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EXISTENCE OF SOLUTIONS TO STEADY NAVIER-STOKES EQUATIONS VIA A MINIMAX APPROACH

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Abstract. Our objective in this paper is to develop and utilize a minimax principle for proving the existence of symmetric solutions for the stationary Navier-Stokes equations. Notwithstanding its application to symmetric solutions in this paper, our minimax principle is broad enough to capture other types of solutions provided the equation and the external force are compatible under a family of operations including but not limited to being invariant by compact groups. The subset of functions compatible under this family of operations is not required to be a linear subspace, and being a closed convex set suffices for our purpose.

1. Introduction

We are concerned with the following stationary Navier-Stokes equation with homogeneous boundary condition

$$\begin{gathered}(u\cdot\nabla)u+f(x)=\Delta u-\nabla p_u \quad \forall x\in\Omega,\\ \nabla\cdot u=0 \quad \forall x\in\Omega,\\ u=0 \quad \forall x\in\partial\Omega,\end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^m$ $(m=2,3,4)$, u is the vector-valued velocity function, p_u is the scalar-valued pressure function associated with the velocity u, and $f\in L^2(\Omega)$ is the external force function. We herein develop a minimax machinery to prove the existence of solutions to the above problem with specific properties based on the provided initial data Ω and the external force f. We then apply this machinery to several cases including the stationary Navier-Stokes equations under certain symmetric conditions.

To be precise, for $\Omega\subset\mathbb{R}^m$ $(m=2,3,4)$, set $V=\{u\in H_0^1(\Omega):\nabla\cdot u=0\}$, and define $B:V\times V\to\mathbb{R}^m$ as follows:

$$B(u,v)=(u\cdot\nabla)v=\sum_{j,k=1}^{m}u_k\frac{\partial v_j}{\partial x_k}\mathbf{e}_j,$$

where $\mathbf{e}_j$ is the unit vector along the jth axis. We set $B(u,u)=\Lambda u$. The following theorem is the main abstract result of this paper.

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Theorem 1.1. *Let K be a closed convex subset of V, and assume one of the following two conditions hold:*

(i) *For each $u \in K$, there exists $v \in K$ such that*
$$\Lambda u + f(x) = \Delta v - \nabla p_v \quad \forall x \in \Omega,$$
in a weak sense, that is
$$\int_\Omega \Lambda u \cdot \eta \, dx + \int_\Omega f(x) \cdot \eta \, dx = -\int_\Omega \nabla v \cdot \nabla \eta \, dx \quad \forall \eta \in V.$$

(ii) *For each $u \in K$, there exists $v \in K$ such that*
$$B(u, v) + f(x) = \Delta v - \nabla p_v \quad \forall x \in \Omega,$$
in a weak sense;

then there exists $\bar{u} \in K$ such that
$$\begin{gathered}\Lambda \bar{u} + f(x) = \Delta \bar{u} - \nabla p_{\bar{u}} \quad \forall x \in \Omega, \\ \nabla \cdot \bar{u} = 0 \quad \forall x \in \Omega, \\ \bar{u} = 0 \quad \forall x \in \partial\Omega.\end{gathered}$$

It is worthwhile emphasizing here that the primary consequence of this theorem centers on the choice of K, i.e., by choosing an appropriate K, one is able to establish the existence of a solution enjoying all the properties induced by the set K. For instance, in the case of the 3D stationary Navier-Stokes equations (1.1), choose K to be a subset of V containing all $u = (u_1, u_2, u_3) \in V$ with the following properties:
$$\begin{gathered}u_1(x_1, x_2, x_3) = -u_1(-x_1, x_2, x_3), \\ u_2(x_1, x_2, x_3) = u_2(-x_1, x_2, x_3), \\ u_3(x_1, x_2, x_3) = u_3(-x_1, x_2, x_3).\end{gathered}$$
Correspondingly, let us define the maps $\pi_1, \pi_2, \pi_3 : \Omega \to \Omega$ as follows
$$\begin{gathered}\pi_1(x_1, x_2, x_3) = (-x_1, x_2, x_3), \\ \pi_2(x_1, x_2, x_3) = (x_1, -x_2, x_3), \\ \pi_3(x_1, x_2, x_3) = (x_1, x_2, -x_3).\end{gathered}$$
We shall show that if the domain $\Omega \subset \mathbb{R}^3$ and the external force f are invariant under the maps $\pi_1, \pi_2, \pi_3 : \Omega \to \Omega$, then the Navier-Stokes equations have a solution belonging to the set K. To illustrate our methodology, we have provided more examples throughout the paper.

