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EXISTENCE OF NONTRIVIAL SOLUTIONS FOR SCHRÖDINGER-KIRCHHOFF EQUATIONS WITH INDEFINITE POTENTIALS

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Abstract. We consider a class of Schrödinger-Kirchhoff equations in $\mathbb{R}^3$ with a general nonlinearity g and coercive sign-changing potential V so that the Schrödinger operator $-a\Delta + V$ is indefinite. The nonlinearity considered here satisfies the Ambrosetti-Rabinowitz type condition $g(t)t \geq \mu G(t) > 0$ with $\mu > 3$. We obtain the existence of nontrivial solutions for this problem via Morse theory.

1. Introduction

In this article, we consider the Schrödinger-Kirchhoff type problem

$$-\Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\Big)\Delta u + V(x)u = g(u), \quad u \in H^1(\mathbb{R}^3), \tag{1.1}$$

where a and b are positive constants, $V \in C(\mathbb{R}^3)$ is the potential and $g \in C(\mathbb{R})$ is the nonlinearity.

The above problem is nonlocal as the term $\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x$ implies that (1.1) is not a pointwise identity. This feature causes some mathematical difficulties, which make the investigation of (1.1) particularly interesting. Problem (1.1) arises in an interesting physical context. Indeed, if we set $V(x) = 0$ and replace $\mathbb{R}^3$ by a bounded domain $\Omega \subset \mathbb{R}^3$, problem (1.1) reduces to the Dirichlet problem

$$\begin{gathered} -\Big(a + b\int_{\Omega} |\nabla u|^2 \mathrm{d}x\Big)\Delta u = g(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

which is related to the stationary analogue of the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 \mathrm{d}x\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.3}$$

proposed by Kirchhoff [10] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings, where ρ, P_0, h, E and L are positive

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constants. Lions [15] introduced an abstract functional analysis framework for the equation

$$u_{tt} - \Big(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x\Big)\Delta u = g(x,u). \tag{1.4}$$

Since then, problem (1.4) have received much attention.

Problem (1.1) has been studied extensively in recent years (in bounded or unbounded domain), see, for example, [1, 7, 14, 22, 23, 26, 28] and references therein. We emphasize that in all these papers, the potential V is assumed to be nonnegative. In this case, the quadratic part of the variational functional Φ given in (2.1) is positively definite, the zero function $u = 0$ is a local minimizer of Φ and the mountain pass theorem [2] can be applied. However, when the potential V is negative somewhere so that the quadratic part of Φ is indefinite, the zero function $u = 0$ is no longer a local minimizer of Φ, the mountain pass theorem is not applicable anymore. For the semilinear problem ($b = 0$)

$$-\Delta u + V(x)u = g(u)$$

with indefinite Schrödinger operator $-\Delta + V$, one usually applies the linking theorem to obtain a solution, see e.g. [11, 19]. For problem (1.1), it seems hard to verify the linking geometry because of the nonlocal term $\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x$, which prevents the functional Φ to be nonpositive on the negative space of the Schrödinger operator, see Remark 1.2 for detail. Hence, the classical linking theorem [25, Lemma 2.12] is also not applicable.

For this reason, there are very few results about (1.1) with indefinite potential. To the best of our knowledge, the first work on this situation is due to Chen and Liu [6]. To overcome the above difficulty, a crucial observation in [6] is that Φ has a local linking at $u = 0$, thus for certain $i \in \mathbb{N}$, the i-th critical group of Φ at $u = 0$ is nontrivial. Assuming that g is 3-superlinear

$$\lim_{|t|\to\infty} \frac{g(t)}{t^3} = +\infty, \tag{1.5}$$

with suitable technical conditions it was shown in [6] that all critical groups of Φ at infinity are trivial, then a nonzero critical point of Φ can be found via Morse theory. See also [13] for a recent result for 3-superlinear nonlinearity.

For other results on the indefinite problem (1.1), we mention [27] and [9], where the nonlinearity g is sublinear and subquadratic, respectively. In these two papers the variational functional is coercive. The three critical points theorem of Liu and Su [16] and the classical Clark theorem are applied to obtain multiple solutions.

Having the above results in mind, it is natural to ask what will happen when g is 2-superlinear (g satisfies (1.5) with 3 replaced by 2)? This is the motivation of the current paper.

The main difficulty under this assumption is that it is not know whether Palais-Smale (or even Cerami) sequences are bounded or not. Motivated by Liu and Mosconi [20] on the study of nonlinear Schrödinger-Poisson systems (see also [8]), we add a dummy variable and consider an augmented functional $\tilde\Phi : \mathbb{R} \times X \to \mathbb{R}$, see (2.4). It turns out that $\tilde\Phi$ satisfies the (PS) condition (see Lemma 2.6), and if $(\bar s, \bar u)$ is a critical point of $\tilde\Phi$, then $\bar u$ is a critical point of Φ (see Lemma 2.5). Moreover, $\tilde\Phi$ has a local linking at zero, and all critical groups of $\tilde\Phi$ at infinity are trivial. Eventually, a nonzero critical point of $\tilde\Phi$ can be obtained by using Morse theory, which give rise to a nonzero critical point of Φ.

Without loss of generality, we assume that $a = b = 1$. Then the problem (1.1) can be rewritten as follows

$$-\Big(1+\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\Big)\Delta u+V(x)u=g(u),\quad u\in H^1(\mathbb{R}^3). \tag{1.6}$$

Let $H^1(\mathbb{R}^3)$ be the standard Sobolev space. If $V \in C(\mathbb{R}^3)$ is bounded from below, we can choose a constant $m > 2$ such that

$$\tilde{V}(x) := V(x) + m \geq \frac{m}{2} \geq 1, \quad \text{for all} \quad x \in \mathbb{R}^3. \tag{1.7}$$

On the linear subspace

$$X = \big\{u \in H^1(\mathbb{R}^3) : \int V(x)u^2 < \infty\big\}, \tag{1.8}$$

where from now on all integrals are taken over $\mathbb{R}^3$ except stated explicitly, we equip with the inner product

$$(u, v) = \int \Big(\nabla u \cdot \nabla v + \tilde{V}(x)uv\Big)$$

and the corresponding norm $\|\cdot\|$. Note that if the assumptions (A1) below holds, then X is a Hilbert space and by Bartsch and Wang [4] we have a compact embedding $X \hookrightarrow L^s(\mathbb{R}^3)$ for $s \in [2, 6)$.

