ in Ω and $\boldsymbol{\lambda} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \in \mathbf{W}^{-1/p,p}(\Gamma)$ then $\boldsymbol{\lambda} \in \mathbf{W}^{1,p}(\Omega)$ referring to [7]. Thus, $\nabla(\operatorname{div} \mathbf{w}) = -\mathbf{F}_\eta + \operatorname{curl} \boldsymbol{\lambda} \in \mathbf{L}^p(\Omega)$. So $\mathbf{w} \in \mathbf{L}^p(\Omega)$, $\operatorname{curl} \mathbf{w} \in \mathbf{W}^{1,p}(\Omega)$ and $\operatorname{div} w \in W^{1,p}(\Omega)$ with $\mathbf{w} \cdot \mathbf{n} = 0$ on Γ , we deduce that \mathbf{w} belongs to $\mathbf{W}^{2,p}(\Omega)$ thanks to (3.1). Therefore, $\mathbf{z} \in \mathbf{W}^{2,p}(\Omega)$ and consequently \mathbf{u} belongs to $\mathbf{W}^{2,p}(\Omega)$ satisfying (4.29). \square

Also, Problem (4.16) can be considered in the case of non-homogeneous boundary conditions and can be solved with the same tools as above.

Theorem 4.15. Assume that $\partial \Sigma \subset \Gamma_N$.

(i) Let $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$ with $3/2 \leq p < +\infty$, $\mathbf{a} \times \mathbf{n} \in \mathbf{W}^{-1/p,p}(\Gamma_N)$, $g \in W^{-1/p,p}(\Gamma_N)$, $b \in W^{1-1/p,p}(\Gamma_D)$ and $\mathbf{h} \times \mathbf{n} \in \mathbf{W}^{-1/p,p}(\Gamma_D)$ satisfying the compatibility condition (4.27). Then the problem

$$\begin{aligned} -\Delta \mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} &= \mathbf{a} \times \mathbf{n}, \quad \operatorname{div} \mathbf{u} = g \quad \text{on } \Gamma_N, \\ \mathbf{u} \cdot \mathbf{n} &= b, \quad \operatorname{curl} \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma_D, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} &= 0, \quad 1 \leq \ell \leq L_N, \end{aligned} \tag{4.35}$$

has a unique solution $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ satisfying (4.28).

(ii) If $\mathbf{f} \in \mathbf{L}^1(\Omega)$ then there exists a unique $\mathbf{u} \in \mathbf{W}^{1,s}(\Omega)$ solution of Problem (4.35)

(iii) Furthermore, if Ω is of class $\mathcal{C}^{2,1}$, $\mathbf{f} \in \mathbf{L}^p(\Omega)$, $\mathbf{a} \times \mathbf{n} \in \mathbf{W}^{2-1/p,p}(\Gamma_N)$, $g \in W^{1-1/p,p}(\Gamma_N)$, $b \in W^{2-1/p,p}(\Gamma_D)$ and $\mathbf{h} \times \mathbf{n} \in \mathbf{W}^{-1/p,p}(\Gamma_D)$ then \mathbf{u} belongs to $\mathbf{W}^{2,p}(\Omega)$ and the estimate (4.29) holds.

We consider now a more general right-hand side than Problem (2.1). This choice is motivated by the generalized Neumann problem which allows to have a right-hand side in negative spaces

and this explains the boundary condition on Γ_D .

$$\begin{aligned} -\Delta \mathbf{u} &= \mathbf{f} + \operatorname{curl} \mathbf{F} \quad \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} &= \mathbf{0}, \quad \operatorname{div} \mathbf{u} = g \quad \text{on } \Gamma_N, \\ \mathbf{u} \cdot \mathbf{n} &= 0, \quad (\operatorname{curl} \mathbf{u} - \mathbf{F}) \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma_D, \\ \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} &= 0, \quad 1 \leq \ell \leq L_N, \quad \langle \mathbf{u} \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J. \end{aligned} \tag{4.36}$$

Theorem 4.16. *Assume that $\partial \Sigma \subset \Gamma_D$.*

(i) *Let $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$, $\mathbf{F} \in \mathbf{L}^p(\Omega)$ with $1 < p < +\infty$, $g \in W^{-1/p,p}(\Gamma_N)$ and $\mathbf{h} \times \mathbf{n} \in \mathbf{W}^{-1/p,p}(\Gamma_D)$ satisfying the following compatibility condition for any $\varphi \in \widetilde{\mathbf{K}}_0^p(\Omega)$:*

$$\int_{\Omega} \mathbf{f} \cdot \varphi \, dx - \langle \mathbf{h} \times \mathbf{n}, \varphi \rangle_{\Gamma_D} + \langle g, \varphi \cdot \mathbf{n} \rangle_{\Gamma_N} = 0. \tag{4.37}$$

Then, Problem (4.36) has a unique solution $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ and it satisfies

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \left(\|\mathbf{f}\|_{\mathbf{L}^{r(p)}(\Omega)} + \|\mathbf{F}\|_{\mathbf{L}^p(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma_N)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma_D)} \right) \tag{4.38}$$

(ii) *Furthermore, if Ω is of class $\mathcal{C}^{2,1}$, $\mathbf{f} \in \mathbf{L}^p(\Omega)$, $g \in W^{1-1/p,p}(\Gamma_N)$ and $\mathbf{h} \times \mathbf{n} \in \mathbf{W}^{1-1/p,p}(\Gamma_D)$ with $\mathbf{F} = \mathbf{0}$ then \mathbf{u} belongs to $\mathbf{W}^{2,p}(\Omega)$ and the following estimate holds*

$$\|\mathbf{u}\|_{\mathbf{W}^{2,p}(\Omega)} \leq C \left(\|\mathbf{f}\|_{\mathbf{L}^p(\Omega)} + \|g\|_{W^{1-1/p,p}(\Gamma_N)} + \|\mathbf{h} \times \mathbf{n}\|_{\mathbf{W}^{1-1/p,p}(\Gamma_D)} \right). \tag{4.39}$$

Proof. (i) The weak formulation of Problem (4.36) is

$$\begin{aligned} \text{Find } \mathbf{u} &\in \widetilde{\mathbf{W}}_{\Sigma}^p(\Omega) \text{ such that for any } \varphi \in \widetilde{\mathbf{W}}_{\Sigma}^p(\Omega), \\ \int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \varphi + \int_{\Omega} (\operatorname{div} \mathbf{u})(\operatorname{div} \varphi) & \\ = \int_{\Omega} \mathbf{f} \cdot \varphi + \int_{\Omega} \mathbf{F} \cdot \operatorname{curl} \varphi + \langle \mathbf{h} \times \mathbf{n}, \varphi \rangle_{\Gamma_D} + \langle g, \varphi \cdot \mathbf{n} \rangle_{\Gamma_N}. & \end{aligned} \tag{4.40}$$

By use of the inf-sup condition (4.18), then Problem (4.36) admits a unique solution $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ satisfying the estimate (4.38). The rest of the proof is similar to that of Theorem 4.14.

(ii) With $\mathbf{F} = \mathbf{0}$, we can prove that \mathbf{u} belongs to $\mathbf{W}^{2,p}(\Omega)$ using the same arguments of Problem (2.1) in Theorem 4.14. \square

Remark 4.17. In item (ii) of the the above theorem, we have assumed that $\mathbf{f} \in \mathbf{L}^p(\Omega)$ and $\mathbf{F} = \mathbf{0}$ to get a solution in $\mathbf{W}^{2,p}(\Omega)$. In the case where $\mathbf{F} \neq \mathbf{0}$, it suffices to take $\mathbf{f} \in \mathbf{L}^p(\Omega)$ and $\operatorname{curl} \mathbf{F} \in \mathbf{L}^p(\Omega)$ to have the same regularity result.

Remark 4.18. We can also consider the same generalized right-hand side for Problem (4.35) and we obtain the same results as for (4.36).