Historically, symmetry conditions of the form above have been imposed on the solution of the Navier-Stokes equations to address the existence problem of these equations in bounded domains, albeit with non-homogeneous boundary conditions, given by
$$\begin{gathered}(u \cdot \nabla) u + f(x) = \Delta u - \nabla p_u \quad \forall x \in \Omega, \\ \nabla \cdot u = 0 \quad \forall x \in \Omega, \\ u = a(x) \quad \forall x \in \partial\Omega.\end{gathered} \tag{1.2}$$
The bounded domain Ω is defined as
$$\Omega = \Omega_0 \setminus \Big(\cup_{i=1}^N \Omega_i \Big),$$

where $\Omega_i \subset \Omega_0$ for $i = 1, \dots, N$, and the C^2 smooth boundary $\partial\Omega$ is composed of $N+1$ disjoint components $\partial\Omega_i$, i.e.,

$$\partial\Omega = \cup_{i=0}^{N} \partial\Omega_i.$$

Note that the divergence free property of the flow (equation (1.1)) enforces the condition

$$\int_{\partial\Omega} a(x) \cdot n(x)\, ds = \sum_{i=0}^{N} \int_{\partial\Omega_i} a(x) \cdot n(x)\, ds = 0,$$

where $n(x)$ is the unit outer normal to $\partial\Omega$. Proving the existence of a solution for the above-mentioned stationary Navier-Stokes equations is commonly referred to as the *Leray Problem*. Although the 2D case is now solved [10], the general 3D case still remains an open problem. In the very first attempt to solve the problem, Leray in his seminal 1933 paper [13], proved the existence of a solution under the condition

$$\int_{\partial\Omega_i} a(x) \cdot n(x)\, ds = 0.$$

Solving the *Leray Problem* generally where the above condition is removed attracted lots of attention in the research community. For several decades, all the proposed solutions to the 2D case relied on some type of conditions; and this is still the case for the 3D problem [7]. In most attempts, this condition is imposed on $a(x)$ at the boundary, i.e.,

$$\sum_{i=0}^{N} \Big| \int_{\partial\Omega_i} a(x) \cdot n(x)\, ds \Big| < c.$$

For some of the examples pertaining the major contributions to this line of research please refer to [11, 4, 6, 19, 12, 9, 8, 2, 17]. Some researchers, however, have tackled the problem where the required conditions are imposed on the entire domain Ω as symmetry conditions. Most notably, Amick [1] first studied the domain $\Omega \subset \mathbb{R}^2$ invariant under the mapping π_1, defined as:

$$\pi_1(x_1, x_2) = (-x_1, x_2).$$

Using "reduction to absurdity", Amick proved in 1984 that the steady Navier-Stokes equations (1.2) has a solution preserving the following symmetry condition,

$$\begin{aligned} u_1(-x_1, x_2) &= -u_1(x_1, x_2), \\ u_2(-x_1, x_2) &= u_2(x_1, x_2). \end{aligned}$$

In a similar effort, Sazonov [18] provided a proof of the existence problem in the presence of the aforementioned symmetry condition. By introducing the concept of "Virtual drain", Fujita [5] also proved the existence of a symmetric solution through constructing a symmetric solenoidal extension of the boundary value. Furthermore, Morimoto [14] presented a different proof by invoking the concept of stream functions. In extending the previous works to $\mathbb{R}^3$, Punhnachev [16, 15] and subsequently Korobkov et al. [10] proved an existence theorem for the axially symmetric problem in a domain with a multiply connected boundary. Note that the function $h = (h_r, h_\theta, h_z)$ in the cylindrical coordinate is called axially symmetric if $h_\theta = 0$, and h_r and h_z are not dependent on θ.

2. Proof of Theorem 1.1

We shall need some preliminary results before proving our abstract Theorem 1.1. We define $V = \{u \in H_0^1(\Omega) : \nabla \cdot u = 0\}$, and assume K is a closed convex subset of V. Furthermore, define $B : V \times V \to \mathbb{R}^m$ $(m = 2, 3, 4)$ as follows:

$$B(u, v) = (u \cdot \nabla)v = \sum_{j,k=1}^{m} u_k \frac{\partial v_j}{\partial x_k} \mathbf{e}_j, \tag{2.1}$$

where $\mathbf{e}_j$ is the unit vector along the jth axis. Note that in particular $B(u, u) = \Lambda u$.