Now we present our assumptions on the potential $V(x)$ and the nonlinearity $g(u)$.

- (A1) $V \in C(\mathbb{R}^3)$ is bounded from below and $|\{V \leq k\}| < \infty$ for all $k \in \mathbb{R}$, where $|\cdot|$ is the Lebesgue measure on $\mathbb{R}^3$.
- (A2) $V \in C^1(\mathbb{R}^3)$ and there exists $R > 0$ such that $V(x) - \nabla V(x) \cdot x \geq 0$ for $|x| \geq R$.
- (A3) There exists $\kappa > 0$ such that
$$|\nabla V(x) \cdot x| \leq \kappa\,(V(x) + m) =: \kappa\tilde{V}(x)$$
for all $x \in \mathbb{R}^3$, here m is the constant from (1.7).
- (A4) $g \in C(\mathbb{R})$ and there exist $C > 0$ and $p \in (2, 6)$ such that
$$|g(t)| \leq C\left(|t| + |t|^{p-1}\right).$$
- (A5) There exists $\mu > 3$ such that $g(t)t \geq \mu G(t) > 0$ for all $t \in \mathbb{R} \setminus \{0\}$.

Consider the bilinear form

$$Q(u, v) = \frac{1}{2}\int (\nabla u \cdot \nabla v + V(x)uv) \quad u, v \in X, \tag{1.9}$$

then $X = X^+ \oplus X^- \oplus X^0$, where X^+, X^- and X^0 are positive, negative, and null eigenspaces of the Schrödinger operator. It is well known that there exist constants $\eta^\pm > 0$ such that

$$\pm\, Q(u, u) \geq \eta^\pm\|u^\pm\|^2, \quad \text{for } u \in X^\pm \oplus X^0, \text{ respectively.} \tag{1.10}$$

We are now ready to state our main result.

Theorem 1.1. *Suppose that* (A1)–(A5) *hold. If either*

1. $\dim X^- > 0$, $\dim X^0 = 0$, *or*
2. $\dim X^0 > 0$ *and* $G(t) = \int_0^t g(\tau)\mathrm{d}\tau \geq c|t|^\nu$ *for some* $\nu < 4$,

then problem (1.6) *has at least one nontrivial solution.*

Remark 1.2. As we have mentioned, because of the nonlocal term $\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x$, our functional Φ (see (2.1)) does not satisfy the geometric assumption of the linking theorem. For $R>r>0$ set

$$N=\{u\in X^+:\|u\|=r\},\quad M=\{u\in X^-\oplus X^0\oplus\phi\mathbb{R}^+:\|u\|\le R\},$$

where $\phi\in X^+\setminus\{0\}$. Then we have

$$b=\inf_N\Phi>0,\qquad \sup_{u\in\partial M,\|u\|=R}\Phi<0$$

provided R is large enough and r is small enough. However, because the integral $\left(\int|\nabla u|^2\right)^2$ in our functional, Φ may be very large, it is possible that $\Phi(u)>b$ for some $u\in\partial M\cap(X^-\oplus X^0)$. Therefore, the following geometric assumption of the linking theorem

$$\inf_N\Phi>\sup_{\partial M}\Phi$$

can not be satisfied. Thus, the linking theorem is not applicable for problem (1.6).

2. Palais-Smale condition

Now, we investigate the functional Φ. Under the assumptions (A1) and (A4) we can show that the functional $\Phi:X\to\mathbb{R}$,

$$\Phi(u)=\frac12\int\left(|\nabla u|^2+V(x)u^2\right)+\frac14\Big(\int|\nabla u|^2\Big)^2-\int G(u) \tag{2.1}$$

is well defined and of class C^1. The derivative of Φ is given by

$$\langle D\Phi(u),v\rangle=\Big(1+\int|\nabla u|^2\Big)\int\nabla u\cdot\nabla v+\int V(x)uv-\int g(u)v.$$

for $u,v\in X$. Consequently, critical points of Φ are weak solutions of problem (1.6).

Proposition 2.1 ([20, Lemma 3.1]). *Assume* (A3) *holds and let $\tilde V:=V+m$. Then for any $t>0$ and $x\in\mathbb{R}^3$, we have*

$$\tilde V(tx)\le\max\{t^\kappa,t^{-\kappa}\}\tilde V(x). \tag{2.2}$$

For $t>0$ and $u\in X$ we define

$$u_t(x)=tu(\frac{x}{t}), \tag{2.3}$$

and define on the Hilbert space $\mathbb{R}\times X$ (with natural norm $\|(s,u)\|^2=s^2+\|u\|^2$) the augmented functional

$$\tilde\Phi(s,u):=\frac{s^2}{2}+\Phi(u_{e^s}). \tag{2.4}$$

Remark 2.2. Obviously, for $s,t>0$, from (2.3) we have $(u_t)_s=u_{ts}=(u_s)_t$.