5. SECOND ELLIPTIC PROBLEM

We recall the elliptic problem

$$\begin{aligned} -\Delta \mathbf{u} &= \mathbf{f} + \operatorname{div} \mathbb{F} \quad \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} &= \mathbf{a} \times \mathbf{n}, \quad [(\nabla \mathbf{u} + \mathbb{F})\mathbf{n}] \cdot \mathbf{n} = g \quad \text{on } \Gamma_N, \\ \mathbf{u} \cdot \mathbf{n} &= b, \quad [(\nabla \mathbf{u} + \mathbb{F})\mathbf{n}] \times \mathbf{n} = \mathbf{h} \quad \text{on } \Gamma_D, \end{aligned} \tag{5.1}$$

where \mathbf{f} , \mathbf{a} , b , g and \mathbf{h} are given data. We start as the for the first problem by the following special case:

$$\begin{aligned} -\Delta \mathbf{u} &= \mathbf{f} + \operatorname{div} \mathbb{F} \quad \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} &= \mathbf{0}, \quad [(\nabla \mathbf{u} + \mathbb{F})\mathbf{n}] \cdot \mathbf{n} = g \quad \text{on } \Gamma_N, \\ \mathbf{u} \cdot \mathbf{n} &= 0, \quad [(\nabla \mathbf{u} + \mathbb{F})\mathbf{n}] \times \mathbf{n} = \mathbf{h} \quad \text{on } \Gamma_D. \end{aligned} \tag{5.2}$$

We introduce the space

$$\mathbf{V}_{n,\tau}^p(\Omega) = \{ \mathbf{v} \in \mathbf{W}^{1,p}(\Omega), \quad \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_D, \quad \mathbf{v} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_N \}.$$

Remark 5.1. Let Ω be a Lipschitz domain and $\mathbf{v} \in \mathbf{E}^p(\Delta; \Omega)$, then $\frac{\partial \mathbf{v}}{\partial \mathbf{n}} \in W^{-1/p,p}(\Gamma)$ and we have for any $\varphi \in \mathbf{W}^{1,p'}(\Omega)$ the Green formula

$$\int_{\Omega} \varphi \cdot \Delta \mathbf{v} + \int_{\Omega} \nabla \mathbf{v} : \nabla \varphi = \langle \frac{\partial \mathbf{v}}{\partial \mathbf{n}}, \varphi \rangle_{\Gamma}, \tag{5.3}$$

Furthermore, if Ω is of class $\mathcal{C}^{1,1}$, then $(\frac{\partial \mathbf{u}}{\partial \mathbf{n}})_{\tau}$ makes sense in $\mathbf{W}^{-1/p,p}(\Gamma)$ and for any $\varphi \in \mathbf{W}^{1,p'}(\Omega)$ such that $\varphi \cdot \mathbf{n} = 0$ on Γ ,

$$\int_{\Omega} \varphi \cdot \Delta \mathbf{v} + \int_{\Omega} \nabla \mathbf{v} : \nabla \varphi = \langle (\frac{\partial \mathbf{v}}{\partial \mathbf{n}})_{\tau}, \varphi \rangle_{\Gamma}. \tag{5.4}$$

In the next proposition, we prove that the Poincaré inequality holds for all $\mathbf{u} \in \mathbf{V}_{n,\tau}^p(\Omega)$.

Proposition 5.2. For any $\mathbf{v} \in \mathbf{V}_{n,\tau}^p(\Omega)$, we have

$$\|\mathbf{v}\|_{\mathbf{L}^p(\Omega)} \leq C \|\nabla \mathbf{v}\|_{\mathbf{L}^p(\Omega)} \tag{5.5}$$

where $C > 0$ is a constant that only depends on Ω .

Proof. Reasoning by absurdity and assuming that (5.5) is not valid. Then, we can find a sequence $\{\mathbf{v}_k\}_k \in \mathbf{V}_{n,\tau}^p(\Omega)$ such that

$$\|\nabla \mathbf{v}_k\|_{\mathbf{L}^p(\Omega)} < \frac{1}{k} \|\mathbf{v}_k\|_{\mathbf{L}^p(\Omega)}. \tag{5.6}$$

Setting then $\mathbf{u}_k = \frac{\mathbf{v}_k}{\|\mathbf{v}_k\|_{\mathbf{L}^p(\Omega)}}$, we deduce that (\mathbf{u}_k) is bounded in $\mathbf{W}^{1,p}(\Omega)$ with $\|\mathbf{u}_k\|_{\mathbf{L}^p(\Omega)} = 1$. Moreover

$$\nabla \mathbf{u}_k \rightarrow 0 \text{ strongly in } \mathbf{L}^p(\Omega),$$

otherwise $\nabla \mathbf{u} = \mathbf{0}$ in $\mathcal{D}'(\Omega)$. Since Ω is supposed to be a connected set, we have $\mathbf{u} = \mathbf{a}$ where \mathbf{a} is a constant in \mathbb{R}^3 . Because of the weak convergence of \mathbf{u}_k to \mathbf{a} in $\mathbf{W}^{1,p}(\Omega)$ and that $0 = \mathbf{u}_k \cdot \mathbf{n} \rightarrow \mathbf{a} \cdot \mathbf{n}$ in $W^{1-1/p,p}(\Gamma_D)$, also $\mathbf{0} = \mathbf{u}_k \times \mathbf{n} \rightarrow \mathbf{a} \times \mathbf{n}$ in $\mathbf{W}^{1-1/p,p}(\Gamma_N)$ thus $\mathbf{a} = \mathbf{0}$ on Γ . Therefore

$$\mathbf{u}_k \rightarrow \mathbf{0} \text{ in } \mathbf{L}^p(\Omega),$$

which contradicts $\|\mathbf{u}_k\|_{\mathbf{L}^p(\Omega)} = 1$. □

According to Lemma 3.6, we recall that the space $\mathcal{D}(\overline{\Omega})$ is dense in $\mathbf{E}^p(\Delta; \Omega)$ and the mapping $\mathbf{v} \mapsto \frac{\partial \mathbf{v}}{\partial \mathbf{n}}$ is continuous from $\mathbf{E}^p(\Delta; \Omega)$ into $W^{-1/p,p}(\Gamma)$.

Lemma 5.3. For each $\mathbf{v} \in \mathbf{E}^p(\Delta; \Omega)$, the following relations hold

$$\operatorname{div} \mathbf{v} = \operatorname{div}_{\Gamma} \mathbf{v}_{k\tau} - 2K \mathbf{v} \cdot \mathbf{n} + \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \cdot \mathbf{n} \text{ on } \Gamma, \tag{5.7}$$

$$\operatorname{curl} \mathbf{v} \times \mathbf{n} = -\nabla_{\tau}(\mathbf{v} \cdot \mathbf{n}) + \Lambda \mathbf{v} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{n}}\right)_{\tau} \text{ on } \Gamma, \tag{5.8}$$

Proof. Using the above density, there exists a sequence $(\mathbf{v}_k)_k \in \mathcal{D}(\overline{\Omega})$ such that $\mathbf{v}_k \rightarrow \mathbf{v}$ in $\mathbf{E}^p(\Delta; \Omega)$. Thus

$$\operatorname{div} \mathbf{v}_k = \operatorname{div}_{\Gamma} \mathbf{v}_{k\tau} - 2K \mathbf{v}_k \cdot \mathbf{n} + \frac{\partial \mathbf{v}_k}{\partial \mathbf{n}} \cdot \mathbf{n} \text{ on } \Gamma. \tag{5.9}$$

