

## EXISTENCE OF POSITIVE SOLUTIONS FOR SYSTEMS OF QUASILINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. In this article, we study the existence of standing wave solutions for the quasilinear Schrödinger system

$$\begin{aligned} -\varepsilon^2 \Delta u + W(x)u - \kappa \varepsilon^2 \Delta(u^2)u &= Q_u(u, v) \quad \text{in } \mathbb{R}^N, \\ -\varepsilon^2 \Delta v + V(x)v - \kappa \varepsilon^2 \Delta(v^2)v &= Q_v(u, v) \quad \text{in } \mathbb{R}^N, \\ u, v > 0 \quad \text{in } \mathbb{R}^N, \quad u, v &\in H^1(\mathbb{R}^N). \end{aligned}$$

where  $N \geq 3$ ,  $\kappa > 0$ ,  $\varepsilon > 0$ ,  $W, V : \mathbb{R}^N \rightarrow \mathbb{R}$  are continuous functions that fall into two classes of potentials. To overcome the lack of differentiability, we use the dual approach developed by Colin–Jeanjean. The existence of solutions is obtained using Del Pino–Felmer’s penalization technique with an adaptation of Alves’ arguments [1].

### 1. INTRODUCTION

In this article, we examine the existence of solutions to the system of quasilinear Schrödinger equations (QLSE):

$$\begin{aligned} -\varepsilon^2 \Delta u + W(x)u - \kappa \varepsilon^2 \Delta(u^2)u &= Q_u(u, v) \quad \text{in } \mathbb{R}^N, \\ -\varepsilon^2 \Delta v + V(x)v - \kappa \varepsilon^2 \Delta(v^2)v &= Q_v(u, v) \quad \text{in } \mathbb{R}^N, \\ u, v > 0 \quad \text{in } \mathbb{R}^N, \quad u, v &\in H^1(\mathbb{R}^N). \end{aligned} \tag{1.1}$$

where  $N \geq 3$ ,  $\kappa > 0$ ,  $\varepsilon > 0$ ,  $W, V : \mathbb{R}^N \rightarrow \mathbb{R}$  are continuous functions that fall into two classes of potentials introduced in [1]. The functions  $Q_u, Q_v : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  are continuous functions denoting partial derivatives of the function  $Q : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ , which belongs to the class of  $C^1$  and is  $p$ -homogeneous.

Systems of type (1.1) are related to various applications in hydrodynamics, Heidelberg ferromagnetism, Magnus theory, condensed matter theory, and dissipative quantum mechanics.

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By a simple change of variables, system (1.1) is equivalent to the system

$$\begin{aligned} -\Delta u + W(\varepsilon x)u - \kappa \Delta(u^2)u &= Q_u(u, v) \quad \text{in } \mathbb{R}^N, \\ -\Delta v + V(\varepsilon x)v - \kappa \Delta(v^2)v &= Q_v(u, v) \quad \text{in } \mathbb{R}^N, \\ u, v &> 0 \quad \text{in } \mathbb{R}^N \end{aligned} \quad (1.2)$$

Considering the potential values of  $\kappa$ , assumptions regarding potentials, and various nonlinearity types, numerous studies have explored the existence of solutions for system (1.1), particularly when  $\kappa \neq 0$ , as seen in [5, 6, 8, 10, 11, 19].

Recently, numerous articles have examined the scalar equation:

$$-\varepsilon^2 \Delta u + V(x)u - \kappa \varepsilon^2 \Delta(u^2)u = g(u), \quad \text{in } \mathbb{R}^N, \quad (1.3)$$

where  $N \geq 3$ ,  $\kappa \in \mathbb{R}$ ,  $\varepsilon > 0$  are real parameters, and  $V : \mathbb{R}^N \rightarrow \mathbb{R}$  satisfies certain geometries. The function  $g : \mathbb{R} \rightarrow \mathbb{R}$  is continuous. This kind of equation frequently appears in various models, notably in connection with standing wave phenomena within the quasilinear Schrödinger equation.

$$-i\varepsilon \frac{\partial z}{\partial t} = -\varepsilon^2 \Delta z + F(x)z - \kappa \varepsilon^2 \Delta \rho(|z|^2) \rho'(|z|^2)z - f(|z|^2)z, \quad \text{for all } x \in \mathbb{R}^N. \quad (1.4)$$

where  $z : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}$ ,  $F$  represents the potential,  $\kappa$  is a real constant, and  $f, \rho$  are real-valued functions. When  $\rho(s) = s$ , this equation arises in fields such as fluid mechanics, plasma physics, dissipative mechanics, and condensed matter theory. The stationary solutions of (1.4) take the form

$$z(t, x) = \exp\left(-\frac{iEt}{\varepsilon}\right)u(x), \quad E \in \mathbb{R}, \quad (1.5)$$

where  $u$  represents the solution to equation (1.3) with  $V(x) = F(x) - E$  and  $g(u) = f(u^2)u$ . For the physical motivation of equation (1.4), readers are referred to [13, 12, 14] and references therein.

The semilinear scenario, identified by  $\kappa = 0$ , has undergone thorough examination in recent years. For instance, del Pino and Felmer [17] studied the problem:

$$-\varepsilon^2 \Delta u + V(x)u = q(u), \quad \text{in } \mathbb{R}^N, \quad u > 0 \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N). \quad (1.6)$$

where  $\varepsilon > 0$ ,  $N \geq 3$ , and  $q : \mathbb{R} \rightarrow \mathbb{R}$  is a subcritical nonlinearity. The function  $V$  is a locally Hölder continuous potential satisfying

$$0 < \alpha = \inf_{x \in \mathbb{R}^N} V(x) \leq V_0 = \inf_{\Omega} V(x) < \min_{\partial\Omega} V(x). \quad (1.7)$$

In [17], the authors introduced the penalization method and proved that if  $V$  satisfies (1.7), then (1.6) has a solution  $u_\varepsilon$  that concentrates at a minimum of  $V$ . Alves, do Ó, and Souto [3] also studied (1.6) and proved the same result as in [17] for  $V$  satisfying (1.7), with the subcritical nonlinearity perturbed by a critical term.

Alves [1] studied problem (1.6) with the nonlinearity  $q : \mathbb{R} \rightarrow \mathbb{R}$  being continuous and having subcritical or critical growth. Alves [1] introduced for the first time two interesting classes of potential  $V$ , namely:

**Class 1:** The potential  $V$  satisfies the Palais-Smale (PS) condition, with the following conditions:

- (A1) There exists  $V_0 > 0$  such that  $V(x) \geq V_0$ , for all  $x \in \mathbb{R}^N$ , where  $V_0 = \inf_{\mathbb{R}^N} V(x)$ .
- (A2)  $V \in C^2(\mathbb{R}^N)$  and  $V, \frac{\partial V}{\partial x_i}, \frac{\partial^2 V}{\partial x_i \partial x_j}$  are bounded across  $\mathbb{R}^N$ , for all  $i, j \in \{1, 2, 3, \dots, N\}$ .

- (A3)  $V$  adheres to the Palais-Smale (PS) condition, that is, if  $(x_n) \subset \mathbb{R}^N$ , with  $(V(x_n))$  being bounded and  $\nabla V(x_n) \rightarrow 0$ , then  $(x_n)$  possesses a convergent subsequence.

**Class 2:** The potential  $V$  lacks critical points along the boundary of some bounded domain. Within this class of potentials,  $V$  satisfies (A1), (A2) and

- (A4) there exists a domain  $\Lambda \subset \mathbb{R}^N$  where  $\nabla V(x) \neq 0$  for all  $x \in \partial\Lambda$ .

Given that  $V$  falls into either Class 1 or Class 2 and taking into account certain conditions met by the nonlinearity, the author demonstrated the existence of a positive solution for  $\varepsilon > 0$  sufficiently small.

