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INITIAL-BOUNDARY VALUE PROBLEM OF PLASMA-CHARGE MODEL IN THE HALF SPACE

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Abstract. This article studies an initial-boundary value problem for the plasma-charge model in a half-space. We establish the existence of classical local solutions when the point charges are in motion, and the existence of classical global solutions when the point charges remain fixed away from the boundary. In contrast to previous studies of the plasma-charge model in convex domains, our approach does not require the separation of singular sets, owing to the flat boundary. Moreover, we derive classical growth estimates for the support of the plasma distribution and obtain corresponding dispersion estimates.

1. Introduction

In this article, we consider the time evolution of a one species plasma in a half space, coupled with N heavy point charges of same sign. For simplicity, we assume that the charges and the masses of the point charges are unitary. $f(t,x,v)$ denotes the density distribution of electron particles at time $t \geq 0$, position $x \in \mathbb{R}^3_+ = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0\}$, velocity $v \in \mathbb{R}^3$. The heavy point charges have positions $\{\xi_\alpha(t)\}$ and velocities $\{\eta_\alpha(t)\}$ at time t. f satisfies the Vlasov equation

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0, \quad t > 0,\ x \in \mathbb{R}^3_+,\ v \in \mathbb{R}^3, \tag{1.1}$$

where $F = \nabla_x \phi$. $\phi(t,x)$ is the electric potential induced by the plasma particles and the heavy point charges, satisfying the Poisson equation

$$\Delta\phi(t,x) = 4\pi\rho(t,x) + 4\pi \sum_\alpha \delta_{\xi_\alpha(t)}(x), \quad t > 0,\ x \in \mathbb{R}^3_+,$$

where $\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,dv$ is the macroscopic density of the plasma particles. We equip the Poisson equation with the Dirichlet boundary condition, i.e., the reference points are the boundary $\partial\mathbb{R}^3_+$ and the infinity

$$\lim_{x\in\mathbb{R}^3_+,\, |x|\to\infty} \phi(t,x) = \phi(t,x)|_{x\in\partial\mathbb{R}^3_+} = 0, \quad t > 0.$$

Under some suitable assumptions of ρ, e.g. ρ is a tempered distribution, then by Weyl's lemma and the maximum principle, ϕ can be solved uniquely by the Green's representation

$$\phi(t,x) = \phi_\rho(t,x) + \sum_\alpha \Big(\frac{1}{|x - \xi^*_\alpha(t)|} - \frac{1}{|x - \xi_\alpha(t)|} \Big),$$

where ϕ_ρ is the part of electric field induced by the plasma

$$\phi_\rho(t,x) = \int_{\mathbb{R}^3_+} \Big(\frac{1}{|x - y^*|} - \frac{1}{|x - y|} \Big) \rho(t,y)\,dy. \tag{1.2}$$

The symbol * denotes the reflection operator,

$$y^* = (-y_1, y_2, y_3), \quad \text{for } y = (y_1, y_2, y_3)$$

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and we have used that $|x^* - y| = |x - y^*|$ for $x, y \in \mathbb{R}^3_+$. The electric field $F = \nabla_x \phi$ is

$$F(t,x) = F_\rho(t,x) + \sum_\alpha \Big(\frac{x - \xi_\alpha(t)}{|x - \xi_\alpha(t)|^3} - \frac{x - \xi^*_\alpha(t)}{|x - \xi^*_\alpha(t)|^3} \Big). \tag{1.3}$$

where $F_\rho = \nabla_x \phi_\rho$ is

$$F_\rho(t,x) = \int_{\mathbb{R}^3_+} \Big(\frac{x - y}{|x - y|^3} - \frac{x - y^*}{|x - y^*|^3} \Big) \rho(t,y)\, dy. \tag{1.4}$$

Naturally, the force acting on the α-th heavy point charge with position $\xi_\alpha(t)$, velocity $\eta_\alpha(t)$ is

$$F_\alpha(t, \xi_\alpha(t)) = F_\rho(t, \xi_\alpha(t)) + \sum_{\beta:\, \beta \neq \alpha} \Big(\frac{\xi_\alpha(t) - \xi_\beta(t)}{|\xi_\alpha(t) - \xi_\beta(t)|^3} - \frac{\xi_\alpha(t) - \xi^*_\beta(t)}{|\xi_\alpha(t) - \xi^*_\beta(t)|^3} \Big). \tag{1.5}$$

$\xi_\alpha(t), \eta_\alpha(t)$ satisfy the Newton's equations of motion,

$$\dot{\xi}_\alpha(t) = \eta_\alpha(t), \quad \dot{\eta}_\alpha(t) = F_\alpha(t, \xi_\alpha(t)), \tag{1.6}$$

as long as $\xi_\alpha(t)$ remains in $\mathbb{R}^3_+$, where $\dot{}$ denotes the derivative with respect to time.

The initial conditions associated with (1.1), (1.6) are

$$\begin{gathered} f(t,x,v)|_{t=0} = \mathring{f}(x,v), \quad x \in \mathbb{R}^3_+,\ v \in \mathbb{R}^3, \\ (\xi_\alpha, \eta_\alpha)|_{t=0} = (\mathring{\xi}_\alpha, \mathring{\eta}_\alpha), \quad \alpha = 1, \dots, N, \end{gathered} \tag{1.7}$$

where $\mathring{f}$ is non-negative and $\mathring{\xi}_\alpha \in \mathbb{R}^3_+$, $\mathring{\eta}_\alpha \in \mathbb{R}^3$.

We assume that f satisfies the specular reflection boundary condition

$$f(t,x,v) = f(t,x,v^*), \quad t > 0,\ x \in \partial\mathbb{R}^3_+. \tag{1.8}$$

The characteristic flows associated with (1.1) are the solutions $(X(s,t,x,v),\ V(s,t,x,v))$ of the ODEs,

$$\begin{gathered} \frac{dX}{ds} = V, \quad \frac{dV}{ds} = F(s,X), \\ (X(t,t,x,v), V(t,t,x,v)) = (x,v) \in \operatorname{supp} f(t), \end{gathered} \tag{1.9}$$

as long as $X(s,t,x,v)$ remains in $\mathbb{R}^3_+$. We extend this definition to arbitrarily long times which takes the condition (1.8) into account (see [12]). Assuming, at time $s = \bar{s}$, that $X(\bar{s},t,x,v) \in \partial\mathbb{R}^3_+$, we define

$$V^*(\bar{s} + 0, t, x, v) = V(\bar{s} - 0, t, x, v),$$

where $V(\bar{s} \pm 0, t, x, v) = \lim_{\Delta \to 0^+} V(\bar{s} \pm \Delta, t, x, v)$. Moreover we set

$$|V(\bar{s}, t, x, v)| = \lim_{\Delta \to 0^+} |V(\bar{s} \pm \Delta, t, x, v)|. \tag{1.10}$$

Then (1.9)-(1.10) together define the generalize characteristic flows along which f is constant for f satisfying (1.1) and (1.8). Condition (1.10) ensures that $|V(s)|^2$ is continuous whence $|X(s)|^2$ is continuous differentiable for time. Then conservation laws deduced in Proposition 2.8 hold for time.

Similarly, if $\xi_\alpha(\bar{s}) \in \partial\mathbb{R}^3_+$, we define

$$\eta^*_\alpha(\bar{s} + 0) = \eta_\alpha(\bar{s} - 0), \quad |\eta_\alpha(\bar{s})| = \lim_{\Delta \to 0^+} |\eta_\alpha(\bar{s} \pm \Delta)|. \tag{1.11}$$

Now the plasma-charge model in a half space is defined as (1.1), (1.3), (1.5)-(1.8), (1.11), we call it the model with moving point charges. We also consider the model with fixed point charges, i.e., (1.1), (1.3), (1.7)-(1.8) with $\xi_\alpha(t) \equiv \mathring{\xi}_\alpha$ for $1 \leq \alpha \leq N$.

To prove solvability of (1.1)-(1.8), (1.11), we need to assume that $\mathring{f}$ satisfies the compatibility conditions (see [16, p. 138]): for $x \in \partial\mathbb{R}^3_+$

$$\begin{gathered} \mathring{f}(x,v) = \mathring{f}(x,v^*), \\ v_1 \partial_{x_1} \mathring{f}(x,v) + v_1 \partial_{x_1} \mathring{f}(x,v^*) + 2F_1(0,x) \partial_{v_1} \mathring{f}(x,v) = 0. \end{gathered} \tag{1.12}$$

There are two types of singular sets we have to deal with. The first one is the well known grazing set $\{(x,v) \in \partial\mathbb{R}^3_+ \times \mathbb{R}^3 \colon x_1 v_1 = 0\}$ brought by the boundary effect. The second is

$\{(x,v)\in\overline{\mathbb{R}^3_+}\times\mathbb{R}^3 : \exists 1\le\alpha\le N,\, t\ge 0 \text{ s.t. } x=\xi_\alpha(t)\}$ in which the trajectories of plasma particles overlap one of the point charges.

To make sure that the solutions stay away from the singular sets, we also assume that there exists $\delta_0>0$ such that

$$\begin{gathered}\mathring{\xi}_\alpha\in\mathbb{R}^3_+,\mathring{\eta}_\alpha\in\mathbb{R}^3,\quad \min_{1\le\alpha\neq\beta\le N}\{|\mathring{\xi}_\alpha-\mathring{\xi}_\beta|,|\mathring{\xi}_{\alpha,1}|+|\mathring{\eta}_{\alpha,1}|\}>\delta_0,\\ \mathring{f}(x,v)\equiv 0\quad\text{for }\min_{1\le\alpha\le N}\{|x-\mathring{\xi}_\alpha|,|x_1|+|v_1|\}\le\delta_0.\end{gathered}\tag{1.13}$$

Throughout this paper, C will denote a constant depending only on $\|\mathring{f}\|_1,\|\mathring{f}\|_\infty$, $\{|\mathring{\xi}_\alpha|,|\mathring{\eta}_\alpha|\}$, δ_0 and the support of $\mathring{f}$. It may change from line to line. First we have the local well-posedness for the model with moving point charges.