Lemma 2.1. *The function $M_1 : K \times K \to \mathbb{R}$ defined by $M_1(u, v) = \langle \Lambda u, v\rangle$ is weakly lower semi-continuous on $K \times K$ for each $v \in K$.*

Proof. Let $v \in C^1(\Omega) \cap K$ and $u^n \rightharpoonup u$ weakly in V. Using Rellich-Konrachov Compactness Theorem, one can prove that $u^n \to u$ strongly in $L^p(\Omega)$ for $1 \le p < 2m/(m-2)$. Applying Lemma 4.3 in the Appendix results in

$$M_1(u^n, v) = \int_\Omega (u^n \cdot \nabla)u^n \cdot v \, dx = -\int_\Omega (u^n \cdot \nabla)v \cdot u^n dx. \tag{2.2}$$

Therefore,

$$\begin{aligned}
&|M_1(u^n, v) - M_1(u, v)| \\
&= \Big| \sum_{j,k=1}^{m} \int_\Omega \Big(u_k^n \frac{\partial v_j}{\partial x_k} u_j^n - u_k \frac{\partial v_j}{\partial x_k} u_j \Big) dx \Big| \\
&\le \|v\|_{C^1(\Omega)} \sum_{j,k=1}^{m} \int_\Omega \big| u_k^n u_j^n - u_k u_j \big| dx \\
&\le \|v\|_{C^1(\Omega)} \sum_{j,k=1}^{m} \Big(\int_\Omega |u_k^n u_j^n - u_k u_j^n| + \int_\Omega |u_k u_j^n - u_k u_j| \Big) dx \\
&\le \|v\|_{C^1(\Omega)} \sum_{j,k=1}^{m} \Big(\|u_j^n\|_{L^2(\Omega)} \|u_k^n - u_k\|_{L^2(\Omega)} + \|u_k\|_{L^2(\Omega)} \|u_j^n - u_j\|_{L^2(\Omega)} \Big).
\end{aligned}$$

Therefore, $M_1(u^n, v)$ converges strongly to $M_1(u, v)$ on K for every $v \in C^1(\Omega) \cap K$. Using Lemma 4.2 in the Appendix, we know that $M_1(u, v)$ is strongly continuous on $H^1(\Omega)$; hence, by using the density argument we can conclude that $M_1(u, v)$ is weakly lower semi-continuous on $K \times K$ for each $v \in K$. □

Next, we define $M : V \times V \to \mathbb{R}$ as

$$M(u, v) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{1}{2} \int_\Omega |\nabla v|^2 dx + \int_\Omega \Lambda u \cdot (u - v) dx + \int_\Omega f(x) \cdot (u - v) dx,$$

where $f \in L^2(\Omega)$.

Lemma 2.2. *The function $M(u, v)$ is lower semi-continuous on $K \times K$, where K is a convex and closed subset of V.*

Proof. Assume that $u^n \rightharpoonup u$ weakly in V,

- It follows from the lower semi continuity of the norm that

$$\int_\Omega |\nabla u|^2 dx \le \liminf_{n\to\infty} \int_\Omega |\nabla u^n|^2 dx;$$

- we have $\Lambda u \cdot (u) = 0$ resulting from Lemma 4.3 in the Appendix, and $M_1(u,v) = \langle \Lambda u, v\rangle$ is weakly lower semi-continuous as proven in Lemma 2.1;
- since $f \in L^2(\Omega)$, applying the strong convergence of $u^n \to u$ in $L^2(\Omega)$ leads to
$$\int_\Omega f(x)\, u\, dx = \lim_{n\to\infty} \int_\Omega f(x) u^n dx.$$

This proves that $M(u,v)$ is lower semi-continuous on $K \times K$. □

Proof of Theorem 1.1. Part 1: Assume condition (i) holds. Set $M : V \times V \to \mathbb{R}$ as follows:
$$M(u,v) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \int_\Omega \Lambda u \cdot (u-v) dx + \int_\Omega f(x)\cdot (u-v) dx,$$
where $f \in L^2(\Omega)$. Note that $M : K \times K \to \mathbb{R}$ satisfies all the conditions of the Ky Fan's Min-Max Principle presented in Theorem 4.1 in the Appendix:

(1) For each $v \in K$, the map $u \mapsto M(u,v)$ is weakly lower semi-continuous on K as proved in Lemma 2.2.
(2) For each $u \in V$, the map $v \mapsto M(u,v)$ is concave on K: note that $M(u,v)$ is a linear functional with respect to v except for $\frac{1}{2}\int_\Omega |\nabla v|^2 dx$, which is in fact convex.
(3) Note that $M(u,u) = 0 = \gamma$ for every $u \in K$.
(4) As required in Theorem 4.1, we should show that there exists $v_0 \in K$ such that the set $\{u \in K : M(u,v_0) \le \gamma\}$ is bounded. Set $v_0 = 0$, we show that such that $K_0 = \{u \in K : M(u,v_0) \le \gamma\}$ is bounded. Take $u \in K_0$, using Hölder's inequality, we have
$$\frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} = \frac{1}{2}\int_\Omega |\nabla u|^2 dx \le -\int_\Omega f(x)\cdot (u) dx \le \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}.$$
Using Sobolev embedding results $\|u\|_{L^2(\Omega)} \le c\|\nabla u\|_{L^2(\Omega)}$ on the right hand side, we obtain
$$\|\nabla u\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}.$$
Therefore, the set K_0 is bounded under the $\|\cdot\|_{H^1(\Omega)}$.

We now apply the Ky Fan's Min-Max Principle to conclude that there exists $\bar u \in K$ such that
$$M(\bar u, v) \le 0 \quad \forall v \in K;$$
that is
$$\frac{1}{2}\int_\Omega |\nabla \bar u|^2 dx - \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \int_\Omega \Lambda \bar u \cdot (\bar u - v) dx + \int_\Omega f(x)\cdot (\bar u - v) dx \le 0 \quad (2.3)$$
for all $v \in K$. By assumption (i), there exits $\bar v \in K$ such that
$$\int_\Omega \Lambda \bar u \cdot \eta\, dx + \int_\Omega f(x)\cdot \eta\, dx = -\int_\Omega \nabla \bar v \cdot \nabla \eta\, dx \quad \forall \eta \in V. \quad (2.4)$$
Now, choose $\eta = \bar u - \bar v$, we have
$$\int_\Omega \Lambda \bar u \cdot (\bar u - \bar v)\ dx + \int_\Omega f(x)\cdot (\bar u - \bar v)\ dx = -\int_\Omega \nabla \bar v \cdot \nabla (\bar u - \bar v)\ dx. \quad (2.5)$$
On the other hand, equation (2.3) holds for $\bar v \in K$, i.e.,
$$\frac{1}{2}\int_\Omega |\nabla \bar u|^2 dx - \frac{1}{2}\int_\Omega |\nabla \bar v|^2 dx + \int_\Omega \Lambda \bar u \cdot (\bar u - \bar v) dx + \int_\Omega f(x)\cdot (\bar u - \bar v) dx \le 0.$$

Replacing the last two terms of the above inequality with the right-hand side of equation (2.5) results in the inequality

$$\frac{1}{2}\int_\Omega |\nabla \bar{u}|^2 dx - \frac{1}{2}\int_\Omega |\nabla \bar{v}|^2 dx - \int_\Omega \nabla \bar{v}\cdot\nabla\left(\bar{u}-\bar{v}\right)\, dx \le 0.$$

Therefore,

$$\frac{1}{2}\int_\Omega |\nabla \bar{u}-\nabla\bar{v}|^2 dx \le 0.$$

Hence, we have $\nabla\bar{u}=\nabla\bar{v}$, and since $\bar{u}=\bar{v}=0$ on $\partial\Omega$, we conclude that $\bar{u}=\bar{v}$ on Ω. Substituting $\bar{u}=\bar{v}$ in equation (2.4) results in

$$\int_\Omega \Lambda\bar{u}\cdot\eta\, dx + \int_\Omega f(x)\cdot\eta\, dx = -\int_\Omega \nabla\bar{u}\cdot\nabla\eta\, dx \quad \forall \eta\in V.$$

or equivalently

$$\begin{gathered}\Lambda\bar{u} + f(x) = \Delta\bar{u} - \nabla p_{\bar{u}} \quad \forall x\in\Omega,\\ \nabla\cdot\bar{u} = 0 \quad \forall x\in\Omega,\\ \bar{u}=0 \quad \forall x\in\partial\Omega.\end{gathered}$$