Alves [2] explored the presence and concentration of solutions for the system derived from (1.1) with  $\kappa = 0$ :

$$\begin{aligned} -\varepsilon^2 \Delta u + W(x)u &= Q_u(u, v) \quad \text{in } \mathbb{R}^N, \\ -\varepsilon^2 \Delta v + V(x)v &= Q_v(u, v) \quad \text{in } \mathbb{R}^N, \\ u, v > 0 \quad \text{in } \mathbb{R}^N, \quad u, v &\in H^1(\mathbb{R}^N). \end{aligned} \tag{1.8}$$

where the functions  $W, V : \mathbb{R}^N \rightarrow \mathbb{R}$  are Hölder continuous satisfying  $W(x), V(x) \geq \alpha > 0$  in  $\mathbb{R}^N$  and the condition:

- (5) There exists an open and bounded set  $\Lambda \subset \mathbb{R}^N$ , with  $x_0 \in \Lambda$  and  $\rho > 0$ , such that  $W(x), V(x) \geq \rho$ , for all  $x \in \partial\Lambda$  and  $W(x_0), V(x_0) < \rho$ .

Severo and Silva [19] employed the variational approach within an appropriate Orlicz space to examine a system of type (1.1) with  $\kappa = 1$ . Recently, Arruda-Figueiredo and Nascimento [4] considered the two classes of potentials introduced by Alves in [1] and showed the existence of solutions for the system (1.8).

Motivated by these works, and mainly by [1, 2, 4, 17, 19], we study system (1.1) for  $\kappa = 1$ . We shall refer to the potential  $V$  as belonging to Class 1 when it satisfies (A6)–(A8), and as belonging to Class 2 when it satisfies (A6), (A7), (A9) where

- (A6) There exist  $\mathcal{V}_\infty, \mathcal{V}_0 > 0$  such that  $\mathcal{V}_0 \leq \mathcal{V}(x) \leq \mathcal{V}_\infty$ , for all  $x \in \mathbb{R}^N$ , where  $\mathcal{V}_0 = \inf_{\mathbb{R}^N} \mathcal{V}(x)$ .  
 (A7)  $\mathcal{V} \in C^2(\mathbb{R}^N)$  and  $\mathcal{V}, \frac{\partial \mathcal{V}}{\partial x_i}, \frac{\partial^2 \mathcal{V}}{\partial x_i \partial x_j}$  are bounded in  $\mathbb{R}^N$ , for  $i, j \in \{1, 2, \dots, N\}$ .  
 (A8)  $\mathcal{V}$  satisfies the PS-condition, that is, if  $(x_n) \subset \mathbb{R}^N$ , such that  $\mathcal{V}(x_n)$  is bounded and  $\nabla \mathcal{V}(x_n) \rightarrow 0$ , then  $(x_n)$  possesses a convergent subsequence.  
 (A9) There exists a domain  $\Lambda \subset \mathbb{R}^N$  exists where  $\nabla \mathcal{V}(x) \neq 0$  for every  $x \in \partial\Lambda$ .

**Notation** Let  $H^1(\mathbb{R}^N)$  be the Sobolev space with norm

$$\|u\|_{H^1(\mathbb{R}^N)} := \left( \|u\|_{L^2(\mathbb{R}^N)}^2 + \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 \right)^{1/2}.$$

Let  $2^*$  be the Sobolev critical exponent

$$2^* = \frac{2N}{N-2} \quad \text{for } N > 2.$$

Now, we present the assumptions on the function  $Q$ . Let  $\mathbb{R}_+^2 := [0, +\infty) \times [0, +\infty)$ , we assume that the nonlinearity  $Q \in C^1(\mathbb{R}_+^2, \mathbb{R})$  is  $p$ -homogeneous with subcritical growth. More precisely, our hypotheses on  $Q$  are:

- (A10) There exists  $p \in (4, 2.2^*)$ , such that  $Q(tu, tv) = t^p Q(u, v)$  for all  $t > 0$ ,  $(u, v) \in \mathbb{R}_+^2$ , where  $2^* = \frac{2N}{N-2}$  and  $N \geq 3$ .

(A11) There exists  $C > 0$  such that

$$|Q_u(u, v)| + |Q_v(u, v)| \leq C(|u|^{p-1} + |v|^{p-1})$$

for all  $(u, v) \in \mathbb{R}_+^2$ .

(A12)  $Q_u(0, 1) = 0, Q_v(1, 0) = 0$ .

(A13)  $Q_u(1, 0) = 0, Q_v(0, 1) = 0$ .

(A14)  $Q(u, v) > 0$  for each  $u, v > 0$ .

(A15)  $Q_u(u, v), Q_v(u, v) \geq 0$  for each  $(u, v) \in \mathbb{R}_+^2$ .

Since  $Q$  is a homogeneous function of degree  $p > 4$ , it follows that

$$pQ(u, v) = uQ_u(u, v) + vQ_v(u, v).$$

Moreover,  $\nabla Q$  is a homogeneous function of degree  $p - 1$ .

A prototype of function  $Q$  that satisfies (A11)–(A15) is

$$H(u, v) := a|u|^p + \sum_{\alpha_i + \beta_i = p} b_i |u|^{\alpha_i} |v|^{\beta_i} + c|v|^p,$$

where  $a, b_i, c \in \mathbb{R}$ ,  $\alpha_i + \beta_i = p$ ,  $\alpha_i, \beta_i \geq 1$ ,  $i \in I$ , with  $I$  denoting a finite subset of  $\mathbb{N}$ .

**Definition 1.1.** We say that the pair  $(u, v) \in H^1(\mathbb{R}^N) \cap L_{\text{loc}}^\infty(\mathbb{R}^N)$  is a solution to (1.1) if  $u, v > 0$  almost everywhere in  $\mathbb{R}^N$  and satisfies

$$\varepsilon^2 \int_{\mathbb{R}^N} (1 + 2u^2) \nabla u \nabla \varphi + 2 \int_{\mathbb{R}^N} |\nabla u|^2 u \varphi + \int_{\mathbb{R}^N} W(\varepsilon x) u \varphi = \int_{\mathbb{R}^N} Q_u(u, v) \varphi,$$

for all  $\varphi \in C_0^\infty(\mathbb{R}^N)$ , and

$$\varepsilon^2 \int_{\mathbb{R}^N} (1 + 2v^2) \nabla v \nabla \phi + 2 \int_{\mathbb{R}^N} |\nabla v|^2 v \phi + \int_{\mathbb{R}^N} V(\varepsilon x) v \phi = \int_{\mathbb{R}^N} Q_v(u, v) \phi,$$

for all  $\phi \in C_0^\infty(\mathbb{R}^N)$ .

**Theorem 1.2.** Assume that  $W$  and  $V$  satisfy (A6) and that either  $W$  or  $V$  falls into Class 1 or class 2. Furthermore, suppose that  $Q$  satisfies (A10)–(A15). Then system (1.1) has a solution for each  $\varepsilon \in (0, \varepsilon_0)$ . Moreover,  $u_\varepsilon, v_\varepsilon \in C_{\text{loc}}^{1,\alpha}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ , and there exist constants  $C_1, C_2, C_3, C_4 > 0$  satisfying

$$u_\varepsilon(x) \leq C_1 \exp(-C_2|x/\varepsilon|), \quad v_\varepsilon(x) \leq C_3 \exp(-C_4|x/\varepsilon|), \quad \forall x \in \mathbb{R}^N.$$

**Remark 1.3.** Theorem 1.2 extends the findings of Arruda-Figueiredo and Nascimento [4, Theorem 1.1] in at least two ways:

- (1) The first is that we consider  $(\kappa \neq 0)$ , which leads to entirely different estimates (regarding the functional and the solution to the auxiliary problem) when compared to the case of  $(\kappa = 0)$ .
- (2) The second difference is that, unlike [4, Theorem 1.1], we do not require both potentials  $V$  and  $W$  to belong to the same Class 1 or Class 2. We only require that one of the potentials belongs to one of these classes and that the other satisfies condition (A6).

We recall that  $J \in C^1(E, \mathbb{R})$  satisfies the Cerami condition on level  $b$ , denoted by the  $(\text{Ce})_b$  condition, if any sequence  $(u_n) \subset E$  for which

- (i)  $J(u_n) \rightarrow b$ ,
- (ii)  $\|J'(u_n)\|_{E'} (\|u_n\| + 1) \rightarrow 0$  as  $n \rightarrow \infty$ ,

possesses a convergent subsequence.

$J$  satisfies the Cerami condition, denoted by  $(Ce)$ , if it satisfies  $(Ce)_b$  for every  $b \in \mathbb{R}$ . We say that  $(u_n) \subset E$  is a  $(Ce)_b$  sequence if it satisfies (i) and (ii). We also say that  $(u_n) \subset E$  is a  $(Ce)$  sequence if it is a  $(Ce)_b$  sequence for some  $b \in \mathbb{R}$ .