Theorem 1.1. *Let $\mathring{f}\in C^{1,\mu}_c(\mathbb{R}^3_+\times\mathbb{R}^3)$ for some $\mu>0$ and let $\mathring{f}\ge 0$, $\{\mathring{\xi}_\alpha,\mathring{\eta}_\alpha\}$ satisfying (1.12)-(1.13). Then there exists a unique solution $\{f,\xi_\alpha,\eta_\alpha\}$ to the Vlasov-Poisson system (1.1),(1.3), (1.5)-(1.8), (1.11) on some time interval $[0,\mathring{T})$ with $f\in C^1([0,\mathring{T})\times\mathbb{R}^3_+\times\mathbb{R}^3)$, $f\ge 0$, $\phi_\rho\in C^3([0,\mathring{T})\times\mathbb{R}^3_+)$; $\xi_\alpha\in C_b([0,\mathring{T}))\cap C^2([0,\mathring{T})\setminus\mathcal{T})$, $\eta_\alpha\in L^\infty(0,\mathring{T})\cap C^1([0,\mathring{T})\setminus\mathcal{T}))$ for a countable set $\mathcal{T}$. If $\mathring{T}>0$ has been chosen maximal, then for any $T_1<\mathring{T}$, $f\in C^{1,\lambda}_c([0,T_1]\times\mathbb{R}^3_+\times\mathbb{R}^3)$, $\phi_\rho\in C^{3,\lambda}([0,T_1]\times\mathbb{R}^3_+)$ for some $0<\lambda<\mu$; $\mathcal{T}\cap[0,T_1]$ is a finite set; there exists $\delta_1=\delta_1(T_1)$ such that*

$$f(t,x,v)=0\quad\text{for }\min\{|x-\xi_\alpha(t)| : 1\le\alpha\le N,\ 0\le t\le T_1\}<\delta_1.$$

Moreover, if

$$\sup\Big\{|v|+\sum_\alpha\frac{1}{|x-\xi_\alpha(t)|} : (x,v)\in\operatorname{supp}f(t),\ 0\le t<\mathring{T}\Big\}<\infty,$$

then $\mathring{T}=+\infty$.

Now we have global well-posedness for the model with fixed point charges.

Theorem 1.2. *Let $\xi_\alpha(t)\equiv\mathring{\xi}_\alpha$ for $1\le\alpha\le N$. Let $\mathring{f}\in C^{1,\mu}_c(\mathbb{R}^3_+\times\mathbb{R}^3)$ for some $\mu>0$ and let $\mathring{f}\ge 0$, $\{\mathring{\xi}_\alpha\}$ satisfying (1.12)-(1.13). Then for any $T>0$, there exists a unique solution f to the Vlasov-Poisson system (1.1), (1.3), (1.7)-(1.8) with $f\in C^{1,\lambda}_c([0,T]\times\mathbb{R}^3_+\times\mathbb{R}^3)$, $\phi\in C^{3,\lambda}([0,T]\times\mathbb{R}^3_+)$ for some $0<\lambda<\mu$. The solution satisfies the dispersion estimate*

$$\|\rho(t)\|_{5/3}\le C(1+t)^{-3/5}.\tag{1.14}$$

Moreover, for the case $N=1$, the support of f satisfies

$$|x|\le C(t+1)^{22},\quad |v|+\frac{1}{|x-\mathring{\xi}|^{1/2}}\le C(t+1)^{21}\tag{1.15}$$

for $0\le t\le T$.

Remark 1.3. With the help of the dispersion estimate (1.14), the growth estimate (1.15) might be improved by the method developed by Schaeffer [31] and Pallard [26]. However, due to the presence of the point charges, it is more difficult to balance the contributions of each subdomain in the Pfaffelmoser's method and new ideas might be required to improve (1.15) to the same degree as the results of [3].

Remark 1.4. A problem left in this paper and [36] is the global well-posedness for the plasma-charge model with moving point charges. We will consider this problem in the future work.

Without point charges, the plasma-charge model reduces to the well-known Vlasov-Poisson system, which has been widely investigated. In two dimensions, the existence and uniqueness results trace back to Ukai and Okabe [32] and Wollman [33]. The three dimensional problem is more complicated, which was unsolved until Pfaffelmoser's work [27], see also Schaeffer's nice simplified version [30]. Pfaffelmoser's approach is based on a careful analysis of characteristic flows associated with the system, which is also known as the Lagrangian method. A complete different approach was established by Lions and Perthame [20], which is based on the propagation of moments, known as the Eulerian method. We refer the reader to Rein's report [29] for a more complete literature review of this equations.

The problems of the Vlasov-Poisson systems with singular initial data trace back to Majda, Majda and Zheng's series work [23, 24, 37] for one dimension. In two dimensions, when the point charges have the same sign as the plasma so that the interaction is repulsive, global well-posedness of the plasma-charge model has be solved by Caprino and Marchioro [5]. For three dimensional case, Marchioro, Miot and Pulvirenti [25] adopt the Lagrangian method to establish the existence and uniqueness result for three dimensional problem. The analysis in [5, 25] rely on a pointwise energy function of the plasma (see Section 3) and the minimum distance between the plasma particles and point charges is strictly positive. The Eulerian method was then extended to the plasma-charge model by Desvillettes, Miot and Saffirio [9], in which the plasma particles was allowed to overlap the point charges. The small condition in [9] was removed by Wu and Zhang [34] and they extended [9] to the case of multi-point charges and the case of constant density of plasma around the point charges in [35] recently. Benefited from the vigorous development of the Vlasov-Poisson systems, ample results for the repulsive plasma-charge model have been obtained very recently. The Lagrangian solutions, a class of weak solutions which can be represented by the characteristic flows, have been established in [4]. Debye screening for the stationary solutions has been studied in [1]. For small radial data with a point charge fixed at origin, stability has been obtained in [28].

In contrast, the attractive plasma-charge model has been rarely studied. The global existence of classical solutions has been established by [6] in two dimensions, without uniqueness. The existence of classical solutions for three dimensions is totally open.

The initial-boundary value problems are more delicate than the Cauchy problems for the Vlasov type systems. Existence of weak solutions were established by Guo [11] for the Vlasov-Maxwell systems and Mischler [21] for the Vlasov-Poisson-Bolzmann systems, inspired by DiPerna-Lions' breakthroughs [8, 7]. In [12, 13], Guo established the well-known Velocity Lemma and showed that singularity might be propagated from the boundary when studying the Vlasov-Maxwell system in a half line. Nevertheless, with the aid of the Velocity Lemmas, classical solutions to the Vlasov-Poisson system have been obtained by the Eulerian method or the Lagrangian method under suitable conditions on the initial datum near the grazing set, as shown in [12, 16, 17, 18] for half space or convex domain and Dirichlet boundary condition or Neumann boundary condition. More results related to the initial-boundary problems of the kinetic equations can be referred to [22, 14, 15, 2] and the references therein.

The initial-boundary value problem for the plasma-charge model was first studied in [36]. By introducing the separation technique of the singular sets, global well-posedness of the model with fixed point charges has been established in [36].

However, the classical growth estimates as well as the dispersion estimates have not obtained in [36]. The main contribution of this paper is to deduce the classical dispersion estimates and the growth estimates on the support of the plasma distribution for the case $N = 1$, which might be helpful to the study of the large time asymptotic behavior. The novelty is that thanks to the flat boundary, we can make full use of the representation of characteristic flow with the specular reflection boundary condition, which allows us to avoid the separation technique introduced by [36].

We clarify the notation used in this paper. When the index set $\mathcal{I}$ and the index $i \in \mathcal{I}$ are clear, we always denote $\{a_i\}$ as the set $\{a_i\colon i \in \mathcal{I}\}$. The support of a distribution f on $\mathbb{R}^3_+ \times \mathbb{R}^3$ is defined as the complement of the largest open set $U \subset \mathbb{R}^3_+ \times \mathbb{R}^3$ denoted as $\operatorname{supp} f = \mathbb{R}^3_+ \times \mathbb{R}^3 \setminus U$ such that

$$\langle f, \varphi\rangle = 0, \quad \forall \varphi \in C_c^\infty(U).$$

We define the summation signs for short

$$\sum_{\beta\colon \beta\neq\alpha} = \sum_{1\le\beta\le N,\beta\neq\alpha}, \quad \sum_{\alpha\neq\beta} = \sum_{1\le\alpha\le N}\sum_{\beta\colon \beta\neq\alpha}, \quad \sum_{\alpha} = \sum_{1\le\alpha\le N}.$$

This paper is organized as follows. In Section 2, we prove the local existence (Theorem 1.1) as well as energy conservation and dispersion estimates of the solutions to the Vlasov-Poisson system with moving point charges. In Section 3, we establish the global existence (Theorem 1.2) of the classical solutions to the Vlasov-Poisson system with fixed point charges.

2. Local well-posedness of the model with moving point charges

We assume that ρ in (1.2), (1.4) is given. The following regularity results of ϕ_ρ have been established in [17, Lem.1-3].

Lemma 2.1. *Let $T>0$ and $\rho\in C_c^1([0,T]\times\mathbb{R}^3_+)$ given. Assume the support of ρ is contained in $[0,T]\times(B_L(0)\cap\mathbb{R}^3_+)$. Then*

$$|\partial_{x_2}\phi_\rho(t,x)|+|\partial_{x_3}\phi_\rho(t,x)|\le Cx_1(1+|\log x_1|)\quad \text{in } [0,T]\times\mathbb{R}^3_+$$

where $C>0$ depending only on L and $\|\rho\|_{L^\infty([0,T]\times\mathbb{R}^3_+)}$. If moreover, ρ satisfies

$$\partial_t\rho+\nabla\cdot j=0\quad \text{in } [0,T]\times\mathbb{R}^3_+$$

for some function $j\in C^1([0,T]\times\mathbb{R}^3_+;\mathbb{R}^3)$ with its support contained in $[0,T]\times(B_L(0)\cap\mathbb{R}^3_+)$. Then

$$|\partial_t\phi_\rho(t,x)|\le Cx_1(1+|\log x_1|)\quad \text{in } [0,T]\times\mathbb{R}^3_+$$

for some constant $C>0$ depending only on L and $\|j\|_{L^\infty([0,T]\times\mathbb{R}^3_+)}$.

Lemma 2.2. *Let $T>0$ and $\rho\in C_c^1([0,T]\times\mathbb{R}^3_+)$ given. Assume the support of ρ is contained in $[0,T]\times(B_L(0)\cap\mathbb{R}^3_+)$. Let us assume also that $\rho\ge 0$ and satisfies*

$$\int_{\mathbb{R}^3_+}\rho(t,x)\,dx\ge\kappa>0\quad \text{for } t\in[0,T].$$

Then

$$\phi_\rho(t,x)\le-\varepsilon_0x_1\quad \text{for } |x|\le 2L,\ t\in[0,T]$$

where the constant ε_0 depends only on L, κ and $\|\rho\|_{L^\infty([0,T]\times\mathbb{R}^3_+)}$.

The following estimates are well known in the study of the Vlasov-Poisson system [29].