Lemma 2.3. *Suppose that* (A3) *and* (A4) *hold. Then the functional $\tilde\Phi$ is well defined on $\mathbb{R}\times X$, of class C^1, and*

$$\begin{aligned}\tilde\Phi(s,u)=\frac{s^2}{2}+\frac{e^{3s}}{2}\int|\nabla u|^2+\frac{e^{6s}}{4}\Big(\int|\nabla u|^2\Big)^2\\ +\frac{e^{5s}}{2}\int V(xe^s)u^2-e^{3s}\int G(e^su),\end{aligned} \tag{2.5}$$

$$\begin{aligned}\langle \partial_u \tilde{\Phi}(s,u), \varphi\rangle = e^{3s} \int \nabla u \nabla \varphi + e^{6s}\Big(\int |\nabla u|^2\Big) \int \nabla u \nabla \varphi \\ + e^{5s} \int V(xe^s) u\varphi - e^{4s} \int g(e^s u)\varphi,\end{aligned} \tag{2.6}$$

$$\begin{aligned}\partial_s \tilde{\Phi}(s,u) = s + \frac{3e^{3s}}{2} \int |\nabla u|^2 + \frac{3e^{6s}}{2}\Big(\int |\nabla u|^2\Big)^2 \\ + \frac{e^{5s}}{2} \int \left[5V(xe^s) + \nabla V(xe^s)\cdot xe^s\right] u^2 \\ - e^{3s} \int \left[3G(e^s u) + g(e^s u) e^s u\right].\end{aligned} \tag{2.7}$$

Proof. By changing variables, we obtain

$$\Big(\int |\nabla u_t|^2\Big)^2 = t^6 \Big(\int |\nabla u|^2\Big)^2 \le t^6 \|u\|^4.$$

Similarly,

$$\int |\nabla u_t|^2 = t^3 \int |\nabla u|^2 \le t^3 \|u\|^2,$$

$$\big|\int G(u_t)\big| \le C|t|^5 \int u^2 + C|t|^{p+3} \int |u|^p.$$

By the continuity of V, changing variables $x = yt$ on any fixed ball B_R and applying Proposition 2.1, we obtain

$$\int_{B_R} \tilde{V}(x) u_t^2 = t^5 \int_{B_{R/t}} \tilde{V}(tx) u^2 \le t^5 \max\{t^\kappa, t^{-\kappa}\} \int_{B_{R/t}} \tilde{V}(x) u^2 \le c_t \|u\|^2.$$

Letting $R \to +\infty$ we deduce that

$$\int \tilde{V}(x) u_t^2 \le c_t \|u\|^2.$$

From the above estimates and continuous embeddings,

$$\begin{aligned}\Phi(u_t) &= \frac{1}{2} \int \left(|\nabla u_t|^2 + V(x) u_t^2\right) + \frac{1}{4}\Big(\int |\nabla u_t|^2\Big)^2 - \int G(u_t) \\ &\le \frac{t^3}{2}\|u\|^2 + \frac{c_t}{2}\|u\|^2 + \frac{t^6}{4}\|u\|^4 + C|t|^5\|u\|^2 + C|t|^{p+3}\|u\|^p.\end{aligned}$$

Together with (2.4), it is easy to see that $\tilde{\Phi}$ is well defined. Formula (2.5) directly follows from changing variables.

Formula (2.6) can be computed in a standard way, while (2.7) is obtained by differentiating under the integral sign in (2.5). By (A4), we deduce that

$$|3G(e^s u) + g(e^s u) e^s u| \le C(e^{2s}|u|^2 + e^{ps}|u|^6). \tag{2.8}$$

It follows from $|V| \le \tilde{V} + m$, (A3) and Proposition 2.1 that

$$\begin{aligned}|5V(xe^s) + \nabla V(xe^s)\cdot xe^s| &\le 5|V(xe^s)| + |\nabla V(xe^s)\cdot xe^s| \\ &\le 5m + (5+\kappa)\tilde{V}(xe^s) \\ &\le 5m + (5+\kappa) e^{\kappa|s|}\tilde{V}(x).\end{aligned} \tag{2.9}$$

Noting that, by the dominated convergence theorem, both

$$s \mapsto \int |5V(xe^s) + \nabla V(xe^s)\cdot xe^s|, \quad s \mapsto \int |3G(e^s u) + g(e^s u) e^s u|$$

are continuous, standard arguments yields the differentiation formula (2.7). Finally, estimates (2.8) and (2.9) ensure the continuity of the corresponding Nemitskii operators appearing in (2.6) and (2.7), so that $\tilde{\Phi}$ is of class C^1. $\square$

Lemma 2.4 (Pohožaev identity). *Suppose that* (A3) *and* (A4) *hold. Let* $u \in X$ *be a weak solution to problem* (1.6)*, then we have the following Pohožaev identity,*

$$\frac{1}{2}\int |\nabla u|^2 + \frac{3}{2}\int V(x)u^2 + \frac{1}{2}\int (\nabla V(x), x)u^2 + \frac{1}{2}\Big(\int |\nabla u|^2\Big)^2 - 3\int G(u) = 0. \tag{2.10}$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1}\Phi(u_t) = 0.$$

Proof. The proof of Pohožaev identity is standard, so we omit it and refer the reader to [12, Lemma 2.1]. Now, it suffices to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1}\Phi(u_t) = 0.$$

By changing variables and (2.1) , we have

$$\Phi(u_t) = \frac{t^3}{2}\int |\nabla u|^2 + \frac{t^5}{2}\int V(xt)u^2 + \frac{t^6}{4}\left(\int |\nabla u|^2\right)^2 - t^3\int G(tu). \tag{2.11}$$

Then, by (2.10), (2.11), and $D\Phi(u) = 0$, we obtain

$$\begin{aligned}
\frac{\mathrm{d}\Phi(u_t)}{\mathrm{d}t}\Big|_{t=1} &= \frac{3}{2}\int |\nabla u|^2 + \frac{5}{2}\int V(x)u^2 + \frac{1}{2}\int (\nabla V(x), x)u^2 \\
&\quad + \frac{3}{2}\Big(\int |\nabla u|^2\Big)^2 - \int [3G(u) + g(u)u] \\
&= \langle D\Phi(u), u\rangle + \frac{1}{2}\int |\nabla u|^2 + \frac{3}{2}\int V(x)u^2 \\
&\quad + \frac{1}{2}\int (\nabla V(x), x)u^2 + \frac{1}{2}\Big(\int |\nabla u|^2\Big)^2 - 3\int G(u) = 0.
\end{aligned}$$

$\square$

In the following, we denote by $\tilde{D}\tilde{\Phi}$ the total differential of $\tilde{\Phi}$ with respect to both variables s and u.

Lemma 2.5. *Suppose that* (A3) *and* (A4) *hold, then*

$$\tilde{D}\tilde{\Phi}(\bar{s}, \bar{u}) = 0 \quad \Leftrightarrow \quad \bar{s} = 0 \text{ and } D\Phi(\bar{u}) = 0.$$

Proof. The proof is similar to the proof of [20, Lemma 3.5], we include it here for the reader's convenience.