Since $\mathbf{v}_k \rightarrow \mathbf{v}$ in $\mathbf{W}^{1,p}(\Omega)$ and $\Delta \mathbf{v}_k \rightarrow \Delta \mathbf{v}$ in $\mathbf{L}^{r(p)}(\Omega)$ then $\operatorname{div} \mathbf{v}_k \rightarrow \operatorname{div} \mathbf{v}$ in $W^{-1/p,p}(\Gamma)$. Moreover $\mathbf{v}_{k\tau} \rightarrow \mathbf{v}_{\tau}$ in $\mathbf{W}^{1-1/p,p}(\Gamma)$ thus

$$\operatorname{div}_{\Gamma} \mathbf{v}_{k\tau} \rightarrow \operatorname{div}_{\Gamma} \mathbf{v}_{\tau} \text{ in } W^{-1/p,p}(\Gamma).$$

Since $\mathbf{n} \in W^{1,\infty}(\Omega)$, we have

$$2K \mathbf{v}_k \cdot \mathbf{n} \rightarrow 2K \mathbf{v} \cdot \mathbf{n} \text{ in } W^{1-1/p,p}(\Gamma),$$

Now, since $\frac{\partial \mathbf{v}_k}{\partial \mathbf{n}} \rightarrow \frac{\partial \mathbf{v}}{\partial \mathbf{n}}$ in $W^{-1/p,p}(\Gamma)$ so

$$\frac{\partial \mathbf{v}_k}{\partial \mathbf{n}} \cdot \mathbf{n} \rightarrow \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \cdot \mathbf{n} \text{ in } W^{-1/p,p}(\Gamma).$$

Therefore, the relation (5.7) holds. Then we follow the same lines as when proving (5.8). □

Definition 5.4. Given $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$, $\mathbb{F} \in \mathbb{L}^p(\Omega)$, $g \in W^{-1/p,p}(\Gamma_N)$ and $\mathbf{h} \in \mathbf{W}^{-1/p,p}(\Gamma_D)$ such that $\mathbf{h} \cdot \mathbf{n} = 0$ on Γ_N then a function $\mathbf{u} \in \mathbf{V}_{n,\tau}^p(\Omega)$ is called weak solution of Problem (5.2) if it satisfies for any $\varphi \in \mathbf{V}_{n,\tau}'(\Omega)$ that

$$\int_{\Omega} \nabla \mathbf{u} : \nabla \varphi = \int_{\Omega} \mathbf{f} \cdot \varphi - \int_{\Omega} \mathbb{F} : \nabla \varphi + \langle \mathbf{h}, \varphi_{\tau} \rangle_{\Gamma_D} + \langle g, \varphi \cdot \mathbf{n} \rangle_{\Gamma_N}. \tag{5.10}$$

Next, we prove that (5.10) and Problem (5.2) are equivalent.

Proposition 5.5. Let $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$, $\mathbb{F} \in \mathbb{L}^p(\Omega)$, $g \in W^{-1/p,p}(\Gamma_N)$ and $\mathbf{h} \in \mathbf{W}^{-1/p,p}(\Gamma_D)$ such that $\mathbf{h} \cdot \mathbf{n} = 0$ on Γ_N then the following statements are equivalent:

(i) $\mathbf{u} \in \mathbf{V}_{n,\tau}^p(\Omega)$ is solution of (5.2) in the sense of (5.10).

(ii) There exists $(\mathbf{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times \mathbf{L}_0^p(\Omega)$ satisfying

$$\begin{aligned} -\Delta \mathbf{u} &= \mathbf{f} + \operatorname{div} \mathbb{F}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} &= \mathbf{0} \quad \text{on } \Gamma_N, \\ [(\nabla \mathbf{u} + \mathbb{F})\mathbf{n}] \cdot \mathbf{n} &= g \quad \text{in } W^{-1/p,p}(\Gamma_N), \\ \mathbf{u} \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma_D, \\ [(\nabla \mathbf{u} + \mathbb{F})\mathbf{n}]_{\tau} &= \mathbf{h} \quad \text{in } \mathbf{W}^{-1/p,p}(\Gamma_D). \end{aligned} \tag{5.11}$$

Proof. If $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ is solution of (5.11), thanks to Green’s formula (5.3), it is easily checked that \mathbf{u} is a weak solution of (5.2) in the sense of (5.10).

Conversely, we can let $\mathbf{u} \in \mathbf{V}_{n,\tau}^p(\Omega)$ be a weak solution of (5.2). Taking the test function $\varphi \in \mathcal{D}(\Omega)$ in (5.10) leads to

$$\langle -\Delta \mathbf{u}, \varphi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} = \int_{\Omega} \nabla \mathbf{u} : \nabla \varphi = \int_{\Omega} \mathbf{f} \cdot \varphi - \int_{\Omega} \mathbb{F} : \nabla \varphi.$$

So

$$-\Delta \mathbf{u} = \mathbf{f} + \operatorname{div} \mathbb{F} \quad \text{in } \Omega. \tag{5.12}$$

As $\mathbf{u} \in \mathbf{V}_{n,\tau}^p(\Omega)$, we have $\mathbf{u} \times \mathbf{n} = \mathbf{0}$ on Γ_N and $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ_D so it remains to show that $[(\nabla \mathbf{u} + \mathbb{F})\mathbf{n}] \cdot \mathbf{n} = g$ on Γ_N and $[(\nabla \mathbf{u} + \mathbb{F})\mathbf{n}]_{\tau} = \mathbf{h}$ on Γ_D . Taking the dual product of (5.12) with $\varphi \in \mathbf{V}_{n,\tau}'(\Omega)$ and from (5.3), we infer that for any $\varphi \in \mathbf{V}_{n,\tau}'(\Omega)$,

$$\langle [(\nabla \mathbf{u} + \mathbb{F})\mathbf{n}]_{\tau}, \varphi_{\tau} \rangle_{\Gamma_D} + \langle [(\nabla \mathbf{u} + \mathbb{F})\mathbf{n}] \cdot \mathbf{n}, \varphi \cdot \mathbf{n} \rangle_{\Gamma_N} = \langle \mathbf{h}, \varphi_{\tau} \rangle_{\Gamma_D} + \langle g, \varphi \cdot \mathbf{n} \rangle_{\Gamma_N}.$$

Let $\boldsymbol{\mu} \in \mathbf{W}^{1/p,p'}(\Gamma_D)$, so there exists $\varphi \in \mathbf{W}^{1,p'}(\Omega)$ with $\operatorname{div} \varphi = 0$ in Ω and such that $\varphi = \boldsymbol{\mu}_{\tau}$ on Γ_D and $\varphi = \mathbf{0}$ on Γ_N . We can then write

$$\langle [(\nabla \mathbf{u} + \mathbb{F})\mathbf{n}]_{\tau} - \mathbf{h}, \boldsymbol{\mu} \rangle_{\Gamma_D} = \langle [(\nabla \mathbf{u} + \mathbb{F})\mathbf{n}]_{\tau} - \mathbf{h}, \boldsymbol{\mu}_{\tau} \rangle_{\Gamma_D} = \langle [(\nabla \mathbf{u} + \mathbb{F})\mathbf{n}]_{\tau} - \mathbf{h}, \varphi \rangle_{\Gamma_D} = 0.$$

Thus $[(\nabla \mathbf{u} + \mathbb{F})\mathbf{n}]_{\tau} = \mathbf{h}$ in $\mathbf{W}^{-1/p,p}(\Gamma_D)$. Now, let μ be any element of $W^{1/p,p'}(\Gamma_N)$. Then there exists $\psi \in W^{2,p'}(\Omega)$ such that $\Delta \psi = 0$ in Ω , $\frac{\partial \psi}{\partial \mathbf{n}} = \mu$ on Γ_N and $\psi = 0$ on Γ_D . By setting $\nabla \psi = \varphi$ we have

$$\langle [(\nabla \mathbf{u} + \mathbb{F})\mathbf{n}] \cdot \mathbf{n} - g, \mu \rangle_{\Gamma_N} = \langle [(\nabla \mathbf{u} + \mathbb{F})\mathbf{n}] \cdot \mathbf{n} - g, \frac{\partial \psi}{\partial \mathbf{n}} \rangle_{\Gamma_N} = \langle [(\nabla \mathbf{u} + \mathbb{F})\mathbf{n}] \cdot \mathbf{n} - g, \varphi \cdot \mathbf{n} \rangle_{\Gamma_N} = 0,$$

which implies that $[(\nabla \mathbf{u} + \mathbb{F})\mathbf{n}] \cdot \mathbf{n} = g \in W^{-1/p,p}(\Gamma_N)$. □

5.1. Hilbertian case. In the following, we prove the existence and uniqueness of weak and strong solutions associated to Problem (5.2) in L^2 -theory.