Lemma 2.3. *Assume $\rho\in L^p\cap L^\infty(\mathbb{R}^3_+)$ for some $1\le p<3$. Then there exists a constant $C_p>0$ depending only on p for which*

$$\int_{\mathbb{R}^3_+}\frac{|\rho(y)|}{|x-y|^2}\,dy\le C_p\|\rho\|_p^{p/3}\|\rho\|_\infty^{1-p/3}. \tag{2.1}$$

As a consequence, if $\rho(x)=\int_{|v-v_0|<R}f(x,v)\,dv$ for some $v_0\in\mathbb{R}^3$ and $f\in L^\infty(\mathbb{R}^3_+\times\mathbb{R}^3)$, then

$$\int_{\mathbb{R}^3_+}\frac{|\rho(y)|}{|x-y|^2}\,dy\le KR^{4/3}, \tag{2.2}$$

where $K=C_{5/3}\|f\|_\infty^{4/9}\|\rho\|_{5/3}^{5/9}$.

Analogous to [13, 17], we introduce the distance functions toward the grazing set as

$$\beta(s,x,v)=\frac{|v_1|^2}{2}-\phi_\rho(s,x)+\sum_\alpha x_1\xi_{\alpha,1}(s)\Big(\frac{1}{|x-\xi_\alpha(s)|^3}+\frac{1}{|x-\xi^*_\alpha(s)|^3}\Big),$$

$$\beta_\alpha(s,x,v)=\frac{|v_1|^2}{2}-\phi_\rho(s,x)+\sum_{\beta:\ \beta\neq\alpha}x_1\xi_{\beta,1}(s)\Big(\frac{1}{|x-\xi_\beta(s)|^3}+\frac{1}{|x-\xi^*_\beta(s)|^3}\Big).$$

Along the characteristic flows of the plasma as well as the point charges, the distance functions are respectively given by

$$\boldsymbol{\beta}(s,t)=\boldsymbol{\beta}(s,t,x,v):=\beta(s,X(s,t,x,v),V(s,t,x,v)),$$
$$\boldsymbol{\beta}_\alpha(s):=\beta_\alpha(s,\xi_\alpha(s),\eta_\alpha(s)).$$

With the aid of the distance functions defined above, we extend the well-known Velocity Lemmas established in [13, 17] to the plasma-charge model.

Lemma 2.4. *Let $T > 0$. Suppose that the assumptions of Lemmas 2.1-2.2 are satisfied. Suppose that $\{\xi_\alpha(s), \eta_\alpha(s)\}$ satisfy the ODEs (1.6) for $0 \le s \le T$. Suppose that $(X, V) = (X(s,t,x,v), V(s,t,x,v))$ be the characteristic flows defined by (1.9) for $0 \le s \le t \le T$. Suppose also that*

$$|X| + |V| + \sum_\alpha \Big(|\xi_\alpha(s)| + |\eta_\alpha(s)| + \frac{1}{|X - \xi_\alpha(s)|}\Big) \le L \quad \text{for } 0 \le s \le t \le T. \tag{2.3}$$

Then

$$e^{-e^{C(t-s)}}\boldsymbol{\beta}(s,t) \le \boldsymbol{\beta}(t,t) \le e^{e^{C(t-s)}}\boldsymbol{\beta}(s,t) \quad \text{for } 0 \le s \le t \le T, \tag{2.4}$$

$$e^{-e^{C(t-s)}}\boldsymbol{\beta}_\alpha(s) \le \boldsymbol{\beta}_\alpha(t) \le e^{e^{C(t-s)}}\boldsymbol{\beta}_\alpha(s) \quad \text{for } 0 \le s \le t \le T,\ 1 \le \alpha \le N. \tag{2.5}$$

for some C depending on L, $\|\rho\|_{L^\infty([0,T]\times\mathbb{R}^3_+)}$, $\|j\|_{L^\infty([0,T]\times\mathbb{R}^3_+)}$, κ.

Proof. Note by Lemma 2.2 we have

$$\boldsymbol{\beta}(s,t) \ge \frac{|V_1|^2}{2} + \varepsilon X_1, \quad \boldsymbol{\beta}^\alpha(s) \ge \frac{|\eta_{\alpha,1}(s)|^2}{2} + \varepsilon \xi_{\alpha,1}(s). \tag{2.6}$$

Differentiating $\boldsymbol{\beta}$ with respect to s we obtain

$$\begin{aligned}
\partial_s\boldsymbol{\beta} &= V_1\partial_s V_1 - \nabla_x\phi_\rho(s,X)\partial_s X - \partial_s\phi_\rho(s,X) \\
&\quad + \sum_\alpha (V_1\xi_{\alpha,1} + X_1\eta_{\alpha,1})\Big(\frac{1}{|X-\xi_\alpha|^3} + \frac{1}{|X-\xi_\alpha^*|^3}\Big) \\
&\quad - 3\sum_\alpha X_1\xi_{\alpha,1}\Big(\frac{(X-\xi_\alpha)\cdot(V-\eta_\alpha)}{|X-\xi_\alpha|^5} + \frac{(X-\xi_\alpha^*)\cdot(V-\eta_\alpha^*)}{|X-\xi_\alpha^*|^5}\Big).
\end{aligned}$$

From (1.6) and (1.9) it follows that

$$\begin{aligned}
\partial_s\boldsymbol{\beta} &= \sum_\alpha V_1X_1\Big(\frac{1}{|X-\xi_\alpha|^3} - \frac{1}{|X-\xi_\alpha^*|^3}\Big) - \partial_{x_2}\phi_\rho(s,X)V_2 - \partial_{x_3}\phi_\rho(s,X)V_3 \\
&\quad - \partial_s\phi_\rho(s,X) + \sum_\alpha X_1\eta_{\alpha,1}\Big(\frac{1}{|X-\xi_\alpha|^3} + \frac{1}{|X-\xi_\alpha^*|^3}\Big) \\
&\quad - 3\sum_\alpha X_1\xi_{\alpha,1}\Big(\frac{(X-\xi_\alpha)\cdot(V-\eta_\alpha)}{|X-\xi_\alpha|^5} + \frac{(X-\xi_\alpha^*)\cdot(V-\eta_\alpha^*)}{|X-\xi_\alpha^*|^5}\Big).
\end{aligned}$$

Using Lemma 2.1 and (2.3), we obtain

$$|\partial_s\boldsymbol{\beta}| \le CX_1(s,t,x,v)\big(1 + |\log X_1(s,t,x,v)|\big), \quad 0 \le s \le t \le T$$

with C depending on L, $\|\rho\|_{L^\infty([0,T]\times\mathbb{R}^3_+)}$, $\|j\|_{L^\infty([0,T]\times\mathbb{R}^3_+)}$. Integrating the inequality and combining with (2.6) we obtain (2.4). The proof of (2.5) is similar and the lemma follows. ☐

Along the generalized characteristic flows, we define the back-time cycles introduced in [12].

Definition 2.5. Given $C^0_tC^1_x$-fields F, F_α, we will denote as a *back-time ℓ-cycle connecting (t,x,v) and $(0,x^0,v^0)$* the trajectories $(X(s,t,x,v),\ V(s,t,x,v))$ solving (1.9) in $\mathbb{R}_+ \times \mathbb{R}^3_+ \times \mathbb{R}^3$ which connect

$$\begin{gathered}
(t,x,v) = (t^\ell, x^\ell, v^\ell) \quad \text{with } (t^{\ell-1}, x^{\ell-1}, v^{\ell-1}), \\
(t^{\ell-1}, x^{\ell-1}, (v^{\ell-1})^*) \quad \text{with } (t^{\ell-2}, x^{\ell-2}, v^{\ell-2}), \dots, \\
(t^i, x^i, (v^i)^*) \quad \text{with } (t^{i-1}, x^{i-1}, v^{i-1}), \dots, \\
(t^1, x^1, (v^1)^*) \quad \text{with } (0, x^0, v^0),
\end{gathered}$$

where $t^i > t^{i-1}$, $x^i \in \partial\mathbb{R}^3_+$ for $1 \le i \le \ell - 1$, $v^i_1 \ge 0$, $1 \le i \le \ell$. The back-time cycle for the point charges $\{\xi_\alpha(s), \eta_\alpha(s)\}$ satisfying (1.6) can be defined similarly.

According to velocity Lemma 2.4 it is possible to show that the characteristic flows starting their motion at a positive distance from the singular set, remain during their evolution to a distance of the same order of magnitude for times of order one.

Corollary 2.6. *Suppose that the assumptions in Lemma 2.4 are satisfied. Let $(t^i, x^i, v^i)_{i=0}^{\ell}$ be the back-time cycle connecting (t,x,v)-$(0,x^0,v^0)$; and $(t^i, \xi_\alpha^i, \eta_\alpha^i)_{i=0}^m$ be the back-time cycle connecting $(t, \xi_\alpha(t), \eta_\alpha(t))$-$(0, \mathring{\xi}_\alpha, \mathring{\eta}_\alpha)$. Then there exists a constant $C_1 > 0$ such that*

$$\begin{aligned} C_1(x_1 + (v_1)^2) &\le (v_1^i)^2 \le C_1^{-1}(x_1^0 + (v_1^0)^2), \\ C_1(x_1^0 + (v_1^0)^2) &\le (v_1^i)^2 \le C_1^{-1}(x_1 + (v_1)^2), \\ C_1(\xi_{\alpha,1}(t) + (\eta_{\alpha,1}(t))^2) &\le (\eta_{\alpha,1}^i)^2 \le C_1^{-1}(\mathring{\xi}_{\alpha,1} + (\mathring{\eta}_{\alpha,1})^2), \\ C_1(\mathring{\xi}_{\alpha,1} + (\mathring{\eta}_{\alpha,1})^2) &\le (\eta_{\alpha,1}^i)^2 \le C_1^{-1}(\xi_{\alpha,1}(t) + (\eta_{\alpha,1}(t))^2), \end{aligned}$$

where C_1 is independent of the number of bounces ℓ, m and depends only on L, $\|\rho\|_{L^\infty([0,T]\times\mathbb{R}^3_+)}$, $\|j\|_{L^\infty([0,T]\times\mathbb{R}^3_+)}$, κ.

Lemma 2.7 ([17, Lemma 14]). *Let (X,V) and (Y,W) be two trajectories on some interval $[t_1,t_2]$. Suppose that*

$$|Y_1(s_0) - X_1(s_0)| = \min_{s\in[t_1,t_2]} |Y_1(s) - X_1(s)|$$

with $s_0 \in (t_1,t_2)$. Then either both $Y_1(s_0) > 0$, $X_1(s_0) > 0$ or both $Y_1(s_0) = X_1(s_0) = 0$.

Proof of Theorem 1.1. The proof is somewhat standard, which has been established exhaustively, e.g., in [12, 17] for the boundary case and [10, 29] for the whole space case. Hence we only sketch the proof of existence.