To demonstrate our primary finding, we will use the following variant of the Mountain Pass Theorem.

**Theorem 1.4** ([20]). *Let  $E$  be a real Banach space and  $J : E \rightarrow \mathbb{R}$  be a functional of class  $C^1$ . Let  $S$  be a closed subset of  $E$ , which disconnects (arcwise)  $E$  into distinct connected components  $E_1$  and  $E_2$ . Suppose further that  $J(0) = 0$  and*

- (1)  $0 \in E_1$  and there exists  $\alpha > 0$  such that  $J(v) \geq \alpha$  for all  $v \in S$ .
- (2) There exists  $e \in E_2$  such that  $J(e) < 0$ .

Then,  $J$  possesses a  $(Ce)_c$  sequence with  $c \geq \alpha$  given by

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)) \geq \alpha,$$

where

$$\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) \in J^{-1}((-\infty, 0]) \cap E_2\}.$$

This article is structured as follows: In Section 2, we reframe the system and introduce an auxiliary system. Initially, we propose an equivalent system through an appropriate variable transformation as discussed in [7, 15]. To address the issue of compactness, we then define the auxiliary system (2.10), following the methodology presented in [8]. Section 3 focuses on the analysis of the positive solution of the auxiliary system (2.10), employing a variant of the Mountain Pass Theorem (Theorem 1.4) that does not require the (PS) condition. This is used to generate a Cerami sequence at the mountain-pass level. Subsequently, we adapt Del Pino's strategies to identify a solution for the auxiliary problem (2.10) and examine certain solution properties of the auxiliary system. Finally, Section 4 is dedicated to the proof of Theorem 1.2.

## 2. REFORMULATION OF THE SYSTEM AND THE AUXILIARY SYSTEM

Assuming (A6), we consider the closed subspace of  $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ ,

$$X = \left\{ (w, z) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} [W(\varepsilon x)w^2 + V(\varepsilon x)z^2] < \infty \right\}$$

which is a Hilbert space when endowed with the norm

$$\|(w, z)\|^2 = \int_{\mathbb{R}^N} [|\nabla w|^2 + |\nabla z|^2 + W(\varepsilon x)w^2 + V(\varepsilon x)z^2].$$

The natural functional associated with (1.2) is

$$\begin{aligned} J_\varepsilon(u, v) &= \frac{1}{2} \int_{\mathbb{R}^N} [(1 + 2u^2)|\nabla u|^2 + (1 + 2v^2)|\nabla v|^2 + W(\varepsilon x)u^2 + V(\varepsilon x)v^2] \\ &\quad - \int_{\mathbb{R}^N} Q(\varepsilon x, u, v), \end{aligned}$$

which is not well-defined in  $X$ . To address this challenge, we adopt the variable transformation proposed by Colin and Jeanjean [7], and by Liu, Wang, and Wang [15].

For this we consider  $w = f^{-1}(u)$  and  $z = f^{-1}(v)$ , where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$f'(t) = \begin{cases} \frac{1}{\sqrt{1+2f^2(t)}}, & \text{in } [0, +\infty), \\ -f'(-t), & \text{in } (-\infty, 0]. \end{cases} \quad (2.1)$$

**Lemma 2.1.** *The function  $f$  satisfies the following properties:*

- (1)  $f$  is uniquely defined,  $C^\infty$ , and invertible.
- (2)  $|f'(t)| \leq 1$  and  $|f(t)| \leq |t|$ , for  $t \in \mathbb{R}$ .
- (3)  $\frac{f(t)}{t} \rightarrow 1$  as  $t \rightarrow 0$ .
- (4)  $\frac{f(t)}{\sqrt{t}} \rightarrow \sqrt[4]{2}$  as  $t \rightarrow \infty$ .
- (5)  $\frac{|f(t)|}{2} \leq |t|f'(t) \leq |f(t)|$ .
- (6)  $|f(t)| \leq 2^{1/4}|t|^{1/2}$ , for all  $t \in \mathbb{R}$ .
- (7)  $\frac{f^2(t)}{2} \leq tf(t)f'(t) \leq f^2(t)$ , for all  $t \in \mathbb{R}$ .
- (8) There exist constants  $C_1, C_2 > 0$ , such that:

$$|f(t)| \geq C_1|t|, \quad \text{if } |t| \leq 1;$$

$$|f(t)| \geq C_2|t|^{1/2}, \quad \text{if } |t| \geq 1.$$

- (9)  $|f(t)f'(t)| \leq \frac{1}{\sqrt{2}}$ , for all  $t \in \mathbb{R}$ .

- (10) The function  $t \rightarrow f^q(s)f'(s)$  is increasing on  $(0, \infty)$  for each  $q > 1$ .

With the exception of property (10), all other properties are derived from [9, Lemma 2.1] (see also [7, 16, 15]). For property (10), refer to Reference [8, Remark 3.1].

Following the variable transformation

$$I_\varepsilon(w, z) := J_\varepsilon(f(w), f(z)),$$

we obtain the functional

$$\begin{aligned} I_\varepsilon(w, z) &= \frac{1}{2} \int_{\mathbb{R}^N} [|\nabla w|^2 + |\nabla z|^2 + W(\varepsilon x)f(w)^2 + V(\varepsilon x)f(z)^2] \\ &\quad - \int_{\mathbb{R}^N} Q(\varepsilon x, f(w), f(z)), \end{aligned}$$

which is well-defined in  $X$ . More precisely,  $I_\varepsilon$  is of class  $C^1(X, \mathbb{R})$  (because of (A6), (A11), and the properties of  $f$ ). The Gateaux derivative is

$$\begin{aligned} &I'_\varepsilon(w, z)(\phi, \varphi) \\ &= \int_{\mathbb{R}^N} [\nabla w \nabla \phi + \nabla z \nabla \varphi] + \int_{\mathbb{R}^N} [W(\varepsilon x)f(w)f'(w)\phi + V(\varepsilon x)f(z)f'(z)\varphi] \\ &\quad - \int_{\mathbb{R}^N} [Q_w(\varepsilon x, f(w), f(z))f'(w)\phi + Q_z(\varepsilon x, f(w), f(z))f'(z)\varphi]. \end{aligned} \quad (2.2)$$

for all  $(w, z), (\phi, \varphi) \in X$ .

Let  $(w, z) \in X$  be a critical point of the functional  $I_\varepsilon$ . Then  $(w, z)$  constitutes a weak solution to the reformulated system below:

$$\begin{aligned} -\Delta w + W(\varepsilon x)f(w)f'(w) &= Q_w(\varepsilon x, f(w), f(z))f'(w), & \text{in } \mathbb{R}^N, \\ -\Delta z + V(\varepsilon x)f(z)f'(z) &= Q_z(\varepsilon x, f(w), f(z))f'(z), & \text{in } \mathbb{R}^N, \\ w, z > 0, & \quad w, z \in H^1(\mathbb{R}^N). \end{aligned} \quad (2.3)$$

**Proposition 2.2.** *If  $(w, z) \in X \cap [L^\infty_{\text{loc}}(\mathbb{R}^N)]^2$  is a critical point of  $I_\varepsilon$ , then  $(u, v) = (f(w), f(z))$  is a solution for (1.2).*

For a poof of the above propostion, see [19, Proposition 2.5].

To apply the variational method and find a solution to (1.2), we will use the *Penalization Method* developed by del Pino and Felmer [17], following the ideas of Alves [2]. Given our interest in securing a positive solution for (1.2), we assume that

$$Q(u, v) = 0 \quad \text{if } u \leq 0 \text{ or } v \leq 0. \tag{2.4}$$

Let us fix  $a > 0$  and let  $\eta : \mathbb{R} \rightarrow \mathbb{R}$  be a non-increasing  $C^1$  function satisfying

$$\eta \equiv 1 \quad \text{in } (-\infty, a], \quad \eta \equiv 0 \quad \text{in } [5a, +\infty), \quad \eta' \leq 0, \quad \text{and} \quad |\eta'| \leq \frac{C}{a}, \tag{2.5}$$

where the constant  $C$  is independent of  $a$ .