($\Leftarrow$) From the definition of $\tilde{D}\tilde{\Phi}$, we have

$$\tilde{D}\tilde{\Phi}(\bar{s}, \bar{u})[l, \varphi] = \langle \partial_u\tilde{\Phi}(\bar{s}, \bar{u}), \varphi\rangle + \partial_s\tilde{\Phi}(\bar{s}, \bar{u})l, \quad \text{for all } [l, \varphi] \in \mathbb{R}\times X. \tag{2.12}$$

Note that $D\Phi(\bar{u}) = 0$ implies

$$\langle \partial_u\tilde{\Phi}(0, \bar{u}), \varphi\rangle = \langle D\Phi(\bar{u}), \varphi\rangle = 0.$$

Therefore, it suffices to prove that $\partial_s\tilde{\Phi}(0, \bar{u}) = 0$. From $D\Phi(\bar{u}) = 0$, Lemma 2.4 gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1}\Phi(\bar{u}_t) = 0. \tag{2.13}$$

The map $t \mapsto \Phi(\bar{u}_t)$ is C^1 by Lemma 2.3 and

$$\Phi(\bar{u}_t) = \tilde{\Phi}(\log t, \bar{u}) - \frac{\log^2 t}{2}.$$

Now (2.13) gives

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1}\Big(\tilde{\Phi}(\log t, \bar{u}) - \frac{\log^2 t}{2}\Big) = \Big(\partial_s \tilde{\Phi}(\log t, \bar{u})\frac{1}{t} - \frac{\log t}{t}\Big)\Big|_{t=1} = \partial_s \tilde{\Phi}(0, \bar{u}).$$

($\Rightarrow$) If $\tilde{D}\tilde{\Phi}(\bar{s}, \bar{u}) = 0$, combining (2.4), we immediately infer that

$$0 = \partial_u \tilde{\Phi}(\bar{s}, \bar{u}) = \partial_u \Phi(\bar{u}_{e^{\bar{s}}}) = D\Phi(\bar{u}_{e^{\bar{s}}}).$$

Thus, we only have to prove that $\bar{s} = 0$. Applying Lemma 2.4 to $v := \bar{u}_{e^{\bar{s}}}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1}\Phi(v_t) = 0.$$

The map $t \mapsto \Phi(v_t)$ is C^1 by Lemma 2.3 and

$$\Phi(v_t) = \Phi(\bar{u}_{te^{\bar{s}}}) = \tilde{\Phi}(\bar{s} + \log t, \bar{u}) - \frac{(\bar{s} + \log t)^2}{2}.$$

By the chain rule and $\partial_s \tilde{\Phi}(\bar{s}, \bar{u}) = 0$ we have

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1}\Phi(v_t) \\ &= \Big(\partial_s \tilde{\Phi}(\bar{s} + \log t, \bar{u})\frac{1}{t} - \frac{(\bar{s} + \log t)}{t}\Big)\Big|_{t=1} \\ &= \partial_s \tilde{\Phi}(\bar{s}, \bar{u}) - \bar{s} = -\bar{s}. \end{aligned}$$

□

Lemma 2.6. *Suppose that* (A1)–(A5) *hold. Then* $\tilde{\Phi}$ *satisfies the* (PS) *condition.*

Proof. Let $\{(s_n, u_n)\}$ be a (PS) sequence for $\tilde{\Phi}$ in $\mathbb{R} \times X$, that is

$$\sup_n |\tilde{\Phi}(s_n, u_n)| < \infty, \quad \tilde{D}\tilde{\Phi}(s_n, u_n) \to 0.$$

Choose $\lambda \in (3, \mu)$, where $\mu > 3$ is given by (A5). Then

$$\begin{aligned} &(\lambda + 3)\tilde{\Phi}(s, u) - \partial_s \tilde{\Phi}(s, u) \\ &= \frac{\lambda + 3}{2}s^2 - s + \frac{\lambda}{2}e^{3s}\int |\nabla u|^2 + \frac{\lambda - 3}{4}e^{6s}\Big(\int |\nabla u|^2\Big)^2 \\ &\quad + \frac{e^{5s}}{2}\int [(\lambda - 2)V(xe^s) - \nabla V(xe^s) \cdot xe^s]\, u^2 + e^{3s}\int [g(e^s u)e^s u - \lambda G(e^s u)] \\ &\geq \frac{\lambda + 3}{2}s^2 - s + \frac{\lambda}{2}e^{3s}\int |\nabla u|^2 + \frac{\lambda - 3}{4}e^{6s}\Big(\int |\nabla u|^2\Big)^2 \\ &\quad + \frac{e^{5s}}{2}\int [(\lambda - 2)V(xe^s) - \nabla V(xe^s) \cdot xe^s]\, u^2 + (\mu - \lambda)e^{3s}\int G(e^s u). \end{aligned} \tag{2.14}$$

The third integral is bounded from below through (A1)–(A3), and Hölder's inequality. Indeed, we set

$$v(x) = e^s u\left(\frac{x}{e^s}\right), \quad W_\lambda(x) = (\lambda - 2)V(x) - \nabla V(x) \cdot x.$$

Then, by changing variables,

$$e^{5s}\int [(\lambda - 2)V(xe^s) - \nabla V(xe^s) \cdot xe^s]\, u^2 = \int W_\lambda v^2. \tag{2.15}$$

As W_λ is bounded on bounded sets, we let $C_\lambda \in \mathbb{R}$ be such that $W_\lambda \geq -C_\lambda$ in B_R, where R is given in (A2), then

$$\int_{B_R} W_\lambda v^2 \geq -C_\lambda \int_{B_R} v^2 \geq -C_\lambda \Big(\int |v|^\mu \Big)^{2/\mu}. \tag{2.16}$$

We write the integral over $\mathbb{R}^3 \setminus B_R$ as the sum of the integrals over the intersections of $\mathbb{R}^3 \setminus B_R$ with $\{V \geq 0\}$ and $\{V < 0\}$. Obviously, assumption (A1) implies that $\{V < 0\}$ has finite measure. Because $\lambda - 2 > 1$, assumption (A2) implies that

$$W_\lambda(x) = (\lambda - 2)V(x) - \nabla V(x) \cdot x \geq V(x) - \nabla V(x) \cdot x \geq 0$$

for $x \in \{V \geq 0\} \setminus B_R$.