We define iteratively a sequence of functions $\{f^n, \xi_\alpha^n, \eta_\alpha^n\}$ as follows. Thanks to the velocity Lemma 2.4, such sequence are well defined (see [17, Prn.1]). Let $(f^0, \xi_\alpha^0, \eta_\alpha^0) = (\mathring{f}, \mathring{\xi}_\alpha, \mathring{\eta}_\alpha)$. For $n \ge 1$, let $\rho^{n-1} = \int f^{n-1}\,dv$ and

$$\begin{gathered} F^{n-1}(t,x) = F_{\rho^{n-1}}(t,x) + \sum_\alpha \Big(\frac{x - \xi_\alpha^n(t)}{|x-\xi_\alpha^n(t)|^3} - \frac{x - (\xi_\alpha^n(t))^*}{|x - (\xi_\alpha^n(t))^*|^3} \Big), \\ F_\alpha^{n-1}(t,x) = F_{\rho^{n-1}}(t,x) + \sum_{\beta:\ \beta\ne\alpha} \Big(\frac{x - \xi_\beta^n(t)}{|x-\xi_\beta^n(t)|^3} - \frac{x - (\xi_\beta^n(t))^*}{|x - (\xi_\beta^n(t))^*|^3} \Big), \\ \partial_t f^n + v\cdot\nabla_x f^n + F^{n-1}(t,x)\cdot\nabla_v f^n = 0, \\ \dot{\xi}_\alpha^n(t) = \eta_\alpha^n(t), \quad \dot{\eta}_\alpha^n(t) = F_\alpha^{n-1}(t, \xi_\alpha^n(t)), \end{gathered}$$

with initial-boundary conditions

$$\begin{gathered} f^n|_{t=0} = \mathring{f}, \quad (\xi_\alpha^n, \eta_\alpha^n)|_{t=0} = (\mathring{\xi}_\alpha, \mathring{\eta}_\alpha), \\ f^n(t,x,v) = f^n(t,x,v^*), \quad x \in \partial\mathbb{R}^3_+. \end{gathered}$$

For $n \ge 1$, the characteristic flows $(X^n, V^n)(t,0,x,v)$ are given by

$$\begin{gathered} \frac{dX^n}{dt} = V^n, \quad \frac{dV^n}{dt} = F^{n-1}(t, X^n), \\ (X^n, V^n)(0) = (x,v) \in \operatorname{supp}\mathring{f}. \end{gathered}$$

It can be deduced that $\|f^n(t)\|_p = \|\mathring{f}\|_p$ for $1 \le p \le \infty$, $n \ge 1$, see Proposition 2.8 below.

For $n \ge 0$, we introduce a function which contains all the information we need

$$e^n(t,x,v) = \frac{|v|^2}{2} + \sum_\alpha \frac{1}{|x - \xi_\alpha^n(t)|} + \frac{1}{2}\sum_\alpha |\eta_\alpha^n(t)|^2 + \frac{1}{2}\sum_{a\ne\beta} \frac{1}{|\xi_\alpha^n(t) - \xi_\beta^n(t)|}$$

and the associated quantity for $n \ge 0$,

$$\begin{aligned} Q^n(t) &:= \sup\{\sqrt{e^n}(s,x,v) \colon (x,v) \in \operatorname{supp} f^n(s),\ 0 \le s \le t\} \\ (\text{for } n \ge 1) &= \sup\{\sqrt{e^n}(s, X^n(s), V^n(s)) \colon (x,v) \in \operatorname{supp}\mathring{f},\ 0 \le s \le t\}. \end{aligned}$$

Notice that for $n \ge 1$,

$$Q^0(t) = Q^n(0) = \sup\{\sqrt{e}(0,x,v) \colon (x,v) \in \operatorname{supp}\mathring{f}\} := \mathring{Q}.$$

We denote $\mathsf{e}^n(t) = e^n(t, X^n(t), V^n(t))$. Differentiating it gives

$$\begin{aligned}
\frac{d}{dt}\mathsf{e}^n(t) &= \sum_\alpha V^n(t)\cdot F_{\rho^{n-1}}(t, X^n(t)) - \sum_\alpha \frac{V^n(t)\cdot(X^n(t) - (\xi_\alpha^n(t))^*)}{|X^n(t) - (\xi_\alpha^n(t))^*|^3} \\
&\quad + \sum_\alpha \frac{\eta_\alpha^n(t)\cdot(X^n(t) - \xi_\alpha^n(t))}{|X^n(t) - \xi_\alpha^n(t)|^3} + \sum_\alpha \eta_\alpha^n(t)\cdot F_{\rho^{n-1}}(t, \xi_\alpha^n(t)) \\
&\quad - \sum_{\alpha\neq\beta} \frac{\eta_\alpha^n(t)\cdot(\xi_\alpha^n(t) - (\xi_\beta^n(t))^*)}{|\xi_\alpha^n(t) - (\xi_\beta^n(t))^*|^3} \\
&\le C\sqrt{\mathsf{e}^n(t)}\|F_{\rho^{n-1}}\|_\infty + C(\mathsf{e}^n(t))^{5/2} \\
&\le C\sqrt{\mathsf{e}^n(t)}\big(Q^{n-1}(t)\big)^2 + C(\mathsf{e}^n(t))^{5/2},
\end{aligned}$$

where we have used (2.1) for $\rho^{n-1}(t) = \int_{|v|\le 2Q^{n-1}(t)} f^{n-1}(t)\,dv$. Integrating the inequality from 0 to s we have

$$\sqrt{\mathsf{e}^n(t)} \le \sqrt{\mathsf{e}^n(0)} + C\int_0^t \big(Q^{n-1}(s)\big)^2 + (\mathsf{e}^n(s))^2\,ds.$$

Then

$$Q^n(t) \le \mathring{Q} + \mathring{C}\int_0^t \big(Q^{n-1}(s)\big)^2 + \big(Q^n(s)\big)^4\,ds,$$

where $\mathring{C}$ depends on $\|\mathring{f}\|_1$ and $\|\mathring{f}\|_\infty$. We claim that there exist constants $\mathring{T}, \mathring{K}$ depending on $\mathring{Q}, \mathring{C}$ such that

$$Q^n(t) \le \mathring{K}, \quad n \ge 0,\ t\in[0,\mathring{T}]. \tag{2.7}$$

Indeed, $\mathring{T}, \mathring{K}$ can be taken as

$$\mathring{K} = 2\mathring{Q} + 2, \quad \mathring{T} = \frac{1}{48\mathring{C}(\mathring{K}^2 + \mathring{Q}^3)},$$

then it is no difficulty to deduce the claim by induction.

With the aid of (2.7) and Corollary 2.6, it is sufficient to obtain a solution $\{f, \xi_\alpha, \eta_\alpha\}$ on the interval $[0, \mathring{T}]$ by adapting the argument in [17, Sec.3]. ☐

Next, we deduce the conservation laws and the dispersion estimates of the solutions obtained in Theorem 1.1.

We define the total energy of the plasma-charge model (1.1)-(1.6) by

$$\mathcal{E}(t) = \mathcal{E}_{\rm kin}(t) + \mathcal{E}_{\rm pot}(t)$$

with

$$\begin{aligned}
\mathcal{E}_{\rm kin}(t) &= \frac12\int_{\mathbb{R}^3_+\times\mathbb{R}^3} |v|^2 f(t,x,v)\,dx\,dv + \frac12\sum_\alpha |\eta_\alpha(t)|^2, \\
\mathcal{E}_{\rm pot}(t) &= \frac{1}{8\pi}\int_{\mathbb{R}^3_+} |F_\rho(t,x)|^2\,dx + \sum_\alpha\int_{\mathbb{R}^3_+}\Big(\frac{1}{|x-\xi_\alpha(t)|} - \frac{1}{|x-\xi_\alpha^*(t)|}\Big)\rho(t,x)\,dx \\
&\quad + \frac12\sum_{\alpha\neq\beta}\Big(\frac{1}{|\xi_\alpha(t)-\xi_\beta(t)|} - \frac{1}{|\xi_\alpha(t)-\xi_\beta^*(t)|}\Big).
\end{aligned}$$

Proposition 2.8. *Suppose that $\{f, \xi_\alpha, \eta_\alpha\}$ is a solution given by Theorem 1.1. Then*

$$\mathcal{E}(t) = \mathcal{E}(0), \quad \|f(t)\|_p = \|\mathring{f}\|_p \quad t < \mathring{T},\ 1\le p\le\infty. \tag{2.8}$$

As a consequence, we have

$$\int |v|^2 f(t,x,v)\,dx\,dv \le \mathcal{E}(0), \quad \|\rho(t)\|_{5/3} \le C\|\mathring{f}\|_\infty^{2/5}\mathcal{E}(0)^{3/5}. \tag{2.9}$$

Proof. We denote $n_x = (-1,0,0)$ as the outward normal on $\partial\mathbb{R}^3_+$. Firstly, we prove the first equality in (2.8). From (1.1)-(1.6) we derive

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3_+\times\mathbb{R}^3}|v|^2 f\,dx\,dv &= \int_{\mathbb{R}^3_+\times\mathbb{R}^3} F\cdot vf\,dx\,dv + \frac{1}{2}\int_{\partial\mathbb{R}^3_+\times\mathbb{R}^3} n_x\cdot v|v|^2 f\,dx_2\,dx_3\,dv,\\
\frac{1}{8\pi}\frac{d}{dt}\int_{\mathbb{R}^3_+}|F_\rho|^2\,dx &= -\int_{\mathbb{R}^3_+\times\mathbb{R}^3}\phi_\rho\partial_t\rho\,dx\,dv + \frac{1}{4\pi}\int_{\partial\mathbb{R}^3_+}\phi_\rho n_x\cdot\partial_t\nabla_x\phi_\rho\,dx_2\,dx_3\\
&= -\int_{\mathbb{R}^3_+\times\mathbb{R}^3} F_\rho\cdot vf\,dx\,dv + \int_{\partial\mathbb{R}^3_+\times\mathbb{R}^3}\phi_\rho n_x\cdot vf\,dx_2\,dx_3\,dv\\
&\quad + \frac{1}{4\pi}\int_{\partial\mathbb{R}^3_+}\phi_\rho n_x\cdot\partial_t\nabla_x\phi_\rho\,dx_2\,dx_3,\\
\frac{1}{2}\frac{d}{dt}\sum_\alpha|\eta_\alpha(t)|^2 &= \sum_\alpha\eta_\alpha(t)\cdot F_\alpha(t,\xi_\alpha(t)),
\end{aligned}$$

$$\begin{aligned}
&\frac{d}{dt}\sum_\alpha\int_{\mathbb{R}^3_+}\Big(\frac{1}{|x-\xi_\alpha(t)|}-\frac{1}{|x-\xi^*_\alpha(t)|}\Big)\rho(t,x)\,dx\\
&= -\sum_\alpha\eta_\alpha(t)\cdot F_\rho(t,\xi_\alpha(t)) + \sum_\alpha\int_{\mathbb{R}^3_+\times\mathbb{R}^3}\Big(\frac{x-\xi^*_\alpha}{|x-\xi^*_\alpha|^3}-\frac{x-\xi_\alpha}{|x-\xi_\alpha|^3}\Big)\cdot vf\,dx\,dv\\
&\quad -\sum_\alpha\int_{\partial\mathbb{R}^3_+\times\mathbb{R}^3}\Big(\frac{1}{|x-\xi_\alpha|}-\frac{1}{|x-\xi^*_\alpha|}\Big)n_x\cdot vf\,dx_2\,dx_3\,dv,
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{\alpha\neq\beta}\Big(\frac{1}{|\xi_\alpha(t)-\xi_\beta(t)|}-\frac{1}{|\xi_\alpha(t)-\xi^*_\beta(t)|}\Big)\\
&= \sum_{\alpha\neq\beta}\Big(\frac{\xi_\alpha-\xi^*_\beta}{2|\xi_\alpha-\xi^*_\beta|^3}\cdot(\eta_\alpha-\eta^*_\beta)-\frac{\xi_\alpha-\xi_\beta}{|\xi_\alpha-\xi_\beta|^3}\cdot\eta_\alpha\Big).
\end{aligned}$$