Using the function  $\eta$ , we define  $\widehat{Q} : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$\widehat{Q}(s, t) = \eta(|(s, t)|)Q(s, t) + [1 - \eta(|(s, t)|)]A(s^2 + t^2), \tag{2.6}$$

where

$$A := \max \left\{ \frac{Q(s, t)}{s^2 + t^2} : (s, t) \in \mathbb{R}^2, a \leq |(s, t)| \leq 5a \right\}. \tag{2.7}$$

Note that  $A > 0$  and  $A \rightarrow 0$  as  $a \rightarrow 0^+$ . Thus, we can assume that:

$$A < \frac{1}{4} \min\{W_0, V_0\}, \quad W(x) \geq W_0 > 0, \quad V(x) \geq V_0 > 0, \tag{2.8}$$

where  $W_0, V_0$  are obtained from (A6).

Now, fixing a bounded domain  $\Omega \subset \mathbb{R}^N$ , we define the function  $H : \mathbb{R}^N \times \mathbb{R}^2 \rightarrow \mathbb{R}$  by:

$$H(x, s, t) = \chi_\Omega(x)Q(s, t) + [1 - \chi_\Omega(x)]\widehat{Q}(s, t), \tag{2.9}$$

where  $\chi_\Omega$  denotes the characteristic function of  $\Omega$ .

**Lemma 2.3.** *The function  $H$  and its derivatives  $H_s$  and  $H_t$  satisfy the following properties:*

- (A16)  $pH(x, s, t) = sH_s(x, s, t) + tH_t(x, s, t)$  for each  $x \in \Omega$ .
- (A17)  $2H(x, s, t) \leq sH_s(x, s, t) + tH_t(x, s, t)$  for each  $x \in \mathbb{R}^N \setminus \Omega$ .
- (A18) Fixing  $k = \frac{4p}{p-2}$ , we can choose  $a > 0$  sufficiently small such that

$$sH_s(x, s, t) + tH_t(x, s, t) \leq \frac{1}{k}[W(x)s^2 + V(x)t^2], \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

$$\frac{|H_s(x, s, t)|}{a}, \quad \frac{|H_t(x, s, t)|}{a} \leq \frac{1}{4} \min\{W_0, V_0\}, \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

For a proof of the above lemma see [2, Lemma 2.2]. Now, our objective is to study the existence of solutions for the auxiliary system

$$\begin{aligned} -\Delta w + W(\varepsilon x)f(w)f'(w) &= H_w(\varepsilon x, f(w), f(z))f'(w), & \text{in } \mathbb{R}^N, \\ -\Delta z + V(\varepsilon x)f(z)f'(z) &= H_z(\varepsilon x, f(w), f(z))f'(z), & \text{in } \mathbb{R}^N, \\ w, z > 0, & \quad w, z \in H^1(\mathbb{R}^N). \end{aligned} \tag{2.10}$$

**Remark 2.4.** If  $(w, z)$  is a solution of (2.10) satisfying

$$|(f(w(x)), f(z(x)))| \leq a, \quad \forall x \in \mathbb{R}^N \setminus \Omega_\varepsilon,$$

then  $(w, z)$  will be a solution of (2.3), where  $\Omega_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Omega\}$ . Thus, our goal is to obtain solutions of (2.10) with this property.

Associated with the system (2.10), we define on  $X$  the functional

$$\begin{aligned} \Phi_\varepsilon(w, z) &= \frac{1}{2} \int_{\mathbb{R}^N} [|\nabla w|^2 + |\nabla z|^2 + W(\varepsilon x) f^2(w) + V(\varepsilon x) f^2(z)] \\ &\quad - \int_{\mathbb{R}^N} H(\varepsilon x, f(w), f(z)). \end{aligned} \quad (2.11)$$

Under conditions (A7) and (A11), it is possible to show that the functional  $\Phi_\varepsilon$  is of class  $C^1$  with Gateaux derivative

$$\begin{aligned} \Phi'_\varepsilon(w, z)(\phi, \varphi) &= \int_{\mathbb{R}^N} [\nabla w \nabla \phi + \nabla z \nabla \varphi] + \int_{\mathbb{R}^N} [W(\varepsilon x) f(w) f'(w) \phi + V(\varepsilon x) f(z) f'(z) \varphi] \\ &\quad - \int_{\mathbb{R}^N} [H_w(\varepsilon x, f(w), f(z)) f'(w) \phi + H_z(\varepsilon x, f(w), f(z)) f'(z) \varphi], \end{aligned} \quad (2.12)$$

for any  $(w, z), (\phi, \varphi) \in X$ . Therefore, the critical points of  $\Phi_\varepsilon$  are precisely the weak solutions of (2.10).

For each  $\rho > 0$ , consider the set

$$\Sigma_\rho = \{(w, z) \in X : \Psi(w, z) = \rho^2\}, \quad (2.13)$$

where

$$\Psi(w, z) = \int_{\mathbb{R}^N} [|\nabla w|^2 + |\nabla z|^2 + W(\varepsilon x) f^2(w) + V(\varepsilon x) f^2(z)]. \quad (2.14)$$

Since  $\Psi$  is continuous,  $\Sigma_\rho$  is a closed subset in  $X$  that disconnects  $X$  into

$$X_1 := \{(w, z) \in X : \Psi(w, z) > \rho^2\}, \quad X_2 := \{(w, z) \in X : \Psi(w, z) < \rho^2\}.$$

The following lemma guarantees that the functional  $\Phi_\varepsilon$  meets the geometric conditions required by the Mountain Pass Theorem.

**Lemma 2.5.** *The functional  $\Phi_\varepsilon$  satisfies the following conditions:*

- (i) *There exist constants  $\rho, \alpha > 0$  such that  $\Phi_\varepsilon(w, z) \geq \alpha$ , for all  $(w, z) \in \Sigma_\rho$ .*
- (ii) *For every  $\varepsilon \in (0, 1]$ , there exist  $(e_1, e_2) \in X_2$  such that  $\Phi_\varepsilon(e_1, e_2) \leq 0$ .*

*Proof.* Using (A16)–(A18), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} H(\varepsilon x, f(w), f(z)) &\leq \int_{\Omega_\varepsilon} H(\varepsilon x, f(w), f(z)) \\ &\quad + \frac{1}{2k} \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} [W(\varepsilon x) f(w)^2 + V(\varepsilon x) f(z)^2]. \end{aligned}$$

By (2.9) and (A12), we have

$$\begin{aligned} \int_{\mathbb{R}^N} H(\varepsilon x, f(w), f(z)) &\leq C \int_{\mathbb{R}^N} (|f^2(w)|^{p/2} + |f^2(z)|^{p/2}) \\ &\quad + \frac{1}{2k} \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} [W(\varepsilon x) f(w)^2 + V(\varepsilon x) f(z)^2]. \end{aligned} \quad (2.15)$$

Applying Hölder’s inequality and the embedding of  $D^{1,2}(\mathbb{R}^N)$  into  $L^{2^*}(\mathbb{R}^N)$ , we obtain

$$\begin{aligned}
 & \int_{\mathbb{R}^N} H(\varepsilon x, f(w), f(z)) \, dx \\
 & \leq C \left( \int_{\mathbb{R}^N} |f^2(w)|^2 \, dx \right)^{\sigma p/2} \left( \int_{\mathbb{R}^N} |f^2(w)|^{2^*} \, dx \right)^{1-\frac{\sigma p}{2}} \\
 & \quad + C \left( \int_{\mathbb{R}^N} |f^2(z)|^2 \, dx \right)^{\sigma p/2} \left( \int_{\mathbb{R}^N} |f^2(z)|^{2^*} \, dx \right)^{1-\frac{\sigma p}{2}} \\
 & \quad + \frac{1}{2k} \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} [W(\varepsilon x)f(w)^2 + V(\varepsilon x)f(z)^2] \, dx \\
 & \leq C\rho^{\sigma p} \left( \int_{\mathbb{R}^N} |\nabla(f^2(w))|^2 \, dx \right)^{\frac{(1-\frac{\sigma p}{2})2^*}{2}} \\
 & \quad + C\rho^{\sigma p} \left( \int_{\mathbb{R}^N} |\nabla(f^2(z))|^2 \, dx \right)^{\frac{(1-\frac{\sigma p}{2})2^*}{2}} \\
 & \quad + \frac{1}{2k} \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} [W(\varepsilon x)f(w)^2 + V(\varepsilon x)f(z)^2] \, dx
 \end{aligned} \tag{2.16}$$

where  $\sigma := \frac{2 \cdot 2^* - p}{(2^* - 1)p}$  and  $C > 0$ .