Part 2: Assume condition (ii) holds. Using Lemma 4.3 in the Appendix, we have

$$\Lambda u\cdot(u-v) = (u\cdot\nabla)u\cdot(u-v) = (u\cdot\nabla)v\cdot(u-v) = B(u,v)\cdot(u-v). \tag{2.6}$$

The rest of the proof is identical to Part 1. □

3. Applications

In this section, we demonstrate how Theorem 1.1 can be used for proving the existence of symmetric solutions to the Navier-Stokes equations in dimension three. The less involved two dimensional cases can be addressed using a similar approach; thus, they are not repeated here. In light of this objective, let us define the maps $\pi_1,\pi_2,\pi_3:\Omega\to\Omega$ as follows

$$\begin{gathered}\pi_1(x_1,x_2,x_3) = (-x_1,x_2,x_3),\\ \pi_2(x_1,x_2,x_3) = (x_1,-x_2,x_3),\\ \pi_3(x_1,x_2,x_3) = (x_1,x_2,-x_3).\end{gathered}$$

Theorem 3.1. *Consider the 3D stationary Navier-Stokes equations presented in equation (1.1). Assume that Ω is invariant under the map $\pi_1:\Omega\to\Omega$. Moreover, assume that K is a subset of V containing all $u\in V$ with the following properties:*

$$\begin{gathered}u_1(x_1,x_2,x_3) = -u_1(-x_1,x_2,x_3),\\ u_2(x_1,x_2,x_3) = u_2(-x_1,x_2,x_3),\\ u_3(x_1,x_2,x_3) = u_3(-x_1,x_2,x_3).\end{gathered} \tag{3.1}$$

Furthermore, assume that $f(x)\in L^2(\Omega)$ also holds the same properties; i.e.,

$$\begin{gathered}f_1(x_1,x_2,x_3) = -f_1(-x_1,x_2,x_3),\\ f_2(x_1,x_2,x_3) = f_2(-x_1,x_2,x_3),\\ f_3(x_1,x_2,x_3) = f_3(-x_1,x_2,x_3).\end{gathered}$$

Then, the Navier-Stokes equation has a solution in K.

Proof. Step 1: It can be shown that the set K is convex and closed in V. To be precise, since $K \subset V$, the identities in (3.1) are to be understood almost every where in Ω. If $\{u^n\}$ is a sequence in K such that u_n converges weakly in V to a function $u \in V$, then u_n converges strongly in $L^2(\Omega)$. Therefore, up to a subsequence, $u^n(x) \to u(x)$ for a.e. $x \in \Omega$. This implies that u satisfies the identities in (3.1) almost every where in Ω. On the other hand since K is a linear subset of V it is clearly convex.

Step 2: Fix $u \in K$. We now show that there exits $v \in V$ such that

$$\Lambda u - f(x) = \Delta v - \nabla p_v \quad \forall x \in \Omega, \tag{3.2}$$

in a weak sense. To this end, define the the functional $I : V \to \mathbb{R}$ as follows:

$$I(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 dx + \int_\Omega \Lambda u \cdot w\, dx + \int_\Omega f(x) \cdot w\, dx.$$

The functional I is coercive, lower semi-continuous and strictly convex; thus, there exist a unique $v \in V$ such that

$$I(v) = \inf_{w \in V} I(w),$$

and satisfies equation (3.2).

Step 3: We then need to show that $v \in K$. Define $\bar v(x)$ as follows:

$$\begin{aligned} \bar v_1(x_1, x_2, x_3) &= -v_1(-x_1, x_2, x_3),\\ \bar v_2(x_1, x_2, x_3) &= v_2(-x_1, x_2, x_3),\\ \bar v_3(x_1, x_2, x_3) &= v_3(-x_1, x_2, x_3). \end{aligned} \tag{3.3}$$

Now by calculations, we have

$$I(\bar v) = \frac{1}{2}\int_\Omega |\nabla \bar v(x)|^2 dx + \int_\Omega \Lambda u(x) \cdot \bar v(x)\, dx + \int_\Omega f(x) \cdot \bar v(x)\, dx.$$