All the boundary terms vanish, since $|x-\xi_\alpha| = |x-\xi^*_\alpha|$, $\phi_\rho = 0$ for $x\in\partial\mathbb{R}^3_+$ and

$$\int_{\partial\mathbb{R}^3_+\times\mathbb{R}^3}\psi(x,v)f\,dx_2\,dx_3\,dv = 0$$

for any measurable function ψ satisfying $\psi(x,v) = -\psi(x,v^*)$ on $\partial\mathbb{R}^3_+\times\mathbb{R}^3$, because of the specular boundary condition (1.8). Adding all the equalities we obtain

$$\frac{d}{dt}\mathcal{E}(t) = -\frac{1}{2}\sum_{\alpha\neq\beta}\frac{\xi_\alpha-\xi^*_\beta}{|\xi_\alpha-\xi^*_\beta|^3}(\eta_\alpha+\eta^*_\beta) = 0,$$

the above equality is deduced by the facts $(\xi_\alpha-\xi^*_\beta)\cdot\eta^*_\beta = (\xi^*_\alpha-\xi_\beta)\cdot\eta_\beta$ and $|\xi_\alpha-\xi^*_\beta| = |\xi^*_\alpha-\xi_\beta|$. Thanks to the convenience of the definition (1.11), we have

$$\mathcal{E}(t) = \mathcal{E}(0)\quad\text{for } 0\le t<\mathring{T}.$$

The second equality in (2.8) can be proved by multiplying (1.1) by $(f)^{n-1}$, and by integrating by parts using the the boundary conditions.

Finally (2.9) can be obtained with minor adaptations of the ideas used in the proof of a similar estimate in the whole space (cf. [29, Prn.1.9]). □

Note that in the proof above, we used that

$$\frac{1}{8\pi}\int_{\mathbb{R}^3_+}|F_\rho(t,x)|^2\,dx = -\frac{1}{2}\int_{\mathbb{R}^3_+}\rho(t,x)\phi_\rho(t,x)\,dx.$$

Now we deduce the classical dispersion estimates (see [19]) for the plasma-charge model. We define

$$\mathcal{G}(t) := \frac{1}{2}\int_{\mathbb{R}^3_+\times\mathbb{R}^3}|vt-x|^2 f(t,x,v)\,dx\,dv + \frac{1}{2}\sum_\alpha|\eta_\alpha(t)t-\xi_\alpha(t)|^2 + t^2\mathcal{E}_{\text{pot}}(t)$$

$$= t^2\mathcal{E}(t) + \frac{1}{2}\int_{\mathbb{R}^3_+\times\mathbb{R}^3} |x|^2 f - t\int_{\mathbb{R}^3_+\times\mathbb{R}^3} x\cdot vf + \frac{1}{2}\sum_\alpha |\xi_\alpha(t)|^2 - t\sum_\alpha \xi_\alpha(t)\cdot\eta_\alpha(t).$$

Corollary 2.9. *Suppose that $\{f, \xi_\alpha, \eta_\alpha\}$ is a solution given by Theorem 1.1. Then*

$$\int_{\mathbb{R}^3_+\times\mathbb{R}^3} |v - x/t|^2 f(t,x,v)\,dx\,dv + \sum_\alpha |\eta_\alpha(t) - \xi_\alpha(t)/t|^2 \le \frac{2\mathcal{G}(t_0)}{t_0}t^{-1}$$

for $0 < t_0 < t < \mathring{T}$. As a consequence, we have

$$\|\rho(t)\|_{5/3} \le C\|\mathring{f}\|_\infty^{2/5}\Big(\frac{\mathcal{G}(t_0)}{t_0}\Big)^{3/5} t^{-3/5}. \tag{2.10}$$

Proof. Differentiating $\mathcal{G}(t)$ and using (1.1)-(1.6) yields

$$\begin{aligned}
\frac{d}{dt}\mathcal{G}(t) &= 2t\mathcal{E}(t) + \frac{1}{2}\int_{\mathbb{R}^3_+\times\mathbb{R}^3} |x|^2\partial_t f - t\int_{\mathbb{R}^3_+\times\mathbb{R}^3} x\cdot v\partial_t f - \int_{\mathbb{R}^3_+\times\mathbb{R}^3} x\cdot vf\\
&\quad + \sum_\alpha \xi_\alpha\cdot\eta_\alpha - \sum_\alpha \xi_\alpha\cdot\eta_\alpha - t\sum_\alpha |\eta_\alpha|^2 - t\sum_\alpha \xi_\alpha\cdot F_\alpha(t,\xi_\alpha)\\
&= 2t\mathcal{E}_{\rm pot}(t) - t\int_{\mathbb{R}^3_+\times\mathbb{R}^3} x\cdot Ff - t\sum_\alpha \xi_\alpha\cdot F_\alpha(t,\xi_\alpha).
\end{aligned}$$

Note that

$$\begin{gathered}
\int_{\mathbb{R}^3_+\times\mathbb{R}^3_+} x\cdot\frac{x-y}{|x-y|^3}\rho(y)\rho(x)\,dx\,dy = \frac{1}{2}\int_{\mathbb{R}^3_+\times\mathbb{R}^3_+} \frac{1}{|x-y|}\rho(y)\rho(x)\,dx\,dy,\\
\int_{\mathbb{R}^3_+\times\mathbb{R}^3_+} x\cdot\frac{x-y^*}{|x-y^*|^3}\rho(y)\rho(x)\,dx\,dy = \frac{1}{2}\int_{\mathbb{R}^3_+\times\mathbb{R}^3_+} \frac{1}{|x-y^*|}\rho(y)\rho(x)\,dx\,dy,\\
\int_{\mathbb{R}^3_+} x\cdot\Big(\frac{x-\xi_\alpha}{|x-\xi_\alpha|^3} - \frac{x-\xi_\alpha^*}{|x-\xi_\alpha^*|^3}\Big)\rho(x) - \xi_\alpha\cdot\Big(\frac{\xi_\alpha - x}{|\xi_\alpha - x|^3} - \frac{\xi_\alpha - x^*}{|\xi_\alpha - x^*|^3}\Big)\rho(x)\,dx\\
= \int_{\mathbb{R}^3_+}\Big(\frac{1}{|x-\xi_\alpha|} - \frac{1}{|x-\xi_\alpha^*|}\Big)\rho(x)\,dx,\\
\sum_{\alpha\ne\beta}\xi_\alpha\cdot\Big(\frac{\xi_\alpha-\xi_\beta}{|\xi_\alpha-\xi_\beta|^3} - \frac{\xi_\alpha-\xi_\beta^*}{|\xi_\alpha-\xi_\beta^*|^3}\Big) = \frac{1}{2}\sum_{\alpha\ne\beta}\Big(\frac{1}{|\xi_\alpha-\xi_\beta|} - \frac{1}{|\xi_\alpha-\xi_\beta^*|}\Big).
\end{gathered}$$

Then we have

$$t\int_{\mathbb{R}^3_+\times\mathbb{R}^3} x\cdot Ff + t\sum_\alpha \xi_\alpha\cdot F_\alpha(t,\xi_\alpha) = t\mathcal{E}_{\rm pot}(t)$$

and hence

$$\frac{d}{dt}\mathcal{G}(t) = t\mathcal{E}_{\rm pot}(t) \le \frac{\mathcal{G}(t)}{t}.$$

Then for any $0 < t_0 < t < \mathring{T}$, we obtain

$$\mathcal{G}(t) \le \mathcal{G}(t_0)\frac{t}{t_0}.$$

So that (2.10) follows immediately by the classical kinetic interpolation. □

3. Global well-posedness of the model with fixed point charges

As we have seen in the proof of the local existence, the derivative of the function $\mathsf{e}^n(t)$ does not eliminate all the principal singular terms produced by the point charges. Hence for the case of moving point charges, it seems to fail to adapt the argument in [25] to establish the global existence of solutions. Nevertheless, when the point charges are fixed and away from the boundary $\partial\mathbb{R}^3_+$, the pointwise energy function introduced in [5, 25] is still a powerful tool. Since we can adapt directly the argument in [25, Section 4] to prove Theorem 1.2 for $N > 1$, for simplicity, we only deal with the case of single point charge, i.e., $N = 1$. We denote $\mathring{\xi} = \mathring{\xi}_1 = (\mathring{\xi}_{1,1}, \mathring{\xi}_{1,2}, \mathring{\xi}_{1,3})$ with $\mathring{\xi}_{1,1} > \delta_0$, then the energy function is given by

$$h(x,v) = \frac{|v|^2}{2} + \frac{1}{|x-\mathring{\xi}|} + K_1,$$

where $K_1 > 1$ is a large constant depending on the energy $\mathcal{E}(0)$, as in [25, p7, (22)]. K_1 depends also on $\mathring{\xi}_{1,1}$ such that $l \le \frac{\mathring{\xi}_{1,1}}{8}$ where l is defined by (3.1). Along the characteristic flows, we define

$$\begin{aligned} Q_{t,\delta} &= \sup\{\sqrt{h}(x,v) \colon (x,v) \in \operatorname{supp} f(s),\ t-\delta \le s \le t\} \\ &= \sup\{\sqrt{h}(X(s),V(s)) \colon (x,v) \in \operatorname{supp} f(t-\delta), t-\delta \le s \le t\} \end{aligned}$$

where $(X(s),V(s)) = (X,V)(s,t-\delta,x,v)$ is the characteristic flows given by (1.9). Note by the assumption (1.13), we have $|x-\mathring{\xi}^*| \ge \delta_0$ for $x \in \mathbb{R}^3_+$, hence

$$\begin{aligned} \frac{d}{ds}\sqrt{h}(X(s),V(s)) &= \frac{1}{2\sqrt{h}(X(s),V(s))}\Big(V(s)\cdot F(s,X(s)) - \frac{(X(s)-\mathring{\xi})\cdot V(s)}{|X(s)-\mathring{\xi}|^3}\Big) \\ &\le C\int_{\mathbb{R}^3_+} \frac{\rho(s,y)}{|X(s)-y|^2}\,dy + \frac{1}{|X(s)-\mathring{\xi}^*|^2} \\ &\le C\int_{\mathbb{R}^3_+\times\mathbb{R}^3} \frac{f(s,y,w)}{|X(s)-y|^2}\,dy\,dw + \delta_0^{-2}. \end{aligned}$$