Noting that for each  $(w, z) \in \Sigma_\rho$ ,

$$\begin{aligned}
 \int_{\mathbb{R}^N} |\nabla(f^2(w))|^2 & \leq 2 \int_{\mathbb{R}^N} |\nabla w|^2 \leq 2\rho^2, \\
 \int_{\mathbb{R}^N} |\nabla(f^2(z))|^2 & \leq 2 \int_{\mathbb{R}^N} |\nabla z|^2 \leq 2\rho^2.
 \end{aligned} \tag{2.17}$$

From (2.11), (2.17), and (2.16), we obtain

$$\begin{aligned}
 \Phi_\varepsilon(w, z) & \geq \frac{k-1}{2k} \int_{\mathbb{R}^N} [|\nabla w|^2 + |\nabla z|^2 + W(\varepsilon x)f(w)^2 + V(\varepsilon x)f(z)^2] \, dx \\
 & \quad - \frac{C}{V_0^{\sigma p/2}} \rho^{\sigma p} C \rho^{(1-\frac{\sigma p}{2})\frac{2^*}{2}} - \frac{C}{W_0^{\sigma p/2}} \rho^{\sigma p} C \rho^{(1-\frac{\sigma p}{2})\frac{2^*}{2}}.
 \end{aligned}$$

Thus,

$$\Phi_\varepsilon(w, z) \geq \frac{k-1}{2k} \rho^2 C \rho^{(2N+2p)/(N+2)}, \quad \forall (w, z) \in \Sigma_\rho.$$

Since  $\frac{2N+2p}{N+2} > 2$  because  $p > 4$ , we can choose  $\alpha > 0$  sufficiently small, such that

$$\Phi_\varepsilon(w, z) \geq \alpha > 0, \quad \forall (w, z) \in \Sigma_\rho,$$

which proves (i). Now, note that, by condition **(H<sub>1</sub>)**, there exist constants  $C_3, C_4 > 0$  such that

$$H(\varepsilon x, s, t) \geq C_3 |(s, t)|^p - C_4, \quad \forall (x, s, t) \in \Omega \times \mathbb{R}^2. \tag{2.18}$$

Assume without loss of generality that  $0 \in \Omega$ . Let  $\phi \in C_0^\infty(\mathbb{R}^N, [0, 1])$  be such that  $\text{supp } \phi \subset B_r(0) \subset \Omega$ , for some  $r > 0$ . Since  $B_r(0) \subset \Omega_\varepsilon$  for all  $\varepsilon \in (0, 1]$ ,  $f(t\phi) \geq 0$  for all  $t \geq 0$ , and  $W(\varepsilon x) \leq W_\infty, V(\varepsilon x) \leq V_\infty$  in  $\mathbb{R}^N$ , by (2.18), for all  $t \geq 0$ , we have

$$\Phi_\varepsilon((t\phi, t\phi)) \leq \frac{t^2}{2} \int_{B_r(0)} [2|\nabla\phi|^2 + \frac{1}{2} \int_{B_r(0)} [W_\infty + V_\infty] f^2(t\phi)$$

$$- C_3 \int_{B_r(0)} |(f(t\phi), f(t\phi))|^p + C_4 |B_r(0)| \Big].$$

Thus,

$$\begin{aligned} \Phi_\varepsilon((t\phi, t\phi)) &\leq t^2 \left[ \frac{1}{2} \int_{B_r(0)} (2|\nabla\phi|^2 + [W_\infty + V_\infty]\phi^2) - C_3 \int_{B_r(0)} |f(t\phi)|^p \right] \\ &\quad + C_4 |B_r(0)|. \end{aligned}$$

From property (6) of Lemma 2.1, it follows that the function  $\frac{f(t)}{t}$  is decreasing for  $t > 0$ . Since  $0 \leq t\varphi \leq t$ , for all  $x \in \Omega_\varepsilon$  and  $t > 0$ , we obtain  $f(t)\varphi(x) \leq f(t\varphi(x))$ . Hence, for all  $\varepsilon \in (0, 1]$  and  $t \geq 0$ , we have

$$\begin{aligned} &\Phi_\varepsilon((t\phi, t\phi)) \\ &\leq t^2 \left[ \frac{1}{2} \int_{B_r(0)} (2|\nabla\phi|^2 + [W_\infty + V_\infty]\phi^2) - C_3 \frac{f^p(t)}{t^2} \int_{B_r(0)} |\phi|^p \right] + C_4 |B_r(0)|. \end{aligned} \tag{2.19}$$

From property (5) of Lemma 2.1 and since  $p > 4$ , we conclude that

$$\lim_{t \rightarrow +\infty} \frac{f^p(t)}{t^2} = \lim_{t \rightarrow +\infty} \left( \frac{f(t)}{\sqrt{t}} \right)^p t^{\frac{p}{2}-2} = +\infty. \tag{2.20}$$

Hence, by (2.19) and (2.20), the proof of (ii) is complete.  $\square$

### 3. EXISTENCE OF POSITIVE SOLUTIONS FOR THE AUXILIARY SYSTEM

By Theorem 1.4 and Lemma 2.5, there exists a Cerami sequence for  $\Phi_\varepsilon$  at the level

$$c_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} \Phi_\varepsilon(\gamma(t)), \tag{3.1}$$

where

$$\begin{aligned} \Gamma_\varepsilon &= \{ \gamma \in C([0, 1], X) : \gamma(0) = 0, \quad \gamma(1) \in \Phi_\varepsilon^{-1}((-\infty, 0]) \cap X_2 \}, \\ X_2 &= \{ (w, z) \in X : \Psi(w, z) < \rho^2 \}. \end{aligned}$$

That is, there exists  $(w_n, z_n) \subset X$ , such that

$$\Phi_\varepsilon(w_n, z_n) = c_\varepsilon + o_n(1), \quad \text{and} \quad (1 + \|(w_n, z_n)\|) \|\Phi'_\varepsilon(w_n, z_n)\|_* = o_n(1). \tag{3.2}$$

**Lemma 3.1.** *Every Cerami sequence  $(w_n, z_n)$  for  $\Phi_\varepsilon$  is bounded in  $X$ .*

*Proof.* Indeed, let

$$(\phi_n, \varphi_n) := \left( \frac{f(w_n)}{f'(w_n)}, \frac{f(z_n)}{f'(z_n)} \right).$$

As

$$\nabla\phi_n = \left( 1 + \frac{2f^2(w_n)}{1 + 2f^2(w_n)} \right) \nabla w_n, \quad \nabla\varphi_n = \left( 1 + \frac{2f^2(z_n)}{1 + 2f^2(z_n)} \right) \nabla z_n,$$

we have

$$\|\phi_n\| \leq 2\|w_n\|, \quad \|\varphi_n\| \leq 2\|z_n\|.$$

Hence, by (3.2),  $\Phi'_\varepsilon(w_n, z_n)(\phi_n, \varphi_n) = o_n(1)$ ; thus,

$$\begin{aligned} c_\varepsilon + o_n(1) &= \Phi_\varepsilon(w_n, z_n) - \frac{1}{p} \Phi'_\varepsilon(w_n, z_n)(\phi_n, \varphi_n) \\ &= \int_{\mathbb{R}^N} \left[ \frac{1}{2} - \frac{1}{p} \left( 1 + \frac{2f^2(w_n)}{1 + 2f^2(w_n)} \right) \right] |\nabla w_n|^2 \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{R}^N} \left[ \frac{1}{2} - \frac{1}{p} \left( 1 + \frac{2f^2(z_n)}{1 + 2f^2(z_n)} \right) \right] |\nabla z_n|^2 \\
& + \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} [W(\varepsilon x) f^2(w_n) + V(\varepsilon x) f^2(z_n)] \\
& + \frac{1}{p} \int_{\mathbb{R}^N} [f(w_n) H_w(\varepsilon x, f(w_n), f(z_n)) + f(z_n) H_z(\varepsilon x, f(w_n), f(z_n))] \\
& - \frac{1}{p} \int_{\mathbb{R}^N} p H(\varepsilon x, f(w_n), f(z_n)).
\end{aligned}$$