Theorem 5.6. (i) Assume that $\mathbf{f} \in \mathbf{L}^{6/5}(\Omega)$, $\mathbb{F} \in \mathbb{L}^2(\Omega)$, $g \in H^{-1/2}(\Gamma_N)$ and $\mathbf{h} \in \mathbf{H}^{-1/2}(\Gamma_D)$ such that $\mathbf{h} \cdot \mathbf{n} = 0$ on Γ_N . Then, Problem (5.2) admits a unique solution $\mathbf{u} \in \mathbf{V}_{n,\tau}^2(\Omega)$ satisfying

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \leq C (\|\mathbf{f}\|_{\mathbf{L}^{6/5}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^2(\Omega)} + \|g\|_{H^{-1/2}(\Gamma_N)} + \|\mathbf{h}\|_{\mathbf{H}^{-1/2}(\Gamma_D)}). \tag{5.13}$$

(ii) Furthermore, if Ω is of class $\mathcal{C}^{2,1}$, $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $g \in \mathbf{H}^{1/2}(\Gamma_N)$ and $\mathbf{h} \in \mathbf{H}^{1/2}(\Gamma_D)$ then the solution of Problem (5.2), with $\mathbb{F} = 0$ belongs to $\mathbf{H}^2(\Omega)$ and satisfies the estimate

$$\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \leq C (\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma_N)} + \|\mathbf{h}\|_{\mathbf{H}^{1/2}(\Gamma_D)}). \tag{5.14}$$

Proof. (i) The bilinear form

$$a(\mathbf{u}, \varphi) = \int_{\Omega} \nabla \mathbf{u} : \nabla \varphi \tag{5.15}$$

is continuous and coercive on $\mathbf{V}_{n,\tau}^2(\Omega)$ due to Poincaré’s inequality (5.5). Moreover, the second-hand side of (5.10) is a linear and continuous form on $\mathbf{V}_{n,\tau}^2(\Omega)$, one may then apply the Lax Milgram theorem in order to get the existence and uniqueness of a solution $\mathbf{u} \in \mathbf{V}_{n,\tau}^2(\Omega)$ to (5.10) satisfying (5.13).

(ii) Regularity. Assume that $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and Ω is of class $\mathcal{C}^{2,1}$ so there exists a unique $\mathbf{u} \in H^1(\Omega)$ solution of (5.2). The idea of the proof is basically similar to that of Theorem 4.1, with a partition of unity where $\mathbf{v} = \theta \mathbf{u}$ and $\mathbf{w} = \eta \mathbf{u}$, we look for the regularity of $\mathbf{v} \in \mathbf{H}^1(\Omega)$ satisfying

$$\begin{aligned} -\Delta \mathbf{v} &= \mathbf{F}_{\theta} \quad \text{in } \Omega, \\ \mathbf{v} \times \mathbf{n} &= \mathbf{0}, \quad \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \cdot \mathbf{n} = \theta g \quad \text{on } \Gamma, \end{aligned} \tag{5.16}$$

where $\mathbf{F}_{\theta} = \theta \mathbf{f} - 2\nabla \theta \cdot \nabla \mathbf{u} - \mathbf{u} \Delta \theta \in \mathbf{L}^2(\Omega)$. From (5.7) we have

$$\operatorname{div} \mathbf{v} = -2K \mathbf{v} \cdot \mathbf{n} + \theta g \quad \text{on } \Gamma.$$

Since K belongs to $W^{1,\infty}(\Gamma)$ because of the $\mathcal{C}^{2,1}$ regularity of Ω , $\operatorname{div} \mathbf{v}$ belongs to $H^{1/2}(\Gamma)$. Moreover, we have $\Delta \operatorname{div} \mathbf{v} = \operatorname{div} \mathbf{F}_{\theta} \in H^{-1}(\Omega)$ then $\operatorname{div} \mathbf{v}$ belongs to $H^1(\Omega)$. Let $\chi \in H^1(\Omega)$ such that $\Delta \chi = \operatorname{div} \mathbf{F}_{\theta}$ in Ω and $\chi = -2K \mathbf{v} \cdot \mathbf{n} + \theta g$ on Γ . The function $\lambda = \chi - \operatorname{div} \mathbf{v}$ is harmonic and $\lambda = 0$ on Γ then λ is zero [2] and thus $\operatorname{div} \mathbf{v}$ belongs to $H^1(\Omega)$. We know that $\operatorname{curl} \mathbf{v} \in \mathbf{L}^2(\Omega)$ and $\operatorname{curl} \operatorname{curl} \mathbf{v} = -\Delta \mathbf{v} + \nabla(\operatorname{div} \mathbf{v})$ which means that $\operatorname{curl} \operatorname{curl} \mathbf{v} \in \mathbf{L}^2(\Omega)$ and $\operatorname{div} \operatorname{curl} \mathbf{v} = 0$. According to Lemma 3.5, we have $\operatorname{curl} \mathbf{v} \cdot \mathbf{n} = \operatorname{div}_{\Gamma}(\mathbf{v} \times \mathbf{n}) = 0$ on Γ since $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ on Γ . Consequently $\operatorname{curl} \mathbf{v} \in \mathbf{X}_{\Gamma}^2(\Omega) \leftrightarrow \mathbf{H}^1(\Omega)$. Because $\mathbf{v} \in \mathbf{L}^2(\Omega)$, $\operatorname{curl} \mathbf{v} \in \mathbf{H}^1(\Omega)$, $\operatorname{div} \mathbf{v} \in H^1(\Omega)$ and $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ on Γ , we infer that \mathbf{v} belongs to $\mathbf{H}^2(\Omega)$ from (3.2).

Now, we look for the regularity of $\mathbf{w} \in \mathbf{H}^1(\Omega)$ satisfying

$$\begin{aligned} -\Delta \mathbf{w} &= \mathbf{F}_{\eta} \quad \text{in } \Omega, \\ \mathbf{w} \cdot \mathbf{n} &= 0, \quad \frac{\partial \mathbf{w}}{\partial \mathbf{n}} \times \mathbf{n} = \eta \mathbf{h} \quad \text{on } \Gamma. \end{aligned} \tag{5.17}$$

Setting $\mathbf{z} = \operatorname{curl} \mathbf{w}$ so $\operatorname{div} \mathbf{z} = 0$ in Ω and $-\Delta \mathbf{z} = \operatorname{curl} \mathbf{F}_{\eta}$ in Ω . Since $\mathbf{w} \cdot \mathbf{n} = 0$ and $\frac{\partial \mathbf{w}}{\partial \mathbf{n}} \times \mathbf{n} = \eta \mathbf{h}$ on Γ , we deduce from (5.8) that

$$\mathbf{z} \times \mathbf{n} = \Lambda \mathbf{w} + \eta \mathbf{h} \quad \text{on } \Gamma,$$

thus $\mathbf{z} \times \mathbf{n} \in \mathbf{H}^{1/2}(\Gamma)$ so \mathbf{z} satisfies

$$\begin{aligned} -\Delta \mathbf{z} &= \operatorname{curl} \mathbf{F}_{\eta} \quad \text{and} \quad \operatorname{div} \mathbf{z} = 0 \quad \text{in } \Omega, \\ \mathbf{z} \times \mathbf{n} &= \Lambda \mathbf{w} + \eta \mathbf{h} \quad \text{on } \Gamma, \\ \langle \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} &= 0, \quad \forall 1 \leq i \leq I, \end{aligned} \tag{5.18}$$

as in the proof of Theorem 4.1 point (ii), we can show that $\mathbf{z} \in \mathbf{H}^1(\Omega)$. Since $\nabla(\operatorname{div} \mathbf{w}) = -\operatorname{curl} \operatorname{curl} \mathbf{w} + \Delta \mathbf{w} = -\operatorname{curl} \mathbf{z} - \mathbf{F}_{\eta} \in \mathbf{L}^2(\Omega)$; therefore, $\operatorname{div} \mathbf{w} \in H^1(\Omega)$. Moreover, $\mathbf{w} \in \mathbf{L}^2(\Omega)$ and $\mathbf{w} \cdot \mathbf{n} = 0$ on Γ , we infer that \mathbf{w} belongs to $\mathbf{H}^2(\Omega)$ by (3.1). As a consequence, $\mathbf{u} \in \mathbf{H}^2(\Omega)$. \square