To rewrite $I(\bar v)$ in terms of v, we set $\bar x = (-x_1, x_2, x_3)$. We first show that

$$\Lambda u(x) \cdot \bar v(x) = \Lambda u(\bar x) \cdot v(\bar x). \tag{3.4}$$

For simplicity of notation, D_i denotes derivative with respect to the ith variable of a given function $u(x)$. Therefore,

$$\begin{aligned} Du(x) &= \begin{bmatrix} D_1u_1(x) & D_2u_1(x) & D_3u_1(x)\\ D_1u_2(x) & D_2u_2(x) & D_3u_2(x)\\ D_1u_3(x) & D_2u_3(x) & D_2u_3(x) \end{bmatrix}\\ &= \begin{bmatrix} +D_1u_1(\bar x) & -D_2u_1(\bar x) & -D_3u_1(\bar x)\\ -D_1u_2(\bar x) & +D_2u_2(\bar x) & +D_3u_2(\bar x)\\ -D_1u_3(\bar x) & +D_2u_3(\bar x) & +D_2u_3(\bar x) \end{bmatrix}. \end{aligned} \tag{3.5}$$

Now we expand the left-hand side of (3.4) as follows:

$$\begin{aligned} \Lambda u(x) \cdot \bar v(x) = {} & [u_1(x)D_1u_1(x) + u_2(x)D_2u_1(x) + u_3D_3u_1(x)]\, \bar v_1(x)\\ & + [u_1(x)D_1u_2(x) + u_2(x)D_2u_2(x) + u_3D_3u_2(x)]\, \bar v_2(x)\\ & + [u_1(x)D_1u_3(x) + u_2(x)D_2u_3(x) + u_3D_3u_3(x)]\, \bar v_3(x). \end{aligned}$$

Using the relationships in (3.1), (3.3) and (3.5), we have

$$\begin{aligned} \Lambda u(x) \cdot \bar v(x) = {} & \big[(-u_1(\bar x))(+D_1u_1(\bar x)) + (+u_2(\bar x))(-D_2u_1(\bar x))\\ & + (+u_3(\bar x))(-D_3u_1(\bar x))\big](-v_1(\bar x)) + \big[(-u_1(\bar x))(-D_1u_2(\bar x)) \end{aligned}$$

$$
\begin{aligned}
&+ (+u_2(\bar{x}))(+D_2u_2(\bar{x})) + (+u_3(\bar{x}))(+D_3u_2(\bar{x}))\big](+v_2(\bar{x})) \\
&+ \big[(-u_1(\bar{x}))(-D_1u_3(\bar{x})) + (+u_2(\bar{x}))(+D_2u_3(\bar{x})) \\
&+ (+u_3(\bar{x}))(+D_3u_3(\bar{x}))\big](+v_3(\bar{x})) \\
&= \Lambda u(\bar{x}) \cdot v(\bar{x}).
\end{aligned}
$$

Moreover, one can similarly prove that

$$
\begin{gathered}
f(x) \cdot \bar{v}(x) = f(\bar{x}) \cdot v(\bar{x}) \\
|\nabla \bar{v}(x)|^2 = |\nabla v(\bar{x})|^2.
\end{gathered}
$$

Since $|J| = |\partial x/\partial \bar{x}| = 1$, we can equivalently write

$$
I(\bar{v}) = \frac{1}{2}\int_\Omega |\nabla v(\bar{x})|^2 d\bar{x} + \int_\Omega \Lambda u(\bar{x}) \cdot v(\bar{x})\, d\bar{x} + \int_\Omega f(\bar{x}) \cdot v(\bar{x})\, d\bar{x}.
$$

Finally, we conclude that $I(\bar{v}) = I(v)$.

Step 4: Note that

$$
\nabla \cdot \bar{v}(x) = \nabla \cdot v(x) = 0.
$$

Therefore, $\bar{v}(x) \in V$. Since v is the unique minimizer of I, we can conclude that $\bar{v}(x) = v(x)$; therefore, there exits $v \in K$ such that equation (3.2) is satisfied for a fixed $u \in K$.