Since

$$1 + \frac{2f^2(t)}{1 + 2f^2(t)} \leq 2,$$

by (A16), we have

$$\begin{aligned}
c_\varepsilon + o_n(1) & \geq \left( \frac{1}{2} - \frac{2}{p} \right) \int_{\mathbb{R}^N} (|\nabla w_n|^2 + |\nabla z_n|^2) \\
& + \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} [W(\varepsilon x) f^2(w_n) + V(\varepsilon x) f^2(z_n)] \\
& + \frac{1}{p} \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} [f(w_n) H_w(\varepsilon x, f(w_n), f(z_n)) + f(z_n) H_z(\varepsilon x, f(w_n), f(z_n))] \\
& - \frac{1}{p} \int_{\mathbb{R}^N} p H(\varepsilon x, f(w_n), f(z_n)).
\end{aligned}$$

By (A17) and (A18), it follows that:

$$\begin{aligned}
c_\varepsilon + o_n(1) & \geq \frac{p-4}{2p} \int_{\mathbb{R}^N} (|\nabla w_n|^2 + |\nabla z_n|^2) \\
& + \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} [W(\varepsilon x) f^2(w_n) + V(\varepsilon x) f^2(z_n)] \\
& + \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} H(\varepsilon x, f(w_n), f(z_n)) \\
& \geq \frac{p-4}{2p} \int_{\mathbb{R}^N} (|\nabla w_n|^2 + |\nabla z_n|^2) \\
& + \left( \frac{1}{2} - \frac{1}{p} - \frac{1}{2k} \right) \int_{\mathbb{R}^N} [W(\varepsilon x) f^2(w_n) + V(\varepsilon x) f^2(z_n)].
\end{aligned}$$

Using  $k = \frac{4p}{p-2}$ , we obtain

$$c_\varepsilon + o_n(1) \geq c_p \int_{\mathbb{R}^N} [ (|\nabla w_n|^2 + |\nabla z_n|^2) + W(\varepsilon x) f^2(w_n) + V(\varepsilon x) f^2(z_n) ], \quad (3.3)$$

where

$$c_p := \min \left\{ \frac{p-4}{2p}, \frac{p-2}{4p} \right\} > 0, \quad \text{since } 4 < p < 2.2^*.$$

Thus, to show that  $(w_n, z_n)$  is bounded, it suffices to demonstrate the existence of a constant  $C > 0$  such that

$$\int_{\mathbb{R}^N} (W(\varepsilon x) w_n^2 + V(\varepsilon x) z_n^2) \leq C, \quad \forall n \in \mathbb{N}.$$

Using Lemma 2.1 and (3.3), we obtain

$$\int_{|w_n| \leq 1} W(\varepsilon x) w_n^2 \leq \frac{1}{C} \int_{|w_n| \leq 1} W(\varepsilon x) f(w_n)^2 \leq c + o_n(1).$$

Then

$$\begin{aligned} \int_{\{|w_n| > 1\}} W(\varepsilon x) w_n^2 &\leq W_\infty \int_{\mathbb{R}^N} w_n^{2^*} \\ &\leq W_\infty S \left( \int_{\mathbb{R}^N} |\nabla w_n|^2 \right)^{2^*/2} \\ &\leq W_\infty S(c_\varepsilon + o_n(1))^{2^*/2}. \end{aligned}$$

Similarly, we conclude that  $(V(\varepsilon x) z_n^2)$  is bounded in  $L^1(\mathbb{R}^N)$ . Therefore,  $(w_n, z_n)$  is bounded in  $X$ .  $\square$

**Lemma 3.2.** *Suppose that  $(w_n, z_n) \rightharpoonup (w, z)$  in  $X$ . Then:*

(i) *Given  $\xi > 0$ , there exists  $R > 0$  such that*

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N \setminus B_R(0)} [|\nabla w_n|^2 + |\nabla z_n|^2 + W(\varepsilon x) f^2(w_n) + V(\varepsilon x) f^2(z_n)] < \xi.$$

(ii)

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [W(\varepsilon x) f(w_n) f'(w_n) w_n + V(\varepsilon x) f(z_n) f'(z_n) z_n] \\ = \int_{\mathbb{R}^N} [W(\varepsilon x) f(w) f'(w) w + V(\varepsilon x) f(z) f'(z) z]. \end{aligned}$$

(iii)

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [H_w(\varepsilon x, f(w_n), f(z_n)) f'(w_n) w_n + H_z(\varepsilon x, f(w_n), f(z_n)) f'(z_n) z_n] \\ = \int_{\mathbb{R}^N} [H_w(\varepsilon x, f(w), f(z)) f'(w) w + H_z(\varepsilon x, f(w), f(z)) f'(z) z]. \end{aligned}$$

(iv) *If  $(w_n(x), z_n(x)) \rightarrow (w(x), z(x))$  a.e. in  $\mathbb{R}^N$  and*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [W(\varepsilon x) f^2(w_n) + V(\varepsilon x) f^2(z_n)] = \int_{\mathbb{R}^N} [W(\varepsilon x) f^2(w) + V(\varepsilon x) f^2(z)],$$

*then*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [W(\varepsilon x) f^2(w_n - w) + V(\varepsilon x) f^2(z_n - z)] = 0.$$

*Proof.* Consider the cutoff function  $\phi_R \in C^\infty(\mathbb{R}^N)$ , such that  $\phi_R = 0$  in  $B_{R/2}(0)$ ,  $\phi_R = 1$  in  $\mathbb{R}^N \setminus B_R(0)$ ,  $0 \leq \phi_R \leq 1$ , and  $|\nabla \phi_R| \leq C/R$ , where the constant  $C > 0$  is independent of  $R$ .

Since  $(w_n, z_n)$  is bounded, we have

$$\Phi'_\varepsilon(w_n, z_n) \left( \phi_R \frac{f(w_n)}{f'(w_n)}, \phi_R \frac{f(z_n)}{f'(z_n)} \right) = o_n(1).$$

Thus,

$$\int_{\mathbb{R}^N} \left( 1 + \frac{2f^2(w_n)}{1 + 2f^2(w_n)} \right) |\nabla w_n|^2 \phi_R + \int_{\mathbb{R}^N} \left( 1 + \frac{2f^2(z_n)}{1 + 2f^2(z_n)} \right) |\nabla z_n|^2 \phi_R$$

$$\begin{aligned}
& + \int_{\mathbb{R}^N} \left( \frac{f(w_n)}{f'(w_n)} \nabla w_n + \frac{f(z_n)}{f'(z_n)} \nabla z_n \right) \nabla \phi_R \\
& + \int_{\mathbb{R}^N} W(\varepsilon x) f^2(w_n) \phi_R + \int_{\mathbb{R}^N} V(\varepsilon x) f^2(z_n) \phi_R \\
& = \int_{\mathbb{R}^N} [H_w(\varepsilon x, f(w_n), f(z_n)) f(w_n) \phi_R + H_z(\varepsilon x, f(w_n), f(z_n)) f(z_n) \phi_R] + o_n(1).
\end{aligned}$$

Choosing  $R > 0$  such that  $\Omega_\varepsilon \subset B_{R/2}(0)$ , by (A17) and (A18), we have

$$\begin{aligned}
& \int_{\mathbb{R}^N} (|\nabla w_n|^2 + |\nabla z_n|^2) \phi_R + (1 - \frac{1}{k}) \int_{\mathbb{R}^N} [W(\varepsilon x) f^2(w_n) + V(\varepsilon x) f^2(z_n)] \phi_R \\
& \leq - \int_{\mathbb{R}^N} \left( \frac{f(w_n)}{f'(w_n)} \nabla w_n + \frac{f(z_n)}{f'(z_n)} \nabla z_n \right) \nabla \phi_R + o_n(1).
\end{aligned}$$

Using Lemma 2.1, the Cauchy-Schwarz inequality, the boundedness of  $(w_n, z_n)$ , and  $|\nabla \phi_R| \leq C/R$ , we obtain:

$$\begin{aligned}
& (1 - \frac{1}{k}) \int_{\mathbb{R}^N \setminus B_R(0)} [|\nabla w_n|^2 + |\nabla z_n|^2 + W(\varepsilon x) f^2(w_n) + V(\varepsilon x) f^2(z_n)] \\
& \leq 2 (\|w_n\|_{L^2(\mathbb{R}^N)} \|\nabla w_n \nabla \phi_R\| + \|z_n\|_{L^2(\mathbb{R}^N)} \|\nabla z_n \nabla \phi_R\|) \\
& \leq \frac{Mc}{R} + o_n(1),
\end{aligned}$$

which completes the proof of (i).