On the other hand, by (A3) and $V \geq -m$ we have

$$W_\lambda(x) \geq (\lambda - 2)V - \kappa(V + m) \geq -\left((\lambda - 2) + \kappa\right) m \quad \text{on } \{V < 0\}.$$

For some possibly larger C_λ, we have

$$\begin{aligned} \int_{\mathbb{R}^3 \setminus B_R} W_\lambda v^2 &\geq \int_{\{V<0\} \setminus B_R} W_\lambda v^2 \geq -C_\lambda \int_{\{V<0\}} v^2 \\ &\geq -C_\lambda |\{V < 0\}|^{(\mu-2)/\mu} \Big(\int |v|^\mu \Big)^{2/\mu}. \end{aligned} \tag{2.17}$$

Combining (2.15), (2.16), and (2.17), we obtain

$$e^{5s} \int \left[(\lambda - 2)V(xe^s) - \nabla V(xe^s) \cdot xe^s\right] u^2 \geq -C_\lambda \Big(\int |v|^\mu \Big)^{2/\mu}. \tag{2.18}$$

Condition (A5) implies that for some $C > 0$, $G(t) \geq C|t|^\mu$. We deduce

$$(\mu - \lambda)e^{3s} \int G(e^s u) = (\mu - \lambda) \int G(v) \geq (\mu - \lambda)C \int |v|^\mu. \tag{2.19}$$

From (2.14), (2.18), and (2.19), for our (PS) sequence $\{(s_n, u_n)\}$, we have

$$\begin{aligned} O(1) &\geq (\lambda + 3)\tilde{\Phi}(s_n, u_n) - \partial_s \tilde{\Phi}(s_n, u_n) \\ &\geq \frac{\lambda + 3}{2} s_n^2 - s_n + \frac{\lambda}{2} e^{3s_n} \int |\nabla u_n|^2 + \frac{\lambda - 3}{4} e^{6s_n} \Big(\int |\nabla u_n|^2 \Big)^2 \\ &\quad + (\mu - \lambda)e^{3s_n} \int G(e^{s_n} u_n) + \frac{e^{5s_n}}{2} \int \left[(\lambda - 2)V(xe^{s_n}) - \nabla V(xe^{s_n}) \cdot xe^{s_n}\right] u_n^2 \\ &\geq \frac{\lambda + 3}{2} s_n^2 - s_n + \frac{\lambda}{2} e^{3s_n} \int |\nabla u_n|^2 + \frac{\lambda - 3}{4} e^{6s_n} \Big(\int |\nabla u_n|^2 \Big)^2 \\ &\quad + (\mu - \lambda)C \int |v_n|^\mu - C_\lambda \Big(\int |v_n|^\mu \Big)^{2/\mu}, \end{aligned}$$

where $v_n(x) = e^s u_n\left(\frac{x}{e^s}\right)$. From $\mu > \lambda$, $2/\mu < 1$ we infer $(\mu - \lambda)C\xi_n - C_\lambda {\xi_n}^{2/\mu} \to +\infty$ if $\xi_n = |v_n|_\mu^2 \to +\infty$, we deduce from the previous estimate that

$$|s_n|, \quad \int |\nabla u_n|^2, \quad \int |v_n|^\mu, \quad \int G(e^{s_n} u_n) \quad \text{are bounded;} \tag{2.20}$$

recalling (2.5) and $\tilde{\Phi}(s_n, u_n) = O(1)$ we obtain

$$\int V(xe^{s_n}) u_n^2 \leq O(1). \tag{2.21}$$

To complete the proof of the boundedness of $\|u_n\|$, let $S>0$ be such that $|s_n|\le S$. Since $V\ge -m$ on $\{V\le k\}$, we deduce

$$\begin{aligned}\int V(xe^{s_n})u_n^2 &= e^{-2s_n}\int V(xe^{s_n})v_n^2(xe^{s_n}) = e^{-5s_n}\int V(x)v_n^2\\ &\ge e^{-5S}\Big(\int_{\{V>k\}}V(x)v_n^2 - m\int_{\{V\le k\}}v_n^2\Big)\\ &\ge e^{-5S}\Big(\int_{\{V>k\}}V(x)v_n^2 - m|\{V\le k\}|^{(\mu-2)/\mu}\Big(\int |v_n|^\mu\Big)^{2/\mu}\Big)\\ &\ge e^{-5S}\int_{\{V>k\}}V(x)v_n^2 - O(1),\end{aligned}$$

where we used (A1) and (2.20) in the last inequality. From (2.21) we thus infer

$$\int_{\{V>k\}}V(x)v_n^2\le O(1).$$

Next, $k>m$ implies $V+m\le 2V$ on $\{V>k\}$ and $V+m\le 2k$ on $\{V\le k\}$. Using Proposition 2.1 we obtain

$$\begin{aligned}\|u_n\|^2 &= \int\Big(|\nabla u_n|^2+\tilde V(x)u_n^2\Big)\\ &= \int|\nabla u_n|^2 + e^{-5s_n}\int \tilde V(xe^{-s_n})v_n^2\\ &\le \int|\nabla u_n|^2 + e^{(|\kappa|-5)S}\int \tilde V(x)v_n^2\\ &\le \int|\nabla u_n|^2 + 2e^{(|\kappa|-5)S}\Big(\int_{\{V>k\}}V(x)v_n^2 + k\int_{\{V\le k\}}v_n^2\Big)\\ &\le \int|\nabla u_n|^2 + 2e^{(|\kappa|-5)S}\Big(\int_{\{V>k\}}V(x)v_n^2\\ &\quad + k|\{V\le k\}|^{(\mu-2)/\mu}\Big(\int|v_n|^\mu\Big)^{2/\mu}\Big)\le O(1)\end{aligned}$$

by (2.20), proving the boundedness of $\{u_n\}$ in X. Using the compact embedding $X\hookrightarrow L^s$ for $s\in[2,6)$, by standard argument as in [9, Lemma 3.2], we see that $\{u_n\}$ has a convergent subsequence. Hence $\{(s_n,u_n)\}$ has a convergent subsequence. □

3. Critical groups and proof of Theorem 1.1

Having established the (PS) condition for $\tilde\Phi$, we are now ready to present the proof of Theorem 1.1. We start by recalling some concepts and results from infinite-dimensional Morse theory (see e.g., Chang [5] and Mawhin and Willem [21, Chapter 8]).