Step 5: Note that the existence of $v \in K$ (as proved above) satisfies condition (i) of Theorem 1.1; therefore, a solution of the Navier-Stokes equations exist in the set K; i.e., there exists $\bar{u} \in K$ that satisfies the following equations:

$$
\begin{gathered}
\Lambda \bar{u} + f(x) = \Delta \bar{u} - \nabla p_{\bar{u}} \quad \forall x \in \Omega, \\
\nabla \cdot \bar{u} = 0 \quad \forall x \in \Omega, \\
\bar{u} = 0 \quad \forall x \in \partial\Omega.
\end{gathered}
$$

□

One can generalize the aforementioned theorem to encompass a variety of problems that follow the same structure. In order to achieve this, let us define the maps $\gamma_1, \gamma_2, \gamma_3 : L^2(\Omega) \to L^2(\Omega)$ as follows:

$$
\begin{gathered}
\gamma_1(u_1(x), u_2(x), u_3(x)) = (-u_1(x), u_2(x), u_3(x)), \\
\gamma_2(u_1(x), u_2(x), u_3(x)) = (u_1(x), -u_2(x), u_3(x)), \\
\gamma_3(u_1(x), u_2(x), u_3(x)) = (u_1(x), u_2(x), -u_3(x)).
\end{gathered}
$$

We denote the the group generated by γ_1, γ_2 and γ_3 as G_γ and it isomorphic counterpart by G_π which is generated by the elements π_1, π_2 and π_3. The two groups correspond to each other by the isomorphism $g : G_\pi \to G_\gamma$, as follows:

$$
\begin{gathered}
g(\pi_i) = \gamma_i, \\
g(\pi_i \circ \pi_j) = \gamma_i \circ \gamma_j, \\
g(\pi_i \circ \pi_j \circ \pi_k) = \gamma_i \circ \gamma_j \circ \gamma_k,
\end{gathered}
$$

where $i, j, k = 1, 2, 3$.

Theorem 3.2. *Consider the 3D stationary Navier-Stokes equations presented in equation* (1.1)*. Define the groups G_π and G_γ and their isomorphism g as above. Assume that Ω is invariant under the map $\bar{\pi}_1, \dots, \bar{\pi}_m \in G_\pi$, and K is a subset of V containing all $u \in V$ with the property that when $g(\bar{\pi}_1) = \bar{\gamma}_1, \dots, g(\bar{\pi}_m) = \bar{\gamma}_m$ we have $u(x) = \bar{\gamma}_1(u(\bar{\pi}_1(x))), \dots, u(x) = \bar{\gamma}_m(u(\bar{\pi}_m(x)))$. Furthermore, assume that*

$f(x) \in H_0^1(\Omega)$ also holds the same property; i.e., $f(x) = \bar{\gamma}_1(f(\bar{\pi}_1(x))), \dots, f(x) = \bar{\gamma}_m(f(\bar{\pi}_m(x)))$. Then, the Navier-Stokes equation has a solution in K.

Proof. The proof of this theorem follows the steps presented in the previous example except that Step 3 needs to be verified for the pair of functions $\bar{\pi}_1, \bar{\gamma}_1$ to $\bar{\pi}_m, \bar{\gamma}_m$ instead of π_1, γ_1. □

The following two corollaries, whose 2D versions have been solved in the literature using other techniques, are also worthwhile pointing out herein.

Corollary 3.3. *Consider the 3D stationary Navier-Stokes equations presented in equation (1.1). Assume that Ω is invariant under the map $\pi : \Omega \to \Omega$, which is defined as follows:*

$$\pi(x) = \pi(x_1, x_2, x_3) = (-x_1, -x_2, -x_3) = -x. \tag{3.6}$$

Moreover, assume that K is a subset of V containing all $u \in V$ with the property

$$u(x_1, x_2, x_3) = -u(-x_1, -x_2, -x_3).$$

Furthermore, assume that $f(x) \in H_0^1(\Omega)$ also holds the same property; i.e.,

$$f_1(x_1, x_2, x_3) = -f(-x_1, -x_2, -x_3). \tag{3.7}$$

Then, the Navier-Stokes equation has a solution in K.

Proof. Applying Theorem 3.2 for the case $m = 1$, we set $\bar{\pi}_1 = \pi_3 \circ \pi_2 \circ \pi_1$ and $\bar{\gamma}_1 = \gamma_1 \circ \gamma_2 \circ \gamma_3$. □

Corollary 3.4. *Consider the 3D stationary Navier-Stokes equations presented in equation (1.1). Assume that Ω is invariant under the maps $\pi_1, \pi_2, \pi_3 : \Omega \to \Omega$. Moreover, assume that K is a subset of V containing all $u \in V$ with the property*

$$u_i(x) = \begin{cases} -u_i(\pi_j(x)) & i = j, \\ u_i(\pi_j(x)) & \text{otherwise}, \end{cases}$$

where $u(x) = (u_1(x), u_2(x), u_3(x))$. Furthermore, assume that $f(x) \in H_0^1(\Omega)$ also holds the same property; i.e.,

$$f_i(x) = \begin{cases} -f_i(\pi_j(x)) & i = j, \\ f_i(\pi_j(x)) & \text{otherwise}, \end{cases}$$

where $f(x) = (f_1(x), f_2(x), f_3(x))$. Then, the Navier-Stokes equation has a solution in K.