(ii) The proof of (ii) follows from (i) and the property  $f(t)f'(t) \leq f^2(t)$ , we have

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N \setminus B_R(0)} W(\varepsilon x) f(w_n) f'(w_n) w_n + V(\varepsilon x) f(z_n) f'(z_n) z_n = o_R(1). \quad (3.4)$$

Since  $w_n \rightarrow w$  and  $z_n \rightarrow z$  in  $L^s_{\text{loc}}(\mathbb{R}^N)$ , we have

$$\begin{aligned}
& w_n(x) \rightarrow w(x), \quad z_n(x) \rightarrow z(x) \quad \text{a.e. in } \mathbb{R}^N, \\
& |w_n(x)| \leq g_1(x), \quad |z_n(x)| \leq g_2(x), \quad g_1, g_2 \in L^s(B_R(0)), \quad s \in [1, 2^*).
\end{aligned}$$

By the Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \int_{B_R(0)} [W(\varepsilon x) f(w_n) f'(w_n) w_n + V(\varepsilon x) f(z_n) f'(z_n) z_n] \\
& = \int_{B_R(0)} [W(\varepsilon x) f(w) f'(w) w + V(\varepsilon x) f(z) f'(z) z].
\end{aligned} \quad (3.5)$$

From (3.4) and (3.5), we obtain item (ii).

(iii) Now, using item (i) and that  $\Phi'_\varepsilon(w_n, z_n)(w_n, z_n) = o_n(1)$ , we have

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N \setminus B_R(0)} [H_w(\varepsilon x, f(w_n), f(z_n)) f'(w_n) w_n \\
& + H_z(\varepsilon x, f(w_n), f(z_n)) f'(z_n) z_n] = o_R(1).
\end{aligned} \quad (3.6)$$

Using that  $\Omega_\varepsilon \subset B_R(0)$ , hypothesis (A18), the convergence  $w_n \rightarrow w$ ,  $z_n \rightarrow z$  in  $L^s_{\text{loc}}(\mathbb{R}^N)$ , and applying the Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{B_R(0)} [H_w(\varepsilon x, f(w_n), f(z_n))f'(w_n)w_n + H_z(\varepsilon x, f(w_n), f(z_n))f'(z_n)z_n] \\ &= \int_{B_R(0)} [H_w(\varepsilon x, f(w), f(z))f'(w)w + H_z(\varepsilon x, f(w), f(z))f'(z)z]. \end{aligned} \quad (3.7)$$

Thus, from (3.6) and (3.7), we obtain item (iii).  $\square$

**Proposition 3.3.** *The functional  $\Phi_\varepsilon$  satisfies the Cerami condition for each level  $c_\varepsilon$ .*

*Proof.* Let  $(w_n, z_n) \subset X$  be such that

$$\Phi_\varepsilon(w_n, z_n) = c_\varepsilon + o_n(1) \quad \text{and} \quad (1 + \|(w_n, z_n)\|)\|\Phi'_\varepsilon(w_n, z_n)\|_* = o_n(1).$$

By Lemma 3.1,  $(w_n, z_n)$  is bounded in  $X$ . Thus, up to a subsequence,  $(w_n, z_n) \rightharpoonup (w, z)$  in  $X$ . Furthermore, since

$$\Phi'_\varepsilon(w_n, z_n)(w_n, z_n) = o_n(1),$$

using (iii) from Lemma 3.2, we obtain

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [|\nabla w_n|^2 + |\nabla z_n|^2 + W(\varepsilon x)f(w_n)f'(w_n)w_n + V(\varepsilon x)f(z_n)f'(z_n)z_n] \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [H_w(\varepsilon x, f(w), f(z))f'(w)w + H_z(\varepsilon x, f(w), f(z))f'(z)z]. \end{aligned} \quad (3.8)$$

Now, using that  $\Phi'_\varepsilon(w_n, z_n)(\phi, \varphi) = o_n(1)$  for every  $\phi, \varphi \in C_0^\infty(\mathbb{R}^N)$ , by passing to the limit combined with the Lebesgue Dominated Convergence Theorem, we obtain  $\Phi'_\varepsilon(w, z)(\phi, \varphi) = 0$ . It follows that  $\Phi'_\varepsilon(w, z)(\phi, \varphi) = 0$  for every  $\phi, \varphi \in X$ . In particular,  $\Phi'_\varepsilon(w, z)(w, z) = 0$ , i.e.,

$$\begin{aligned} & \int_{\mathbb{R}^N} [|\nabla w|^2 + |\nabla z|^2 + W(\varepsilon x)f(w)f'(w)w + V(\varepsilon x)f(z)f'(z)z] \\ &= \int_{\mathbb{R}^N} [H_w(\varepsilon x, f(w), f(z))f'(w)w + H_z(\varepsilon x, f(w), f(z))f'(z)z]. \end{aligned} \quad (3.9)$$

Combining (3.8) and (3.9), we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [|\nabla w_n|^2 + |\nabla z_n|^2 + W(\varepsilon x)f(w_n)f'(w_n)w_n + V(\varepsilon x)f(z_n)f'(z_n)z_n] \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [|\nabla w|^2 + |\nabla z|^2 + W(\varepsilon x)f(w)f'(w)w + V(\varepsilon x)f(z)f'(z)z]. \end{aligned} \quad (3.10)$$

Using (ii) from Lemma 3.2 and (3.10), it follows that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [|\nabla w_n|^2 + |\nabla z_n|^2] = \int_{\mathbb{R}^N} [|\nabla w|^2 + |\nabla z|^2].$$

Since  $(w_n, z_n) \rightharpoonup (w, z)$  in  $X$ , we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [\nabla w_n \cdot \nabla w + \nabla z_n \cdot \nabla z] = \int_{\mathbb{R}^N} [|\nabla w|^2 + |\nabla z|^2].$$

Thus,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [|\nabla(w_n - w)|^2 + |\nabla(z_n - z)|^2] = 0. \tag{3.11}$$

Observing that

$$\begin{aligned} \|w_n - w\| &\leq C \left[ \Psi_W(w_n - w) + \Psi_W(w_n - w)^{2^*/2} \right], \\ \|z_n - z\| &\leq C \left[ \Psi_V(z_n - z) + \Psi_V(z_n - z)^{2^*/2} \right], \end{aligned}$$

where

$$\Psi_V(w_n - w) := \int_{\mathbb{R}^N} |\nabla(w_n - w)|^2 + \int_{\mathbb{R}^N} \mathcal{V}(\varepsilon x) f^2(w_n - w),$$

we conclude that

$$\begin{aligned} \|(w_n, z_n) - (w, z)\|^2 &\leq C \left[ \Psi_W(w_n - w) + \Psi_W(w_n - w)^{2^*/2} + \Psi_V(z_n - z) + \Psi_V(z_n - z)^{2^*/2} \right] \\ &\leq C \left[ \Psi_W(w_n - w) + \Psi_V(z_n - z) + (\Psi_W(w_n - w) + \Psi_V(z_n - z))^{2^*/2} \right]. \end{aligned}$$

Using (iv) from Lemma 3.2 and (3.11), up to a subsequence, we have  $(w_n, z_n) \rightarrow (w, z)$  in  $X$ . This completes the proof of the proposition.  $\square$

**Theorem 3.4.** *Suppose (A6), (A10)-(A15) are satisfied. Then, for every  $\varepsilon \in (0, 1]$ , the auxiliary system (2.10) has a weak solution  $(w_\varepsilon, z_\varepsilon) \in X$ , such that*

$$\Phi_\varepsilon(w_\varepsilon, z_\varepsilon) = c_\varepsilon \quad \text{and} \quad \|(w_\varepsilon, z_\varepsilon)\|^2 \leq C(c_\varepsilon + c_\varepsilon^{2^*/2}), \tag{3.12}$$

where  $C > 0$  is a constant independent of  $\varepsilon$ , and  $c_\varepsilon$  is defined by (3.1).