5.2. L^p -theory. The task now is to introduce a strategy to get a solution in $\mathbf{W}^{1,p}(\Omega)$ for any $1 < p < +\infty$, based on some inf-sup conditions and on where $\partial \Sigma$ lies.

Theorem 5.7. *Assume that $\partial \Sigma \subset \Gamma_D$ or $\partial \Sigma \subset \Gamma_N$. Then for any $\ell \in [\mathbf{V}_{n,\tau}^{p'}(\Omega)]'$ the problem: Find $\mathbf{u} \in \mathbf{V}_{n,\tau}^p(\Omega)$ such that for any $\varphi \in \mathbf{V}_{n,\tau}^{p'}(\Omega)$ such that*

$$a(\mathbf{u}, \varphi) = \langle \ell, \varphi \rangle,$$

where the bilinear form $a(\cdot, \cdot)$ is defined in (5.15), has a unique solution.

Proof. (1) We assume that $\partial\Sigma \subset \Gamma_D$:

Step 1: Case $p \geq 2$. Since $[\mathbf{V}_{n,\tau}^{p'}(\Omega)]' \hookrightarrow [\mathbf{V}_{n,\tau}^2(\Omega)]'$ by the Lax Milgram theorem, there exists a unique solution $\mathbf{u} \in \mathbf{V}_{n,\tau}^2(\Omega)$ such that

$$a(\mathbf{u}, \varphi) = \langle \ell, \varphi \rangle, \quad \forall \varphi \in \mathbf{V}_{n,\tau}^2(\Omega). \tag{5.19}$$

Recall that we proved an inf-sup condition in Lemma 4.7 associated with the form

$$b(\mathbf{u}, \varphi) = \int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \varphi + (\operatorname{div} \mathbf{u})(\operatorname{div} \varphi) \, dx.$$

Let $\mathbf{v} \in \mathbf{H}^1(\Omega)$ with $\Delta \mathbf{v} \in \mathbf{L}^{6/5}(\Omega)$, so we have the following Green formula for any $\varphi \in \mathbf{V}_{n,\tau}^2(\Omega)$,

$$-\langle \Delta \mathbf{v}, \varphi \rangle_{\Omega} = \int_{\Omega} \nabla \mathbf{v} : \nabla \varphi \, dx - \left\langle \frac{\partial \mathbf{v}}{\partial \mathbf{n}}, \varphi \right\rangle_{\Gamma}$$

but we know that $\mathbf{v} = \mathbf{v}_{\tau} + (\mathbf{v} \cdot \mathbf{n})\mathbf{n}$ thus

$$\frac{\partial \mathbf{v}}{\partial \mathbf{n}} = \left(\frac{\partial \mathbf{v}}{\partial \mathbf{n}} \right)_{\tau} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{n}} \cdot \mathbf{n} \right) \mathbf{n}.$$

So we obtain

$$-\langle \Delta \mathbf{v}, \varphi \rangle_{\Omega} = \int_{\Omega} \nabla \mathbf{v} : \nabla \varphi \, dx - \left\langle \left(\frac{\partial \mathbf{v}}{\partial \mathbf{n}} \right)_{\tau}, \varphi_{\tau} \right\rangle_{\Gamma_D} - \left\langle \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \cdot \mathbf{n}, \varphi \cdot \mathbf{n} \right\rangle_{\Gamma_N}.$$

On the other hand, we have the Green formula for any $\varphi \in \mathbf{V}_{n,\tau}^2(\Omega)$,

$$-\langle \Delta \mathbf{v}, \varphi \rangle_{\Omega} = \int_{\Omega} \operatorname{curl} \mathbf{v} \cdot \operatorname{curl} \varphi + (\operatorname{div} \mathbf{v})(\operatorname{div} \varphi) - \langle \operatorname{curl} \mathbf{v} \times \mathbf{n}, \varphi \rangle_{\Gamma_D} - \langle \operatorname{div} \mathbf{v}, \varphi \cdot \mathbf{n} \rangle_{\Gamma_N}.$$

Then, we can write

$$\begin{aligned} \int_{\Omega} \nabla \mathbf{v} : \nabla \varphi \, dx &= \int_{\Omega} \operatorname{curl} \mathbf{v} \cdot \operatorname{curl} \varphi + (\operatorname{div} \mathbf{v})(\operatorname{div} \varphi) - \langle \operatorname{curl} \mathbf{v} \times \mathbf{n}, \varphi \rangle_{\Gamma_D} \\ &\quad - \langle \operatorname{div} \mathbf{v}, \varphi \cdot \mathbf{n} \rangle_{\Gamma_N} + \left\langle \left(\frac{\partial \mathbf{v}}{\partial \mathbf{n}} \right)_{\tau}, \varphi_{\tau} \right\rangle_{\Gamma_D} + \left\langle \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \cdot \mathbf{n}, \varphi \cdot \mathbf{n} \right\rangle_{\Gamma_N}. \end{aligned} \tag{5.20}$$

From the relations (1.4) and (1.5), we deduce that

$$\operatorname{div} \mathbf{v} - \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \cdot \mathbf{n} = -2K\mathbf{v} \cdot \mathbf{n} + \operatorname{div}_{\Gamma} \mathbf{v}_{\tau}, \quad \operatorname{curl} \mathbf{v} \times \mathbf{n} - \left(\frac{\partial \mathbf{v}}{\partial \mathbf{n}} \right)_{\tau} = \Lambda \mathbf{v} - \nabla_{\tau}(\mathbf{v} \cdot \mathbf{n}).$$

Replacing this in (5.20) gives that $\mathbf{v} \in \mathcal{D}(\overline{\Omega})$ satisfies

$$\int_{\Omega} \nabla \mathbf{v} : \nabla \varphi \, dx = \int_{\Omega} \operatorname{curl} \mathbf{v} \cdot \operatorname{curl} \varphi + (\operatorname{div} \mathbf{v})(\operatorname{div} \varphi) - \langle \Lambda \mathbf{v}, \varphi \rangle_{\Gamma_D} + 2\langle K\mathbf{v} \cdot \mathbf{n}, \varphi \cdot \mathbf{n} \rangle_{\Gamma_N}. \tag{5.21}$$

From the density of $\mathcal{D}(\overline{\Omega})$ into $H^1(\Omega)$, the formula (5.21) holds for any $\mathbf{v} \in \mathbf{V}_{n,\tau}^2(\Omega)$ and $\varphi \in \mathbf{V}_{n,\tau}^2(\Omega)$. Consequently, (5.19) becomes

$$\int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \varphi + (\operatorname{div} \mathbf{u})(\operatorname{div} \varphi) = \langle \ell, \varphi \rangle + \langle \Lambda \mathbf{u}, \varphi \rangle_{\Gamma_D} - 2\langle K\mathbf{u} \cdot \mathbf{n}, \varphi \cdot \mathbf{n} \rangle_{\Gamma_N}.$$

We discuss the regularity of the solution \mathbf{u} case by case.