Let $\tau \in [t-\delta, t]$. Integrating the inequality from $t-\delta$ to τ and by (1.10) to obtain

$$|\sqrt{h}(X(\tau),V(\tau)) - \sqrt{h}(x,v)| \le C\int_{t-\delta}^{\tau}\int_{\mathbb{R}^3_+\times\mathbb{R}^3} \frac{f(s,y,w)}{|X(s)-y|^2}\,dy\,dw\,ds + \delta_0^{-2}(\tau-t+\delta).$$

Now we define the three parameters

$$l = Q_{t,\delta}^{-a}, \quad R = Q_{t,\delta}^{b}, \quad L = Q_{t,\delta}^{-c} \quad \text{with } a = \frac{2}{7},\ b = \frac{5}{7},\ c = \frac{43}{21} \tag{3.1}$$

and set $\delta = c_0 Q_{t,t}^{-1}$ where c_0 will be chosen later small enough depending only on $\mathcal{E}(0), \|\mathring{f}\|_1, \|\mathring{f}\|_\infty$ and $\mathring{\xi}_{1,1}$. Note that

$$(l^{-2} + Q_{t,\delta}^{4/3})\delta \le 2c_0 R. \tag{3.2}$$

By (2.2) and $b > \frac{1}{3}$, take c_0 small enough such that we have

$$|\sqrt{h}(X(\tau),V(\tau)) - \sqrt{h}(x,v)| \le (KQ_{t,\delta}^{4/3} + \delta_0^{-2})\delta \le \frac{Q_{t,\delta}^{1/3}}{4} \le \frac{R}{4}. \tag{3.3}$$

Without loss of generality, we assume $Q_{t,\delta} \ge K_2 > 1$ where K_2 is a large constant depending only on K_1 and K as in [25, p10, (25)]. Since $\epsilon_0 := \min\{2b - \frac{4}{3} + a, 2b - \frac{4}{3}\} > 0$, we have

$$(l+1)Q_{t,\delta}^{4/3} \le 2K_2^{-\frac{\epsilon_0}{2}} R^2. \tag{3.4}$$

Lemma 3.1. *Let $(y,w) \in \operatorname{supp} f(t-\delta)$ and $(Y,W)(s,t-\delta,y,w)$. Suppose $\sqrt{h}(y,w) > R$. Then the set*

$$J = \{s \in (t-\delta,t) \colon |Y(s)-\mathring{\xi}| < l\}$$

is connected. Moreover, we have $\operatorname{meas}(J) \le CR^{-1}l$.

Proof. We denote $I(s) = \frac{1}{2}|Y(s)-\mathring{\xi}|^2$. by differentiating we obtain

$$\dot{I} = (Y-\mathring{\xi})\cdot W,$$

$$\ddot{I} = |W|^2 + \frac{1}{|Y-\mathring{\xi}|} + (Y-\mathring{\xi})\cdot F_\rho(s,Y) - (Y-\mathring{\xi})\cdot\frac{Y-\mathring{\xi}^*}{|Y-\mathring{\xi}^*|^3}.$$

Assume that J is non-empty and $t_0 \in J$. Then for $s \in (t-\delta,t)$,

$$\sqrt{I(s)} \le \sqrt{I(t_0)} + Q_{t,\delta}\delta \le \frac{l}{\sqrt{2}} + c_0 \le \frac{\mathring{\xi}_{1,1}}{2\sqrt{2}}$$

where we have used that $l \le \frac{\mathring{\xi}_{1,1}}{8}$ and taken $c_0 \le \frac{\mathring{\xi}_{1,1}}{8\sqrt{2}}$. Then we have

$$|Y_1(s) - \mathring{\xi}_{1,1}| \le \frac{\mathring{\xi}_{1,1}}{4},$$

which implies that the trajectory has no bounce in the time interval $(t-\delta, t)$. Hence $I(t)$ is C^2 in $(t-\delta, t)$.

By the assumption $\sqrt{h}(y,w) > R$ and (3.3), we have

$$\sqrt{h}(Y(s), W(s)) \ge \frac{R}{2}.$$

Then by (3.4) and take K_2 large enough (depending only on K_1, K) such that we have

$$\begin{aligned} \ddot{I} &\ge h(Y(s), W(s)) - K_1 - |Y - \mathring{\xi}|\big(|F_\rho(Y)| + \delta_0^{-2}\big) \\ &\ge \frac{1}{4}R^2 - K_1 - C(l+1)Q_{t,\delta}^{4/3} \ge \frac{1}{8}R^2. \end{aligned}$$

Hence $I(s)$ is convex on $(t-\delta, t)$ and the sublevel set J is a convex subset of $(t-\delta, t)$, whence J is connected. Now let $t_0 \in \bar{J}$ be the minimizer for $I(s)$, then $\dot{I}(t_0)(t-t_0) \ge 0$ and we obtain for $s \in J$

$$\frac{l^2}{2} \ge I(s) \ge I(t_0) + \dot{I}(t_0)(t-t_0) + \frac{1}{16}R^2(t-t_0)^2 \ge \frac{1}{16}R^2(t-t_0)^2. \qquad \square$$

Lemma 3.2. *Let $(y,w) \in \operatorname{supp} f(t-\delta)$ and $(Y,W)(s, t-\delta, y, w)$. Suppose $\sqrt{h}(y,w) > Q_{t,\delta}/2$. Then the set*

$$J = \{s \in (t-\delta, t) \colon |Y(s) - \mathring{\xi}| < 2l\}$$

is connected. Moreover, we have $\operatorname{meas}(J) \le CQ_{t,\delta}^{-1} l$.

Proof. By the assumption $\sqrt{h}(y,w) > Q_{t,\delta}/2$ and (3.3), we have

$$\sqrt{h}(Y(s), W(s)) \ge \frac{Q_{t,\delta}}{4}.$$

Then by (3.4) and take K_2 large enough (depending only on K_1, K) such that we have

$$\begin{aligned} \ddot{I} &\ge h(Y(s), W(s)) - K_1 - |Y - \mathring{\xi}|\big(|F_\rho(Y)| + \delta_0^{-2}\big) \\ &\ge \frac{1}{16}Q_{t,\delta}^2 - K_1 - C(l+1)Q_{t,\delta}^{4/3} \ge \frac{1}{32}Q_{t,\delta}^2. \end{aligned}$$

The remaining proof is similar to that of Lemma 3.1. $\square$

We denote $Y_\perp = Y_1$, $Y_\parallel = (Y_2, Y_3)$ and $W_\perp = W_1$, $W_\parallel = (W_2, W_3)$. The following result establishes that characteristic flows bouncing repeatedly in some time interval should have a small value of the normal component $W_\perp$.

Lemma 3.3. *Assume that there exists a time interval*

$$J = (t_1, t_2) \subset (t-\delta, t)$$

such that for $s \in J$ we have, with $(Y,W)(s) = (Y,W)(s, t-\delta, y, w)$, where $(y,w) \in \operatorname{supp} f(t-\delta)$

$$\inf_{s\in J} \min\{|Y(s) - \mathring{\xi}|\} > l. \tag{3.5}$$

If (Y,W) has more than one bounce in the interval J, then we have for $s \in J$

$$|W_\perp| \le \frac{R}{32}.$$

Proof. By (2.2) and (3.5), we have

$$\Big|\frac{d|W_\perp|}{ds}\Big| \le KQ_{t,\delta}^{4/3} + l^{-2} + \delta_0^{-2}, \tag{3.6}$$

If (Y,W) has more than one bounce, then we have $W_\perp(\bar{s}) = 0$ for some $\bar{s} \in J$. Then

$$|W_\perp| \le KQ_{t,\delta}^{4/3}\delta + l^{-2}\delta + \delta_0^{-2}\delta.$$

The lemma follows from (3.2) by taking c_0 small enough. □

The following lemma will play a key role as the separation property in [17, Lemma 15] and [25, Lemma 3].

Lemma 3.4. *Let $L>0$. Assume that there exists a time interval*
$$J=(t_1,t_2)\subset(t-\delta,t)$$
such that for $s\in J$ we have, with $(X,V)(s)=(X,V)(s,t-\delta,x,v)$ and $(Y,W)(s)=(Y,W)(s,t-\delta,y,w)$, where $(x,v),(y,w)\in\operatorname{supp}f(t-\delta)$
$$\inf_{s\in J}\min\{|X(s)-\mathring{\xi}|,|Y(s)-\mathring{\xi}|\}>l. \tag{3.7}$$
If $\min\{|V(t_1)-W(t_1)|,|V(t_1)-W^*(t_1)|\}\le R$, then
$$\min\{|V(s)-W(s)|,|V(s)-W^*(s)|\}\le\frac{3}{2}R,\quad s\in J. \tag{3.8}$$
If $\min\{|V(t_1)-W(t_1)|,|V(t_1)-W^*(t_1)|\}>R$, then
$$\min\{|V(s)-W(s)|,|V(s)-W^*(s)|\}>\frac{1}{2}R,\quad s\in J \tag{3.9}$$
and
$$\int_{t_1}^{t_2}\frac{\mathbf{1}_{\{|X(s)-Y(s)|>L\}}}{|X(s)-Y(s)|^2}\,ds\le\frac{C}{RL}. \tag{3.10}$$

Proof. We split the proof into three steps.