Let X be a Banach space, $\varphi: X\to\mathbb{R}$ be a C^1 functional, u be an isolated critical point of φ and $\varphi(u)=c$. Then

$$C_i(\varphi,u) := H_i(\varphi_c,\varphi_c\setminus\{0\}),\quad i\in\mathbb{N}=\{0,1,2,\dots\},$$

is called the i-th critical group of φ at u, where $\varphi_c:=\varphi^{-1}(-\infty,c]$ and H_* stands for the singular homology with coefficients in $\mathbb{Z}$.

If φ satisfies the (PS) condition and the critical values of φ are bounded from below by α, then following Bartsch and Li [3], we define the i-th critical group of φ at infinity by

$$C_i(\varphi,\infty) := H_i(X,\varphi_\alpha), \quad i \in \mathbb{N}.$$

By the deformation lemma, it is well known that the homology on the right-hand side does not depend on the choice of α.

Proposition 3.1 ([3, Proposition 3.6]). *If $\varphi \in C^1(X,\mathbb{R})$ satisfies the (PS) condition and $C_\ell(\varphi,0) \neq C_\ell(\varphi,\infty)$ for some $\ell \in \mathbb{N}$, then φ has a nonzero critical point.*

Proposition 3.2 ([17, Theorem 2.1]). *Suppose $\varphi \in C^1(X,\mathbb{R})$ satisfies the (PS) condition and has a local linking at 0 with respect to the decomposition $X = Y \oplus Z$, i.e., for some $\varepsilon > 0$,*

$$\varphi(u) \le 0 \quad \text{for } u \in Y \cap B_\varepsilon,$$
$$\varphi(u) > 0 \quad \text{for } u \in (Z \setminus \{0\}) \cap B_\varepsilon,$$

where $B_\varepsilon = \{u \in X : \|u\| \le \varepsilon\}$. If $\ell = \dim Y < \infty$, then $C_\ell(\varphi,0) \neq 0$

Since $\tilde\Phi$ satisfies the (PS) condition, the critical group $C_*(\tilde\Phi,\infty)$ of $\tilde\Phi$ at infinity makes sense. To study $C_*(\tilde\Phi,\infty)$ we need the following lemma.

Lemma 3.3. *Suppose that* (A1)–(A5) *hold. Then there exists a constant $M_\lambda \ge 0$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde\Phi(\tau,u_t) \le \frac{\lambda+3}{t}\Big(\tilde\Phi(\tau,u_t) + M_\lambda\Big) \tag{3.1}$$

for all $t > 0$, $u \in X$ and $\tau \in \mathbb{R}$.

Proof. Let $v = u_{e^\tau}$. By (2.4) and Remark 2.2 we observe that

$$\tilde\Phi(\tau,u_t) = \frac{\tau^2}{2} + \Phi(u_{te^\tau}) = \frac{\tau^2}{2} - \frac{\log^2 t}{2} + \tilde\Phi(\log t, v) \ge \tilde\Phi(\log t, v) - \frac{\log^2 t}{2} \tag{3.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde\Phi(\tau,u_t) = -\frac{\log t}{t} + \frac{1}{t}\partial_s\tilde\Phi(\log t, v). \tag{3.3}$$

We claim that for given $\lambda \in (3,\mu)$, there exists $M_\lambda > 0$ such that

$$\partial_s\tilde\Phi(s,v) - s \le (\lambda+3)\Big(\tilde\Phi(s,v) - \frac{s^2}{2} + M_\lambda\Big) \tag{3.4}$$

for all $s \in \mathbb{R}$ and $v \in X$. Indeed, by (2.14), (2.18), and (2.19) we have

$$(\lambda+3)\tilde\Phi(s,v) - \partial_s\tilde\Phi(s,v) \ge \frac{\lambda+3}{2}s^2 - s + (\mu-\lambda)C\int |v|^\mu - C_\lambda\Big(\int |v|^\mu\Big)^{2/\mu}.$$

The last two terms are bounded from below because $\mu > \lambda$ and $2/\mu < 1$, thus (3.4) holds. Then, using (3.2), (3.3), and (3.4), with $s = \log t$ and $v = u_{e^\tau}$, we deduce that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde\Phi(\tau,u_t) &= -\frac{\log t}{t} + \frac{1}{t}\partial_s\tilde\Phi(\log t, v)\\
&\le \frac{\lambda+3}{t}\Big(\tilde\Phi(\log t, u_{e^\tau}) - \frac{\log^2 t}{2} + M_\lambda\Big) = \frac{\lambda+3}{t}\left(\Phi(u_{te^\tau} + M_\lambda)\right)\\
&\le \frac{\lambda+3}{t}\Big(\tilde\Phi((\tau,u_t) + M_\lambda)\Big).
\end{aligned}$$

□

Lemma 3.4. *Suppose* (A2), (A4), (A5) *hold. Then*

$$\lim_{t\to+\infty} \tilde{\Phi}(s,u_t) = -\infty$$

for all $(s,u) \in \mathbb{R}\times X\setminus\{0\}$.