Proof. Applying Theorem 3.2 for the case $m = 3$, we set $\bar{\pi}_i = \pi_i$ and $\bar{\gamma}_i = \gamma_i$ for $i = 1, 2, 3$. □

4. Appendix

The following is the well-known Ky Fan's Min-Max Principle by Brezis-Nirenberg-Stampacchia [3].

Theorem 4.1. *Let E be a closed convex subset of a reflexive Banach space Z, and consider $M : E \times E \to \bar{\mathbb{R}}$ to be a function such that*

1. *For each $y \in E$, the map $x \to M(x, y)$ is weakly lower semi-continuous on E;*
2. *For each $x \in E$, the map $y \to M(x, y)$ is concave on E;*

(3) *There exists* $\gamma \in \mathbb{R}$ *such that* $M(x,x) \le \gamma$ *for every* $x \in E$;
(4) *There exists a* $y_0 \in E$ *such that* $E_0 = \{x \in E : M(x, y_0) \le \gamma\}$ *is bounded.*

Then, there exits $\bar{x} \in E$ *such that* $M(\bar{x}, y) \le \gamma$ *for all* $y \in E$.

We have made frequent use of the following standard result. Now we provide a short proof, for the convenience of the reader.

Lemma 4.2. *Let* $f(u,v,w)$ *in* $\mathbb{R}^m$ $(m = 2,3,4)$ *be defined as*

$$f(u,v,w) = \langle (u \cdot \nabla) \cdot v, w\rangle = \sum_{j,k=1}^{m} u_k \frac{\partial v_j}{\partial x_k} w_j. \tag{4.1}$$

Then, $f(u,v,w)$ *is continuous on* $H^1 \times H^1 \times H^1$.

Proof. Using Hölder's inequality, we have

$$|f(u,v,w)| \le \|u\|_{L^4} \|\nabla v\|_{L^2} \|w\|_{L^4}. \tag{4.2}$$

Using the Sobolev embedding $H^1(\Omega) \subset L^{\frac{2m}{m-2}}(\Omega)$, we have that

$$|f(u,v,w)| \le C\|u\|_{H^1}\|v\|_{H^1}\|w\|_{H^1}, \tag{4.3}$$

for an appropriate constant C. This proves that that $f(u,v,w)$ is strongly continuous. □

Lemma 4.3. *Let* $u \in V$ *and* $v, w \in H^1$. *Then*

$$f(u,v,w) = \langle (u \cdot \nabla) \cdot v, w\rangle = -\langle (u \cdot \nabla) \cdot w, v\rangle = -f(u,w,v), \tag{4.4}$$

and in particular,

$$f(u,v,v) = \langle (u \cdot \nabla) \cdot v, v\rangle = 0. \tag{4.5}$$

Proof. Assume $u \in C_c^\infty(\Omega) \cap V$ and $v, w \in C^1(\Omega)$. Using integration by parts, we have

$$\begin{aligned}
\langle (u \cdot \nabla) v, w\rangle &= \int_\Omega \sum_{j,k=1}^{m} u_k \frac{\partial v_j}{\partial x_k} w_j dx \\
&= -\int_\Omega \sum_{j,k=1}^{m} \frac{\partial u_k}{\partial x_k} v_j w_j dx - \int_\Omega \sum_{j,k=1}^{m} u_k v_j \frac{\partial w_j}{\partial x_k} dx \\
&= -\langle (u \cdot \nabla) \cdot w, v\rangle.
\end{aligned}$$

Since $f(u,v,w)$ is continuous on $H^1 \times H^1 \times H^1$ (proven in Lemma 4.2), we use the density argument to extend the above conclusion to $u \in V$ and $v, w \in H^1$. Furthermore, note that $\langle (u \cdot \nabla) v, v\rangle = -\langle (u \cdot \nabla) v, v\rangle$, therefore,

$$\langle (u \cdot \nabla) v, v\rangle = 0. \tag{4.6}$$

□

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