(i) For any $p \leq 6$: Since Ω is of class $\mathcal{C}^{1,1}$, we have $K \in L^{\infty}(\Gamma)$ and $\mathbf{u} \cdot \mathbf{n} \in H^{1/2}(\Gamma_N) \hookrightarrow L^4(\Gamma_N)$. Thus $K\mathbf{u} \cdot \mathbf{n} \in L^4(\Gamma_N) \hookrightarrow W^{-1/6,6}(\Gamma_N)$ and similarly $\Lambda \mathbf{u} \in \mathbf{L}^4(\Gamma_D) \hookrightarrow \mathbf{W}^{-1/6,6}(\Gamma_D)$. Since $p \leq 6$ then $K\mathbf{u} \cdot \mathbf{n} \in W^{-1/p,p}(\Gamma_N)$ and $\Lambda \mathbf{u} \in \mathbf{W}^{-1/p,p}(\Gamma_D)$ so that the mapping

$$\langle L, \varphi \rangle = \langle \ell, \varphi \rangle + \langle \Lambda \mathbf{u}, \varphi \rangle_{\Gamma_D} - 2\langle K\mathbf{u} \cdot \mathbf{n}, \varphi \cdot \mathbf{n} \rangle_{\Gamma_N}, \quad \forall \varphi \in \mathbf{V}_{n,\tau}^p(\Omega) \tag{5.22}$$

defines an element in the dual space of $\mathbf{V}_{n,\tau}^{p'}(\Omega)$. Then, according to the inf-sup condition (4.18) and Babùska-Brezzi theorem, there exists a unique $\mathbf{w} \in \mathbf{V}_{n,\tau}^p(\Omega)$ satisfying

$$\int_{\Omega} \operatorname{curl} \mathbf{w} \cdot \operatorname{curl} \varphi + (\operatorname{div} \mathbf{w})(\operatorname{div} \varphi) = \langle L, \varphi \rangle_{[\widetilde{\mathbf{W}}_{\Sigma}^{p'}(\Omega)]' \times \widetilde{\mathbf{W}}_{\Sigma}^{p'}(\Omega)}, \quad \forall \varphi \in \widetilde{\mathbf{W}}_{\Sigma}^{p'}(\Omega). \tag{5.23}$$

Note that $\widetilde{\text{grad}} s_j^\ell \in \mathbf{V}_{n,\tau}^2(\Omega) \hookrightarrow \mathbf{V}_{n,\tau}^{p'}(\Omega)$ and we have

$$\begin{aligned} \langle L, \widetilde{\text{grad}} s_j^\ell \rangle_{[\widetilde{\mathbf{W}}_\Sigma^{p'}(\Omega)]' \times \widetilde{\mathbf{W}}_\Sigma^{p'}(\Omega)} &= \langle \boldsymbol{\ell}, \widetilde{\text{grad}} s_j^\ell \rangle + \langle \Lambda \mathbf{u}, \widetilde{\text{grad}} s_j^\ell \rangle_{\Gamma_D} - 2 \langle K \mathbf{u} \cdot \mathbf{n}, \widetilde{\text{grad}} s_j^\ell \cdot \mathbf{n} \rangle_{\Gamma_N} \\ &= \int_\Omega \text{curl } \mathbf{w} \cdot \text{curl } \widetilde{\text{grad}} s_j^\ell + \int_\Omega (\text{div } \mathbf{w})(\text{div } \widetilde{\text{grad}} s_j^\ell) = 0. \end{aligned}$$

To extend (5.23) to any test function φ in $\mathbf{V}_{n,\tau}^{p'}(\Omega)$, we set

$$\tilde{\varphi} = \varphi - \sum_{\ell=1}^{L_N} \sum_{j=1}^J \left(\frac{1}{L_N} \langle \varphi \cdot \mathbf{n}, 1 \rangle_{\Sigma_j} + \frac{1}{J} \langle \varphi \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} \right) \widetilde{\text{grad}} s_j^\ell$$

which yields $\langle L, \varphi \rangle = \langle L, \tilde{\varphi} \rangle$ and $\tilde{\varphi} \in \widetilde{\mathbf{W}}_\Sigma^{p'}(\Omega)$. Thus, we obtain

$$\int_\Omega \text{curl } \mathbf{w} \cdot \text{curl } \varphi + (\text{div } \mathbf{w})(\text{div } \varphi) = \int_\Omega \text{curl } \mathbf{w} \cdot \text{curl } \tilde{\varphi} dx + \int_\Omega (\text{div } \mathbf{w})(\text{div } \tilde{\varphi}) dx = \langle L, \tilde{\varphi} \rangle = \langle L, \varphi \rangle.$$

We deduce that $\mathbf{w} \in \mathbf{V}_{n,\tau}^p(\Omega)$ satisfies

$$\int_\Omega \text{curl } \mathbf{w} \cdot \text{curl } \varphi + (\text{div } \mathbf{w})(\text{div } \varphi) = \langle L, \varphi \rangle, \quad \forall \varphi \in \mathbf{V}_{n,\tau}^{p'}(\Omega). \tag{5.24}$$

Since $\mathbf{V}_{n,\tau}^2(\Omega) \hookrightarrow \mathbf{V}_{n,\tau}^{p'}(\Omega)$, we infer for any $\varphi \in \mathbf{V}_{n,\tau}^2(\Omega)$ that

$$\int_\Omega \text{curl } \mathbf{w} \cdot \text{curl } \varphi + (\text{div } \mathbf{w})(\text{div } \varphi) = \int_\Omega \text{curl } \mathbf{u} \cdot \text{curl } \varphi + (\text{div } \mathbf{u})(\text{div } \varphi),$$

taking $\varphi = \mathbf{w} - \mathbf{u}$ leads to $\text{curl } \mathbf{w} = \text{curl } \mathbf{u}$ and $\text{div } \mathbf{w} = \text{div } \mathbf{u}$ in Ω . Finally, since $\mathbf{u} \in \mathbf{L}^6(\Omega) \hookrightarrow \mathbf{L}^p(\Omega)$, $\text{curl } \mathbf{u} \in \mathbf{L}^p(\Omega)$ and $\text{div } \mathbf{u} \in L^p(\Omega)$. Moreover $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ_D and $\mathbf{u} \times \mathbf{n} = \mathbf{0}$ on Γ_N , hence $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ according to [4, Corollary 3.5].

(ii) For any $p > 6$: we have $\mathbf{u} \in \mathbf{W}^{1,6}(\Omega)$ thus $\mathbf{u} \cdot \mathbf{n}$ and \mathbf{u}_τ belong respectively to $W^{1-1/6,6}(\Gamma_N)$ and $\mathbf{W}^{1-1/6,6}(\Gamma_D)$. But $\mathbf{W}^{1-1/6,6}(\Gamma) \hookrightarrow \mathbf{W}^{-1/p,p}(\Gamma)$, then the mapping L defined in (5.22) is an element in the dual space of $\mathbf{V}_{n,\tau}^{p'}(\Omega)$. Thus, there exists a unique $\mathbf{w} \in \mathbf{V}_{n,\tau}^p(\Omega)$ such that (5.24) holds for any $\varphi \in \mathbf{V}_{n,\tau}^{p'}(\Omega)$. Then, we infer similarly that \mathbf{u} belongs to $\mathbf{W}^{1,p}(\Omega)$.