Proof. Because $u_{e^s} \neq 0$, and $(u_t)_{e^s} = (u_{e^s})_t$, it suffices to prove that if $v \neq 0$ then $\Phi(v_t) \to -\infty$ as $t \to +\infty$, see (2.4). From the proof of (2.5) we obtain

$$\Phi(v_t) = \frac{t^3}{2}\int |\nabla v|^2 + \frac{1}{2}\int V(x)t^2v^2\left(\frac{x}{t}\right) + \frac{t^6}{4}\Big(\int |\nabla v|^2\Big)^2 - t^3\int G(tv)$$

Assumption (A5) implies

$$G(t) \le C|t|^\mu \quad \text{for } |t|<1, \quad G(t)\ge C|t|^\mu \quad \text{for } |t|\ge 1. \tag{3.5}$$

Since $v \neq 0$, we can assume that for some $\varepsilon > 0$, $|\{|v|\ge\varepsilon\}|$ is finite and positive and by (3.5) we have

$$\int G(tv) \ge C\int_{|v|\ge\varepsilon} t^\mu|v|^\mu \ge C\varepsilon^\mu|\{|v|\ge\varepsilon\}|t^\mu =: C_\nu t^\mu,$$

for some $C_\nu > 0$ and $t \ge 1/\varepsilon$.

As V is bounded on B_R, it follows that

$$\int_{B_R} V(x)t^2v^2\left(\frac{x}{t}\right) \le \|V\|_{L^\infty(B_R)}\int t^2v^2\left(\frac{x}{t}\right) = t^5\|V\|_{L^\infty(B_R)}\int v^2.$$

Assumption (A2) implies that for $|\omega| = R$ and $r \ge 1$,

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{\tilde{V}(r\omega)}{r}\Big) = \frac{\nabla V(r\omega)\cdot r\omega - \tilde{V}(r\omega)}{r^2} \le 0$$

where $\tilde{V}(x) = V(x)+m$, so that

$$H(x) = \frac{\tilde{V}(x)}{|x|}\chi_{\mathbb{R}^3\setminus B_R}(x) \quad \text{is radially non-increasing.}$$

Letting $w(x) = v(x)|x|^{1/2}$, we have

$$\int_{\mathbb{R}^3\setminus B_R} \tilde{V}t^2v^2\left(\frac{x}{t}\right) = |t|^3\int H(x)w^2\left(\frac{x}{t}\right) = |t|^6\int H(xt)w^2,$$

and by the monotonicity of H, $H(xt)w^2 \searrow 0$ as $t \to +\infty$. Therefore, by the monotone convergence theorem,

$$\begin{aligned}\int V(x)t^2v^2\left(\frac{x}{t}\right) &\le \int_{B_R} V(x)t^2v^2\left(\frac{x}{t}\right) + \int_{\mathbb{R}^3\setminus B_R}\tilde{V}t^2v^2\left(\frac{x}{t}\right)\\ &\le t^5\|V\|_{L^\infty(B_R)}\int v^2 + o(t^6).\end{aligned}$$

In summary,

$$\Phi(v_t) \le \frac{t^3}{2}\int|\nabla v|^2 + \frac{t^6}{4}\Big(\int|\nabla v|^2\Big)^2 + \frac{t^5\|V\|_{L^\infty(B_R)}}{2}\int v^2 + o(t^6) - C_\nu t^{\mu+3} \to -\infty,$$

as $t \to +\infty$, because $\mu + 3 > 6$. $\square$

Lemma 3.5. *Suppose* (A1)–(A5) *hold. Then for any sufficiently negative* $a \in \mathbb{R}$,

$$C_i(\tilde{\Phi},\infty) = 0 \quad \textit{for all } i \in \mathbb{N}.$$

Proof. Let $\dot{X} = X \setminus \{0\}$ and consider the continuous map $\mathbb{R}\times\dot{X}\times\mathbb{R}^+ \to \mathbb{R}\times\dot{X}$ given by

$$(s,u,t) \mapsto (s,u_t).$$

Choose $\lambda \in (3,\mu)$ and $a < -M_\lambda$, where $M_\lambda \ge 0$ is given in Lemma 3.3. Then by Lemma 3.4, we have

$$\lim_{t\to+\infty} \tilde{\Phi}(s,u_t) = -\infty, \quad \forall (s,u)\in\mathbb{R}\times\dot{X}.$$

So there is $t_{s,u} > 0$ such that $\tilde{\Phi}(s,u_{t_{s,u}}) = a$. A direct computation and Lemma 3.3 gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_{s,u}} \tilde{\Phi}(s,u_t) \le \frac{\lambda+3}{t_{s,u}}\Big(\tilde{\Phi}(s,u_{t_{s,u}}) + M_\lambda\Big) = \frac{\lambda+3}{t_{s,u}}(a+M_\lambda) < 0.$$

By the implicit function theorem, given $(s,u)\in\mathbb{R}\times\dot{X}$, there is a unique solution $t = T(s,u)$ for the equation $\tilde{\Phi}(s,u_t) = a$, and the map

$$T : \{(s,u)\in\mathbb{R}\times\dot{X} : \tilde{\Phi}(s,u) > a\} \to \mathbb{R}^+$$

is continuous. Using the continuous function T, it is standard (see [24, 18]) to construct a deformation from $\mathbb{R}\times\dot{X}$ to the level set $\tilde{\Phi}_a = \tilde{\Phi}^{-1}(-\infty,a]$, and deduce

$$C_i(\tilde{\Phi},\infty) = H_i(\mathbb{R}\times X, \tilde{\Phi}_a) \cong H_i(\mathbb{R}\times X, \mathbb{R}\times\dot{X}) \cong H_i(\mathbb{R}\times X, \{0\}\times S^\infty) = 0,$$

for all $i\in\mathbb{N}$, where $\mathbb{R}\times\dot{X}$ deformation retracts to $\{0\}\times S^\infty$. $\square$

Recalling that X^+, X^- and X^0 are the negative, positive and null eigenspaces of the bilinear form defined in (1.9). To study the critical group $C_*(\tilde{\Phi},0)$ of $\tilde{\Phi}$ at origin, we consider the decomposition

$$\mathbb{R}\times X = \tilde{X}^- \oplus \tilde{X}^+$$

where $\tilde{X}^- = X^- \oplus X^0$, $\tilde{X}^+ = \mathbb{R}\oplus X^+$.