Step 1. We denote the reflection operator as $\mathcal{R}(\cdot)=(\cdot)^*$. Note $\mathcal{R}$ is linear, isometric and self-inverse on $\mathbb{R}^3$. Let $k(s),\tilde{k}(s)$ be the numbers of bounces of (X,V) and (Y,W) respectively during time $[t_1,s)$. It is no difficult to deduce that
$$\begin{aligned}V(s)&=\mathcal{R}^{k(s)}V(t_1)+\int_{t_1}^{s}\mathcal{R}^{k(s)-k(\tau)}\big(F(\tau,X(\tau))\,d\tau,\\ W(s)&=\mathcal{R}^{\tilde{k}(s)}W(t_1)+\int_{t_1}^{s}\mathcal{R}^{\tilde{k}(s)-\tilde{k}(\tau)}\big(F(\tau,Y(\tau)\big)\,d\tau,\end{aligned} \tag{3.11}$$
and
$$\big||V(s)-W(s)|-|\mathcal{R}^{k(\tau)}V(t_1)-\mathcal{R}^{\tilde{k}(\tau)}W(t_1)|\big|\le\int_{t_1}^{s}|F(\tau,X(\tau))|+|F(\tau,Y(\tau))|\,d\tau,$$
$$\big||V(s)-W^*(s)|-|\mathcal{R}^{k(\tau)}V(t_1)-\mathcal{R}^{\tilde{k}(\tau)+1}W(t_1)|\big|\le\int_{t_1}^{s}|F(\tau,X(\tau))|+|F(\tau,Y(\tau))|\,d\tau.$$
By (2.2), (3.7) and taking c_0 small enough, we have
$$\int_{t_1}^{s}|F(\tau,X(\tau))|+|F(\tau,Y(\tau))|\,d\tau\le 2KQ_{t,\delta}^{\frac{4}{3}}\delta+2l^{-2}\delta+2\delta_0^{-2}\delta\le\frac{R}{2}. \tag{3.12}$$
(3.8), (3.9) follow.

Step 2. We claim that there exist at most three points $\bar{t}_1,\bar{t}_2,\bar{t}_3\in J$ such that
$$|X(s)-Y(s)|\ge\min_{i=1,2,3}\{|s-\bar{t}_i|\}\frac{R}{8},\quad s\in J.$$
Then
$$\begin{aligned}\int_{t_1}^{t_2}\frac{\mathbf{1}_{\{|X(s)-Y(s)|>L\}}}{|X(s)-Y(s)|^2}\,ds&\le\int_{\min_{i=1,2,3}\{|s-\bar{t}_i|\}\le m}+\int_{\min_{i=1,2,3}\{|s-\bar{t}_i|\}>m}\frac{\mathbf{1}_{\{|X(s)-Y(s)|>L\}}}{|X(s)-Y(s)|^2}\,ds\\ &\le\sum_{i=1}^{3}\int_{|s-\bar{t}_i|\le m}L^{-2}\,ds+\sum_{i=1}^{3}\int_{|s-\bar{t}_i|>m}\frac{64}{R^2|s-\bar{t}_i|^2}\,ds\\ &\le 6L^{-2}m+\frac{6\times 64}{R^2m}.\end{aligned}$$
Optimizing m we obtain (3.10).

Step 3. We prove the claim in Step 2. It can be split into three cases.

Case 1. If both trajectories have at most one bounce in $[t_1, t_2]$. Without loss of generality, we consider there are exactly two bounces at time s_1, s_2 with $t_1 < s_1 < s_2 < t_2$. For convenience, we denote $s_0 = t_1, s_3 = t_2$. As usual, we analyze the quantity $D \in C^2([s_i, s_{i+1}])$ defined as $D(s) = X(s) - Y(s)$ on (s_i, s_{i+1}) for $i = 0, 1, 2$ and the values of $D, \dot{D}, \ddot{D}$ at endpoints s_i, s_{i+1} are defined by the right and left limits respectively. Pick $\bar{s}_i \in [s_i, s_{i+1}]$ such that $|D(\bar{s}_i)|$ is minimal in $[s_i, s_{i+1}]$. Set

$$\overline{D}(s) = D(\bar{s}_i) + (s - \bar{s}_i)\dot{D}(\bar{s}_i).$$

Note that

$$(s - \bar{s}_i)D(\bar{s}_i) \cdot \dot{D}(\bar{s}_i) \geq 0, \quad s \in [s_i, s_{i+1}].$$

Thus by (3.9) we obtain

$$|\overline{D}(s)|^2 \geq (s - \bar{s}_i)^2|\dot{D}(\bar{s}_i)|^2 = (s - \bar{s}_i)^2|V(\bar{s}_i) - W(\bar{s}_i)|^2 \geq \frac{R^2}{4}(s - \bar{s}_i)^2, \quad s \in [s_i, s_{i+1}]. \tag{3.13}$$

Now it follows from Taylor's theorem that

$$|\ddot{D}(s) - \ddot{\overline{D}}(s)| = |F(s, X(s)) - F(s, Y(s))|$$

so that similar to (3.12) for $s \in [s_i, s_{i+1}]$ we have

$$|D(s) - \overline{D}(s)| \leq \Big| \int_{\bar{s}_i}^{s} \int_{\bar{s}_i}^{\tau} |F(\zeta, X(\zeta))| + |F(\zeta, Y(\zeta))| \, d\zeta \, d\tau \Big| \leq \frac{R}{4}|s - \bar{s}_i|. \tag{3.14}$$

Therefore, by (3.13) and (3.14) we have

$$|D(s)| \geq |\overline{D}(s)| - |D(s) - \overline{D}(s)| \geq \frac{R}{4}|s - \bar{s}_i|.$$

The claim holds for $\bar{t}_i = \bar{s}_{i-1}$, $i = 1, 2, 3$.

Case 2. Now if at least one of the trajectories has more than one bounce in (t_1, t_2). Without loss of generality, let (X, V) has more than one bounce in (t_1, t_2). We first consider the case

$$|W_\parallel(s) - V_\parallel(s)| \geq \frac{1}{2}|W(s) - V(s)| \quad \text{for } s \in [t_1, t_2], \tag{3.15}$$

Note that the tangential parts of the trajectories are $C^2([t_1, t_2])$, we define $D_\parallel(s) = X_\parallel(s) - Y_\parallel(s)$ on $[t_1, t_2]$. Pick $\underline{s}_1 \in [t_1, t_2]$ such that $|D_\parallel(\underline{s}_1)| = \min_{s \in [t_1, t_2]} |D_\parallel(s)|$. Set

$$\overline{D}_\parallel(s) = D_\parallel(\underline{s}_1) + (s - \underline{s}_1)\dot{D}_\parallel(\underline{s}_1).$$

Note that

$$(s - \underline{s}_1)D_\parallel(\underline{s}_1) \cdot \dot{D}_\parallel(\underline{s}_1) \geq 0, \quad s \in [t_1, t_2].$$

Thus by (3.9) and (3.15), we have

$$|\overline{D}_\parallel(s)|^2 \geq (s - \underline{s}_1)^2|\dot{D}_\parallel(\underline{s}_1)|^2 \geq \frac{R^2}{16}(s - \underline{s}_1)^2, \quad s \in [t_1, t_2]. \tag{3.16}$$

Now it follows from Taylor's theorem that

$$|\ddot{D}_\parallel(s) - \ddot{\overline{D}}_\parallel(s)| = |F_\parallel(s, X(s)) - F_\parallel(s, Y(s))|$$

so that similar to (3.12) we derive

$$|D_\parallel(s) - \overline{D}_\parallel(s)| \leq \Big| \int_{\underline{s}_1}^{s} \int_{\underline{s}_1}^{\tau} |F_\parallel(\zeta, X(\zeta))| + |F_\parallel(\zeta, Y(\zeta))| \, d\zeta \, d\tau \Big| \leq \frac{R}{8}|s - \underline{s}_1|. \tag{3.17}$$

Therefore by (3.16) and (3.17) we have for $s \in [t_1, t_2]$

$$|X(s) - Y(s)| \geq |D_\parallel(s)| \geq |\overline{D}_\parallel(s)| - |D_\parallel(s) - \overline{D}_\parallel(s)| \geq \frac{R}{8}|s - \underline{s}_1|.$$

The claim holds for $\bar{t}_i = \underline{s}_1$, $i = 1, 2, 3$.

Case 3. We now consider the complementary case to (3.15). Then there exists $\bar{s} \in [t_1, t_2]$ such that

$$|W_\perp(\bar{s}) - V_\perp(\bar{s})| \geq \frac{1}{2}|W(\bar{s}) - V(\bar{s})|. \tag{3.18}$$

By (3.6) we have

$$\begin{aligned}\big||W_\perp(s)| - |V_\perp(s)|\big| &\ge \big||W_\perp(\bar s)| - |V_\perp(\bar s)|\big| - 2KQ_{t,\delta}^{4/3}\delta - 2l^{-2}\delta - 2\delta_0^{-2}\delta\\ &\ge \big||W_\perp(\bar s)| - |V_\perp(\bar s)|\big| - \frac{R}{16}.\end{aligned}$$

On the other hand, from Lemma 3.3 and the triangle inequality it follows that

$$\begin{aligned}\big||W_\perp(s)| - |V_\perp(s)|\big| &= \big||W_\perp(s) - V_\perp(s) + V_\perp(s)| - |V_\perp(s)|\big|.\\ &\le |W_\perp(s) - V_\perp(s)| + 2|V_\perp(s)|\\ &\le |W_\perp(s) - V_\perp(s)| + \frac{R}{32}.\end{aligned}$$

Similarly, we obtain

$$||W_\perp(\bar s)| - |V_\perp(\bar s)|| \ge |W_\perp(\bar s) - V_\perp(\bar s)| - \frac{R}{32}.$$

Thus by (3.18) and (3.9), we obtain for $s \in [t_1, t_2]$

$$|W_\perp(s) - V_\perp(s)| \ge |W_\perp(\bar s) - V_\perp(\bar s)| - \frac{R}{8} \ge \frac{1}{2}|W(\bar s) - V(\bar s)| - \frac{R}{8} \ge \frac{R}{8}. \tag{3.19}$$

From Lemma 3.3 it follows that

$$|W_\perp(s)| \ge \frac{R}{16} \tag{3.20}$$

for $s \in [t_1, t_2]$. It follows that $W_\perp(s)$ changes sign, by reflection, at most once in the interval $s \in [t_1, t_2]$. Combining Lemma 3.3, (3.19) and (3.20) it follows that $W_\perp(s) - V_\perp(s)$ changes sign at most once for $s \in [t_1, t_2]$. Suppose that $W_\perp(s)$ changes sign (if any) at $s = \underline{s}_2$, i.e., (Y, W) bounces at $s = \underline{s}_2$. Since $Y_\perp(s) \ge 0$, it follows that $\operatorname{sgn}(W_\perp(s)) = \operatorname{sgn}(W_\perp(s) - V_\perp(s)) = \operatorname{sgn}(s - \underline{s}_2)$ for $s \in [t_1, t_2]$. Pick $s_0 \in [t_1, t_2]$ such that

$$\min_{s\in[t_1,t_2]} |Y_\perp(s) - X_\perp(s)| = |Y_\perp(s_0) - X_\perp(s_0)|.$$

If $s_0 \in (t_1, t_2)$, then by (3.19) and Lemma 2.7, it must be $s_0 = \underline{s}_2$ and

$$|Y_\perp(s) - X_\perp(s)| = \Big|\int_{\underline{s}_2}^{s} |W_\perp(\tau) - V_\perp(\tau)|\, d\tau\Big| \ge \frac{R}{8}|s - \underline{s}_2| \quad s \in [t_1, t_2].$$

If there is no minimal point in (t_1, t_2), then $s_0 = t_1$ or $s_0 = t_2$. We discuss it in several situations.