Step 2: Case $1 < p < 2$. We consider the operator $A \in \mathcal{L}(\mathbf{V}_{n,\tau}^p(\Omega), (\mathbf{V}_{n,\tau}^{p'}(\Omega))')$ associated with the bilinear form

$$a(\mathbf{u}, \varphi) = \int_\Omega \nabla \mathbf{u} : \nabla \varphi dx,$$

such that $\langle A \mathbf{u}, \varphi \rangle = a(\mathbf{u}, \varphi)$. From the first step and by virtue of the Babùska-Brezzi theorem, we deduce that the operator A is an isomorphism for $p > 2$ and its dual A' is an isomorphism from $\mathbf{V}_{n,\tau}^{p'}(\Omega)$ into $(\mathbf{V}_{n,\tau}^p(\Omega))'$ for $p' < 2$. This means that the operator A is an isomorphism for any $p < 2$. Consequently the following inf-sup condition holds for any $1 < p < \infty$,

$$\inf_{\varphi \in \mathbf{V}_{n,\tau}^{p'}(\Omega)} \sup_{\mathbf{u} \in \mathbf{V}_{n,\tau}^p(\Omega)} \frac{|\int_\Omega \nabla \mathbf{u} : \nabla \varphi|}{\|\nabla \mathbf{u}\|_{\mathbf{L}^p(\Omega)} \|\nabla \varphi\|_{\mathbf{L}^{p'}(\Omega)}} > 0.$$

This completes the proof for this first case.

(2) We now assume that $\partial \Sigma \subset \Gamma_N$. The proof is exactly the same as above except the case where $p \leq 6$, in which we recall the inf-sup condition (4.20) and Babùska-Brezzi theorem, to prove the existence and uniqueness of $\mathbf{w} \in \mathbf{V}_{n,\tau}^p(\Omega)$ satisfying

$$\int_\Omega \text{curl } \mathbf{w} \cdot \text{curl } \varphi + (\text{div } \mathbf{w})(\text{div } \varphi) = \langle L, \varphi \rangle_{[\tilde{\mathbf{V}}_0^{p'}(\Omega)]' \times \tilde{\mathbf{V}}_0^{p'}(\Omega)}, \quad \forall \varphi \in \tilde{\mathbf{V}}_0^{p'}(\Omega). \tag{5.25}$$

Note that $\nabla s_\ell \in \mathbf{V}_{n,\tau}^2(\Omega) \hookrightarrow \mathbf{V}_{n,\tau}^{p'}(\Omega)$ and we have

$$\begin{aligned} \langle L, \nabla s_\ell \rangle_{[\tilde{\mathbf{V}}_0^{p'}(\Omega)]' \times \tilde{\mathbf{V}}_0^{p'}(\Omega)} &= \langle \boldsymbol{\ell}, \nabla s_\ell \rangle + \langle \Lambda \mathbf{u}, \nabla s_\ell \rangle_{\Gamma_D} - 2 \langle K \mathbf{u} \cdot \mathbf{n}, \nabla s_\ell \cdot \mathbf{n} \rangle_{\Gamma_N} \\ &= \int_\Omega \text{curl } \mathbf{w} \cdot \text{curl } \nabla s_\ell + \int_\Omega (\text{div } \mathbf{w})(\text{div } \nabla s_\ell) = 0. \end{aligned}$$

To extend (5.25) to any test function φ in $\mathbf{V}_{n,\tau}^{p'}(\Omega)$, we set

$$\tilde{\varphi} = \varphi - \sum_{\ell=1}^{L_N} \langle \varphi \cdot \mathbf{n}, 1 \rangle_{\Gamma_N^\ell} \nabla s_\ell$$

which yields $\langle L, \varphi \rangle = \langle L, \tilde{\varphi} \rangle$ and $\tilde{\varphi} \in \tilde{\mathbf{V}}_0^{p'}(\Omega)$. The remainder of the proof is exactly the same as above. \square

Applying the Babùska-Brezzi theorem, we obtain the following result.

Proposition 5.8. *Assume that $\partial\Sigma \subset \Gamma_D$ or $\partial\Sigma \subset \Gamma_N$ then there exists a constant $\gamma > 0$ depending only on Ω and p such that the following inf-sup condition holds*

$$\inf_{\varphi \in \mathbf{V}_{n,\tau}^{p'}(\Omega), \varphi \neq 0} \left[\sup_{\mathbf{u} \in \mathbf{V}_{n,\tau}^p(\Omega), \mathbf{u} \neq 0} \frac{|\int_{\Omega} \nabla \mathbf{u} : \nabla \varphi|}{\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \|\varphi\|_{\mathbf{W}^{1,p'}(\Omega)}} \right] \geq \gamma. \tag{5.26}$$

Now, we are in position to show that the solution of Problem (5.2) belongs to $\mathbf{W}^{1,p}(\Omega)$.

Theorem 5.9. *(i) Assume that $\partial\Sigma \subset \Gamma_D$ or $\partial\Sigma \subset \Gamma_N$. Let $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$, $\mathbb{F} \in \mathbb{L}^p(\Omega)$, $g \in W^{-1/p,p}(\Gamma_N)$ and $\mathbf{h} \in \mathbf{W}^{-1/p,p}(\Gamma_D)$ then Problem (2.2) has a unique solution $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ and it satisfies*

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \left(\|\mathbf{f}\|_{\mathbf{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma_N)} + \|\mathbf{h}\|_{\mathbf{W}^{-1/p,p}(\Gamma_D)} \right).$$

(ii) Furthermore, if Ω is of class $\mathcal{C}^{2,1}$, $\mathbf{f} \in \mathbf{L}^p(\Omega)$, $g \in W^{1-1/p,p}(\Gamma_N)$ and $\mathbf{h} \in \mathbf{W}^{1-1/p,p}(\Gamma_D)$ with $\mathbb{F} = 0$, then \mathbf{u} belongs to $\mathbf{W}^{2,p}(\Omega)$ and the following estimate holds

$$\|\mathbf{u}\|_{\mathbf{W}^{2,p}(\Omega)} \leq C \left(\|\mathbf{f}\|_{\mathbf{L}^p(\Omega)} + \|g\|_{W^{1-1/p,p}(\Gamma_N)} + \|\mathbf{h}\|_{\mathbf{W}^{1-1/p,p}(\Gamma_D)} \right).$$

Proof. (i) The first result is straightforward due to Proposition 5.8.

(ii) Assume that $\mathbf{f} \in \mathbf{L}^p(\Omega) \hookrightarrow \mathbf{L}^{r(p)}(\Omega)$, there exists a unique solution $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ solution of (2.2). We recall the systems (5.16) and (5.17) in Theorem 5.6 as their solutions \mathbf{v} and \mathbf{w} remain in $\mathbf{W}^{1,p}(\Omega)$. We have

$$\operatorname{div} \mathbf{v} = K\mathbf{v} \cdot \mathbf{n} + \theta g \in W^{1-1/p,p}(\Gamma)$$

since $K \in \mathbf{W}^{1,\infty}(\Gamma)$. Moreover, $-\Delta(\operatorname{div} \mathbf{v}) = \operatorname{div} \mathbf{F}_\theta \in \mathbf{W}^{-1,p}(\Omega)$. It follows that $\operatorname{div} \mathbf{v}$ belongs to $W^{1,p}(\Omega)$. As $-\Delta \mathbf{v} = \operatorname{curl} \operatorname{curl} \mathbf{v} - \nabla(\operatorname{div} \mathbf{v})$ then $\operatorname{curl} \mathbf{v} \in \mathbf{X}_T^p(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$. Consequently, $\mathbf{v} \in \mathbf{W}^{2,p}(\Omega)$.

Similarly, taking $\mathbf{z} = \operatorname{curl} \mathbf{w}$ in Problem (5.17) leads to $-\Delta \mathbf{z} = \operatorname{curl} \mathbf{F}_\eta \in \mathbf{W}^{-1,p}(\Omega)$ and $\mathbf{z} \times \mathbf{n} = \Lambda \mathbf{w} + \eta \mathbf{h} \in \mathbf{W}^{1-1/p,p}(\Omega)$. Following the regularity step in the proof of Theorem 5.6, we obtain that $\operatorname{curl} \mathbf{w} \in \mathbf{W}^{1,p}(\Omega)$ and $\nabla(\operatorname{div} \mathbf{w}) \in \mathbf{L}^p(\Omega)$ leading to $\operatorname{div} \mathbf{w} \in W^{1,p}(\Omega)$. Thus $\mathbf{w} \in \mathbf{W}^{2,p}(\Omega)$, we finally infer that \mathbf{u} belongs to $\mathbf{W}^{2,p}(\Omega)$. \square

Remark 5.10. Note that the condition $\partial\Sigma \subset \Gamma_D$ or $\partial\Sigma \subset \Gamma_N$ is not natural due to the boundary conditions of Problem (5.2). However, it gives a generalized solution in $\mathbf{W}^{1,p}(\Omega)$ for any $1 < p < +\infty$.