Lemma 3.6. *Assume* (A1)–(A5) *hold. If either*

(1) $\dim X_- > 0$, $\dim X_0 = 0$,
(2) $\dim X_0 > 0$ *and* $G(t) \ge c|t|^\nu$ *for some* $\nu < 4$,

then the functional $\tilde{\Phi}$ *has a local linking at* 0 *with respect to the decomposition* $\mathbb{R}\times X = \tilde{X}^- \oplus \tilde{X}^+$.

Proof. It suffices to show that for $r > 0$ small enough,

$$\tilde{\Phi}(s,u) > 0 \quad \text{for } (s,u)\in\tilde{X}^+,\ 0 < \|(s,u)\| < r, \tag{3.6}$$
$$\tilde{\Phi}(s,u) \le 0 \quad \text{for } (s,u)\in\tilde{X}^-,\ \|(s,u)\| < r. \tag{3.7}$$

We have

$$\begin{aligned}\int V(xe^s)u^2 &= \int V(x)u^2 + \int\Big(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}V(e^{s\tau}x)\mathrm{d}\tau\Big)u^2\\ &= \int V(x)u^2 + s\int\Big(\int_0^1 \nabla V(e^{s\tau}x)\cdot(e^{s\tau}x)\mathrm{d}\tau\Big)u^2.\end{aligned}$$

Condition (A3) and (2.2) yield

$$
\begin{aligned}
|\int V(e^s x)u^2 - \int V(x)u^2| &\le |s| \int \Big(\int_0^1 |\nabla V(e^{s\tau}x)\cdot(e^{s\tau}x)|\mathrm{d}\tau\Big) u^2 \\
&\le \kappa|s| \int \Big(\int_0^1 \tilde{V}(e^{s\tau}x)\mathrm{d}\tau\Big) u^2 \\
&\le \kappa|s| \int \Big(\int_0^1 e^{\kappa|s|\tau}\tilde{V}(x)\mathrm{d}\tau\Big) u^2 \\
&\le (e^{\kappa|s|}-1)\int \tilde{V}(x)u^2 \\
&\le O(s)\|u\|^2 = o(\|(s,u)\|^2).
\end{aligned}
\tag{3.8}
$$

Moreover, (A4) and (A5) imply that

$$
|G(t)| \le C(|t|^\mu + |t|^{p+1}).
$$

Consequently, for $\|(s,u)\|\to 0$, we have

$$
e^{3s}\Big|\int G(e^s u)\Big| \le Ce^{3s}\int \big(|e^s u|^\mu + |e^s u|^{p+1}\big) \le o(\|u\|^2), \tag{3.9}
$$

while by (1.7) we easily have $|V(x)|\le 3\tilde{V}(x)$, hence

$$
|s|\int |V(x)|u^2 \le 3|s|\int \tilde{V}(x)u^2 = o(\|(s,u)\|^2). \tag{3.10}
$$

Combining (3.8)-(3.10), we obtain

$$
\begin{aligned}
\tilde{\Phi}(s,u) &= \frac{s^2}{2} + \frac{e^{3s}}{2}\int |\nabla u|^2 + \frac{e^{6s}}{4}\Big(\int |\nabla u|^2\Big)^2 + \frac{e^{5s}}{2}\int V(xe^s)u^2 - e^{3s}\int G(e^s u) \\
&= \frac{s^2}{2} + \frac{1+O(s)}{2}\int |\nabla u|^2 + \frac{1+O(s)}{2}\int V u^2 \\
&\quad + O(\|u\|^4) + o(\|(s,u)\|^2) + o(\|u\|^2) \\
&= \frac{1}{2}\int \big(|\nabla u|^2 + V(x)u^2\big) + \frac{s^2}{2} + O(\|u\|^4) + o(\|(s,u)\|^2).
\end{aligned}
$$

Then, it is easy to see that (3.6) is true.

Now we prove (3.7) for $r>0$ small. In item (1) we have $s=0$, it is easy to see that

$$
\tilde{\Phi}(0,u) < 0 \quad \text{for } u\in X^-, \quad \|u\|<r.
$$

In item (2), by simple computations,

$$
\begin{aligned}
\tilde{\Phi}(0,u) &= \frac{1}{2}\int \big(|\nabla u|^2 + V(x)u^2\big) + \frac{1}{4}\Big(\int |\nabla u|^2\Big)^2 - \int G(u) \\
&\le \frac{1}{2}\int \big(|\nabla u| + V(x)u^2\big) + O(\|u\|^4) - c\int |u|^\nu.
\end{aligned}
$$

Since all norms on $X^-\oplus X^0$ are equivalent we deduce that

$$
\tilde{\Phi}(0,u) \le -\eta^-\|u^-\|^2 + O(\|u\|^4) - c\|u\|^\nu \quad \text{for } u\in X^-\oplus X^0.
$$

Because $\nu<4$, we can choose $r>0$ small enough such that

$$
\tilde{\Phi}(0,u) < 0 \quad \text{for } u\in X^-\oplus X^0,\ \|u\|<r.
$$

Hence, (3.7) has also been verified item (2). $\square$

Proof of Theorem 1.1. We have shown that $\tilde{\Phi}$ satisfies (PS) condition and has a local linking at 0 with respect to the decomposition $\mathbb{R} \times X = \tilde{X}^- \oplus \tilde{X}^+$. By Proposition 3.2 we have

$$C_i(\tilde{\Phi}, 0) \neq 0,$$

where $i = \dim \tilde{X}^-$. By Lemma 3.5, $C_i(\tilde{\Phi}, \infty)$=0, therefore

$$C_i(\tilde{\Phi}, 0) \neq C_i(\tilde{\Phi}, \infty).$$

Applying Proposition 3.1, we know that $\tilde{\Phi}$ has a nonzero critical point $(\bar{s}, \bar{u})$. Furthermore, Lemma 2.5 implies that $\bar{s} = 0$, $\bar{u} \neq 0$ and $D\Phi(\bar{u}) = 0$. Thus Φ has a nonzero critical point $\bar{u}$, which is a nontrivial solution of the problem (1.6). □

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