(a) If $s_0 = t_1$, $Y_\perp(t_1) - X_\perp(t_1) \ge 0$, it is obvious that $W_\perp(t_1 + 0) < 0$ is impossible. Then by $W_\perp(t_1 + 0) > 0$, we always have $W_\perp(s) - V_\perp(s) > 0$ and

$$|Y_\perp(s) - X_\perp(s)| = Y_\perp(t_1) - X_\perp(t_1) + \int_{t_1}^{s} W_\perp(\tau) - V_\perp(\tau)\, d\tau \ge \frac{R}{8}|s - t_1| \quad s \in [t_1, t_2].$$

(b). If $s_0 = t_1$, $Y_\perp(t_1) - X_\perp(t_1) < 0$, since (Y, W) bounces at most once, (X, V) bounce at least twice, it must exists $s_1 \in [t_1, t_2]$ such that $Y_\perp(s_1) - X_\perp(s_1) = 0$, which is a contradiction.

(c). If $s_0 = t_2$, $W_\perp(t_2 - 0) < 0$, we always have $W_\perp(s) - V_\perp(s) < 0$ and $Y_\perp(s) - X_\perp(s) \ge 0$. Then

$$|Y_\perp(s) - X_\perp(s)| = Y_\perp(t_2) - X_\perp(t_2) + \int_{t_2}^{s} W_\perp(\tau) - V_\perp(\tau)\, d\tau \ge \frac{R}{8}|s - t_2| \quad s \in [t_1, t_2].$$

(d) If $s_0 = t_2$, $W_\perp(t_2 - 0) > 0$, it is obvious that $Y_\perp(t_2) - X_\perp(t_2) > 0$ is impossible. Since (Y, W) bounces at most once, (X, V) bounce at least twice, it must exists $s_2 \in (t_1, t_2)$ such that $Y_\perp(s_2) - X_\perp(s_2) = 0$, which is a contradiction.

Summing up, the claim holds for $\bar t_i = s_0$, $i = 1, 2, 3$. □

Proof of Theorem 1.2. Select $(\bar x, \bar v) \in \operatorname{supp} f(t - \delta)$ such that there exists $\bar s \in [t - \delta, t]$ such that $\sqrt{h}(X, V)(\bar s, t - \delta, \bar x, \bar v) = Q_{t,\delta}$. Abuse of notation, we denote $(X, V)(s) = (X, V)(s, t - \delta, \bar x, \bar v)$. By

(3.3), we have $\sqrt{h}(\bar{x},\bar{v}) \geq Q_{t,\delta}/2$. By Lemma 3.2, let (t_-,t_+) be the connected interval in which $|X(s)-\mathring{\xi}| < 2l$, combining with (2.2) and the setting of the parameters (3.1) we obtain

$$\int_{t_-}^{t_+}\int_{\mathbb{R}^3_+\times\mathbb{R}^3}\frac{f(s,y,w)}{|X(s)-y|^2}\,dy\,dw\,ds \leq CQ_{t,\delta}^{1/3}l.$$

It remains to control the integrals

$$\int_{t-\delta}^{t_-}\int_{\mathbb{R}^3_+\times\mathbb{R}^3}\frac{f(s,y,w)}{|X(s)-y|^2}\,dy\,dw\,ds \quad\text{and}\quad \int_{t_+}^{t}\int_{\mathbb{R}^3_+\times\mathbb{R}^3}\frac{f(s,y,w)}{|X(s)-y|^2}\,dy\,dw\,ds,$$

which can be handled (using the time reversal) in the same way. Denote $\Pi := (t-\delta,t_-)\times\mathbb{R}^3_+\times\mathbb{R}^3$ we write by the measure-preserving characteristic flows

$$\begin{aligned}\int_{t-\delta}^{t_-}\int_{\mathbb{R}^3_+\times\mathbb{R}^3}\frac{f(s,y,w)}{|X(s)-y|^2}\,dy\,dw\,ds &= \int_{t-\delta}^{t_-}\int_{\mathbb{R}^3_+\times\mathbb{R}^3}\frac{f(t-\delta,y,w)}{|X(s)-Y(s)|^2}\,dy\,dw\,ds\\ &= \sum_{j=1}^{4}\int_{S_j}\frac{f(t-\delta,y,w)}{|X(s)-Y(s)|^2}\,dy\,dw\,ds = \sum_{j=1}^{4}\mathcal{J}_j,\end{aligned}$$

where $(Y,W)(s) = (Y,W)(s,t-\delta,y,w)$ and

$$\begin{gathered}S_1 = \{(s,y,w)\in\Pi\colon \sqrt{h}(y,w)\leq R\},\\ S_2 = \{(s,y,w)\in\Pi\colon |X(s)-Y(s)|\leq L\}\setminus S_1,\\ S_3 = \{(s,y,w)\in\Pi\colon |Y(s)-\mathring{\xi}|\leq l\}\setminus(S_1\cup S_2),\\ S_4 = \Pi\setminus\cup_{i=1}^{3}S_i,\end{gathered}$$

Using (2.2), (3.3) and the measure-preserving property of the characteristic flows, we have

$$\mathcal{J}_1 \leq CKR^{4/3}\delta \leq CQ_{t,\delta}^{4b/3}\delta$$

Using the measure-preserving property of the characteristic flows, we have

$$\mathcal{J}_2 \leq \int_{t-\delta}^{t_-}\int_{|X(s)-y|\leq L}\frac{C\|\mathring{f}\|_\infty Q_{t,\delta}^3}{|X(s)-y|^2}\,dy\,ds \leq CLQ_{t,\delta}^3\delta$$

In S_3, we have $|X(s)-Y(s)| \geq |X(s)-\mathring{\xi}| - |Y(s)-\mathring{\xi}| \geq l$. By Lemma 3.1 and Proposition 2.8, we have

$$\begin{aligned}\mathcal{J}_3 &\leq l^{-2}\int_{\sqrt{h}(y,w)>R} f(t-\delta,y,w)\int_{t-\delta}^{t_-}\mathbf{1}_{\{|Y(s)-\mathring{\xi}|\leq l\}}\,ds\,dy\,dw\\ &\leq l^{-2}R^{-2}\int hf(t-\delta,y,w)\int_{t-\delta}^{t_-}\mathbf{1}_{\{|Y(s)-\mathring{\xi}|\leq l\}}\,ds\,dy\,dw\\ &\leq Cl^{-1}R^{-3}.\end{aligned}$$

By Lemma 3.1, for each (y,w) such that $\sqrt{h}(y,w) > R$, we may split the set

$$\{s\in(t-\delta,t_-)\colon |Y(s)-\mathring{\xi}| > l\}$$

into two at most intervals $J^1(y,w)$, $J^2(y,w)$ such that $\mathcal{J}_4 \leq \sum_{k=1,2}J_{4,k}$, where

$$J_{4,k} = \int_{\sqrt{h}(y,w)>R} f(t-\delta,y,w)\int_{J^k(y,w)}\frac{\mathbf{1}_{\{|Y(s)-X(s)|>L\}}}{|Y(s)-X(s)|^2}\,ds\,dy\,dw.$$

It suffices to control the integral $J_{4,1}$ because $J_{4,2}$ can be handled in the same way. We set $J^1(y,w) = (t_1,t_2)$ and split further the integration domain as follows.

$$\begin{aligned}S_4^1 &= \{(y,w)\colon \sqrt{h}(y,w) > R \text{ and } \min\{|V(t_1)-W(t_1)|,|V(t_1)-W^*(t_1)|\}\leq R\},\\ S_4^2 &= \{(y,w)\colon \sqrt{h}(y,w) > R \text{ and } \min\{|V(t_1)-W(t_1)|,|V(t_1)-W^*(t_1)|\} > R\}.\end{aligned}$$

Then

$$J_{4,1} \leq \sum_{k=1,2} J_{4,1,k},$$

where

$$J_{4,1,k} = \int_{S_4^k} f(t-\delta, y, w) \int_{t_1}^{t_2} \frac{\mathbf{1}_{\{|Y(s)-X(s)|>L\}}}{|Y(s)-X(s)|^2}\, ds\, dy\, dw.$$

By (2.2), (3.8) and the measure-preserving property of the characteristic flows, we have

$$\begin{aligned} \mathcal{J}_{4,1,1} &\leq \int_{\min\{|V(s)-w|,|V(s)-w^*|\}\leq \frac{3}{2}R} f(s,y,w) \int_{t_1}^{t_2} \frac{\mathbf{1}_{\{|y-X(s)|>L\}}}{|y-X(s)|^2}\, ds\, dy\, dw \\ &\leq CR^{4/3}\delta. \end{aligned}$$

By (3.10) and Proposition 2.8, we have

$$\mathcal{J}_{4,1,2} \leq R^{-2} \int_{S_4^2} hf(t-\delta, y, w) \int_{t_1}^{t_2} \frac{\mathbf{1}_{\{|Y(s)-X(s)|>L\}}}{|Y(s)-X(s)|^2}\, ds\, dy\, dw \leq CL^{-1}R^{-3}.$$

Summing up, by the setting of the parameters (3.1) and iteration, we have

$$\begin{aligned} Q_{t,\delta} &\leq Q_{t-\delta,\delta} + C(Q_{t,\delta}^{\frac{4b}{3}-1} + Q_{t,\delta}^{2-c} + Q_{t,\delta}^{a-3b} + Q_{t,\delta}^{c-3b} + Q_{t,\delta}^{\frac{1}{3}-a})Q_{t,\delta}\delta \\ &\leq Q_{t-\delta,\delta} + CQ_{t,\delta}^{20/21}\delta \leq Q_{0,0} + CQ_{t,t}^{20/21}t. \end{aligned}$$

Then we obtain

$$Q_{t,t} \leq Q_{0,0} + Ct^{21}.$$

The estimate (1.15) follows. Now we can verify the continuation criterion in Theorem 1.1 by (1.15). The proof is complete. □

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