Remark 5.11. Without any restriction on $\partial\Sigma$, we can prove that the solution of Problem (5.2) belongs to $\mathbf{W}^{1,p}(\Omega)$ for any $p \geq 2$ using a partition of unity and with $\mathbb{F} = 0$.

Remark 5.12. In the case where $\mathbb{F} = 0$ and $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$ for any $p \geq \frac{3}{2}$, the solution of Problem (5.2) belongs to $\mathbf{W}^{1,p}(\Omega)$ and when $\mathbf{f} \in \mathbf{L}^1(\Omega)$, the solution belongs to $\mathbf{W}^{1,s}(\Omega)$ for any $s < 3/2$.

In the sequel, we generalize the existence, uniqueness and regularity of the solutions of Problem (5.2) to the case of non-homogeneous boundary conditions introduced above and call it (\mathcal{Q}) .

Theorem 5.13. *(i) Assume that $\partial\Sigma \subset \Gamma_D$ or $\partial\Sigma \subset \Gamma_N$. Let $\mathbf{f} \in \mathbf{L}^{r(p)}(\Omega)$, $\mathbb{F} \in \mathbb{L}^p(\Omega)$, $\mathbf{a} \times \mathbf{n} \in \mathbf{W}^{1-1/p,p}(\Gamma_N)$, $g \in W^{-1/p,p}(\Gamma_N)$, $b \in W^{1-1/p,p}(\Gamma_D)$ and $\mathbf{h} \in \mathbf{W}^{-1/p,p}(\Gamma_D)$ then Problem (5.1) has a unique solution $\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$ and it satisfies*

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C \left(\|\mathbf{f}\|_{\mathbf{L}^{r(p)}(\Omega)} + \|\mathbb{F}\|_{\mathbb{L}^p(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma_N)} \right)$$

$$+ \|\mathbf{h}\|_{\mathbf{W}^{-1/p,p}(\Gamma_D)} + \|\mathbf{a} \times \mathbf{n}\|_{\mathbf{W}^{1-1/p,p}(\Gamma_N)} + \|b\|_{W^{1-1/p,p}(\Gamma_D)}).$$

(ii) Furthermore, if Ω is of class $\mathcal{C}^{2,1}$, $\mathbf{f} \in \mathbf{L}^p(\Omega)$, $\mathbf{a} \times \mathbf{n} \in \mathbf{W}^{2-1/p,p}(\Gamma_N)$, $g \in W^{1-1/p,p}(\Gamma_N)$, $b \in W^{2-1/p,p}(\Gamma_D)$ and $\mathbf{h} \times \mathbf{n} \in \mathbf{W}^{1-1/p,p}(\Gamma_D)$ with $\mathbb{F} = 0$, then \mathbf{u} belongs to $\mathbf{W}^{2,p}(\Omega)$ and the following estimate holds

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{W}^{2,p}(\Omega)} \leq C & \left(\|\mathbf{f}\|_{\mathbf{L}^p(\Omega)} + \|g\|_{W^{1-1/p,p}(\Gamma_N)} + \|\mathbf{h}\|_{\mathbf{W}^{1-1/p,p}(\Gamma_D)} \right. \\ & \left. + \|\mathbf{a} \times \mathbf{n}\|_{\mathbf{W}^{2-1/p,p}(\Gamma_N)} + \|b\|_{W^{2-1/p,p}(\Gamma_D)} \right). \end{aligned}$$

We skip the proof in this paper as it can be done with the same arguments as for Problem (2).

REFERENCES

- [1] T. Akiyama, H. Kasai, Y. Shibata, M. Tsutsumi; *On a resolvent estimate of a system of Laplace operators with perfect wall condition*, Funkcial. Ekvac. 47(3) (2004) 361–394
- [2] C. Amrouche, M. Angeles Rodrigues-Bellindo; *Stationary Stokes, Oseen and Navier Stokes equations with singular data*, Arch. Rational Mech. Anal 199 (2011) 597–651
- [3] C. Amrouche, S. Boukassa; *L^p theory for elliptic systems without divergence constraint and with Navier-type boundary condition*, Math. Methods Appl. Sci. 47 (2024), no. 7, 5831–5847
- [4] C. Amrouche, I. Boussetouan; *Vector potentials with mixed boundary conditions. Application to the Stokes problem with pressure and Navier-type boundary conditions*, SIAM J. Math. Anal 53 (2021), no. 2, 1745–1784
- [5] C. Amrouche, V. Girault; *Decomposition of vector space and application to the Stokes problem in arbitrary dimension*, Czechoslovak Math. J. 44 (1994) 109–140.
- [6] C. Amrouche, N. Seloula; *On the Stokes equations with the Navier-type boundary conditions*, Differ. Equ. Appl. 3(4) (2011), 581–607
- [7] C. Amrouche, N. Seloula; *L^p -theory for vector potentials and Sobolev's inequalities for vector fields. Applications to the Stokes equations with pressure boundary conditions*, Math Models Methods Appl Sci. 23 (2013) 37–92.
- [8] H. Brezis, J. V. Schaftingen; *Boundary estimates for elliptic systems with L^1 data*, Calc. Var. 30 (2007) 369–388.
- [9] C. Conca, F. Murat, O. Pironneau; *The Stokes and Navier-Stokes equations with boundary conditions involving the pressure*, Jpn. J. Math. 20 (1994), pp. 279–318.
- [10] M. Costabel; *A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains*, Math. Meth. in the Appl. Sci. 12 (1990) 365–368.
- [11] J.-P. Dias; *A simplified variational model for the bidimensional coupled evolution equations of a nematic liquid cristal*, J. Math. Ana. App. 67 (1979) 524–541.
- [12] P. Grisvard; *Elliptic problems in nonsmooth domains*, Pitman Publishing Inc, 1985.
- [13] J. L. Lions; *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthiers-Villars, Paris, 1969.
- [14] V. G. MAZ'YA, J. ROSSMANN. *Elliptic Equations in Polyhedral Domains, Mathematical Surveys and Monographs*, vol. 162, (2010) American Mathematical Society.
- [15] S. Monniaux, Z. Shen; *Stokes problems in irregular domains with various boundary conditions*, Springer, Cham, 2018, 207–248.
- [16] C.L.M.H. Navier; *Sur les lois d'équilibre et du mouvement des corps élastiques*, Mém. Acad. Sci. 7 (1827) 375–394.
- [17] M.I. Shliomis; *Ferrofluids: Magnetically controllable fluids and their applications*, Oden-bache S (ed.), Lecture Notes in Physics, Springer-Verlag, Heidelberg. 594 (2002) 85–111.
- [18] M. Schoenauer; *Quelques résultats de régularité pour un système elliptique avec des conditions aux limites couplées*, Portugaliae Mathematica. 36 (1977), no. 3-4, 301–310.

IMANE BOUSSETOUAN

NATIONAL HIGHER SCHOOL OF TECHNOLOGY AND ENGINEERING, DEPARTMENT CPST, ANNABA 23005, ALGERIA

Email address: i.boussetouan@ensti-annaba.dz

CHÉRIF AMROUCHE

LMAP, UMR CNRS 5142, BATIMENT IPRA, AVENUE DE L'UNIVERSITÉ BP 1155, PAU CEDEX,64013, FRANCE

Email address: cherif.amrouche@univ-pau.fr