*Proof.* Using Lemma 2.5, Proposition 3.3, and Theorem 1.4, we conclude that the functional  $\Phi_\varepsilon$  has a critical point at

$$c_\varepsilon := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_\varepsilon(\gamma(t)) \geq \alpha,$$

where

$$\Gamma := \{ \gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) \in \Phi^{-1}((-\infty, 0]) \cap X_2 \},$$

and  $\alpha$  is given in Lemma 2.5. Thus, there exists  $(w_\varepsilon, z_\varepsilon) \in X$ , such that

$$\Phi_\varepsilon(w_\varepsilon, z_\varepsilon) = c_\varepsilon, \quad \text{and} \quad \Phi'_\varepsilon(w_\varepsilon, z_\varepsilon) = 0.$$

Therefore,  $(w_\varepsilon, z_\varepsilon)$  is a solution of (2.10).

Consider  $(\bar{w}_\varepsilon, \bar{z}_\varepsilon)$  as a test function and note that  $H(\varepsilon x, s, t) = 0$  for all  $s, t \leq 0$  and

$$W(\varepsilon x) f(w_\varepsilon) f'(w_\varepsilon) \bar{w}_\varepsilon, \quad V(\varepsilon x) f(z_\varepsilon) f'(z_\varepsilon) \bar{z}_\varepsilon \geq 0.$$

Then we obtain

$$\begin{aligned} \|\bar{w}_\varepsilon, \bar{z}_\varepsilon\|_{D^{1,2}(\mathbb{R}^N)}^2 &= \int_{\mathbb{R}^N} [|\nabla \bar{w}_\varepsilon|^2 + |\nabla \bar{z}_\varepsilon|^2] \\ &\leq \int_{\mathbb{R}^N} \nabla w_\varepsilon \nabla \bar{w}_\varepsilon + \int_{\mathbb{R}^N} W(\varepsilon x) f(w_\varepsilon) f'(w_\varepsilon) \bar{w}_\varepsilon \\ &\quad + \int_{\mathbb{R}^N} \nabla z_\varepsilon \nabla \bar{z}_\varepsilon + \int_{\mathbb{R}^N} V(\varepsilon x) f(z_\varepsilon) f'(z_\varepsilon) \bar{z}_\varepsilon \\ &= \int_{\mathbb{R}^N} H(\varepsilon x, f(w_\varepsilon)) f'(w_\varepsilon) \bar{w}_\varepsilon + \int_{\mathbb{R}^N} H(\varepsilon x, f(z_\varepsilon)) f'(z_\varepsilon) \bar{z}_\varepsilon \leq 0. \end{aligned}$$

Therefore,  $\|\bar{w}_\varepsilon, \bar{z}_\varepsilon\|_{D^{1,2}(\mathbb{R}^N)}^2 = 0$ . Hence,  $(\bar{w}_\varepsilon, \bar{z}_\varepsilon) = 0$ , and consequently,  $(w_\varepsilon, z_\varepsilon) = (w_\varepsilon^+, z_\varepsilon^+) \geq 0$  a.e. in  $\mathbb{R}^N$ . By elliptic regularity, we have  $(w_\varepsilon, z_\varepsilon) \in C^{1,\alpha}(\mathbb{R}^N)$  (see the proof of Lemma ??). Thus,  $w_\varepsilon > 0$  in  $\mathbb{R}^N$ .

I did not find Lemma 3.4

Proof of (3.12). Let

$$(\tilde{w}_\varepsilon, \tilde{z}_\varepsilon) = \left( \frac{f(w_\varepsilon)}{f'(w_\varepsilon)}, \frac{f(z_\varepsilon)}{f'(z_\varepsilon)} \right).$$

Since

$$\Phi_\varepsilon(w_\varepsilon, z_\varepsilon) - \frac{1}{p} \Phi'_\varepsilon(w_\varepsilon, z_\varepsilon)(\tilde{w}_\varepsilon, \tilde{z}_\varepsilon) = c_\varepsilon,$$

Using (A16), we obtain

$$\begin{aligned} c_\varepsilon &\geq \int_{\mathbb{R}^N} \left[ \frac{1}{2} - \frac{1}{p} \left( 1 + \frac{2f^2(w_\varepsilon)}{1+2f^2(w_\varepsilon)} \right) \right] |\nabla w_\varepsilon|^2 \\ &\quad + \int_{\mathbb{R}^N} \left[ \frac{1}{2} - \frac{1}{p} \left( 1 + \frac{2f^2(z_\varepsilon)}{1+2f^2(z_\varepsilon)} \right) \right] |\nabla z_\varepsilon|^2 \\ &\quad + \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} [W(\varepsilon x) f^2(w_\varepsilon) + V(\varepsilon x) f^2(z_\varepsilon)] \\ &\quad + \frac{1}{p} \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} [f(w_\varepsilon) H_w(\varepsilon x, f(w_\varepsilon), f(z_\varepsilon)) + f(z_\varepsilon) H_z(\varepsilon x, f(w_\varepsilon), f(z_\varepsilon))] \\ &\quad - \frac{1}{p} \int_{\mathbb{R}^N} p H(\varepsilon x, f(w_\varepsilon), f(z_\varepsilon)). \end{aligned}$$

Using the inequality  $1 + \frac{2f^2(w_\varepsilon)}{1+2f^2(w_\varepsilon)} \leq 2$  and (A18), we obtain:

$$\begin{aligned} c_\varepsilon &\geq \left( \frac{1}{2} - \frac{2}{p} \right) \int_{\mathbb{R}^N} [|\nabla w_\varepsilon|^2 + |\nabla z_\varepsilon|^2] \\ &\quad + \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} [W(\varepsilon x) f^2(w_\varepsilon) + V(\varepsilon x) f^2(z_\varepsilon)] \\ &\quad + \frac{1}{p} \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} [f(w_\varepsilon) H_w(\varepsilon x, f(w_\varepsilon), f(z_\varepsilon)) + f(z_\varepsilon) H_z(\varepsilon x, f(w_\varepsilon), f(z_\varepsilon))] \\ &\quad - \frac{1}{p} \int_{\mathbb{R}^N} p H(\varepsilon x, f(w_\varepsilon), f(z_\varepsilon)). \end{aligned}$$

which gives

$$\begin{aligned} c_\varepsilon &\geq \left( \frac{p-4}{2p} \right) \left[ \int_{\mathbb{R}^N} [|\nabla w_\varepsilon|^2 + |\nabla z_\varepsilon|^2 + W(\varepsilon x) f^2(w_\varepsilon) + V(\varepsilon x) f^2(z_\varepsilon)] \right] \\ &\quad - \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} H(\varepsilon x, f(w_\varepsilon), f(z_\varepsilon)). \end{aligned}$$

Since  $k = \frac{4p}{p-2}$ , by (A18), Lemma 2.1, and the above inequality, we have

$$\begin{aligned} c_\varepsilon &\geq \frac{1}{2k} \left[ \int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2 + |\nabla z_\varepsilon|^2 + C \int_{|w_\varepsilon|, |z_\varepsilon| \leq 1} W(\varepsilon x) w_\varepsilon^2 + V(\varepsilon x) z_\varepsilon^2 \right. \\ &\quad \left. + \int_{|w_\varepsilon|, |z_\varepsilon| > 1} W(\varepsilon x) f^2(w_\varepsilon) + V(\varepsilon x) f^2(z_\varepsilon) \right], \end{aligned} \tag{3.13}$$

for some constant  $C > 0$ , independent of  $\varepsilon$ .

Using (3.13) and the embedding  $H^1(\mathbb{R}^N) \subset D^{1,2}(\mathbb{R}^N)$ , we obtain

$$\int_{|w_\varepsilon|>1} W(\varepsilon x)|w_\varepsilon|^2 \leq W_\infty \int_{|w_\varepsilon|>1} |w_\varepsilon|^{2^*} \leq W_\infty S\left(\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2\right)^{2^*/2}. \tag{3.14}$$

Similarly, the conclusion holds for  $(V(\varepsilon x)z_\varepsilon^2)$ .

Combining (3.13) and (3.14), we obtain (3.12), which completes the proof.  $\square$

**Lemma 3.5.** *Let  $(w_\varepsilon, z_\varepsilon)$  be the solution of (2.10) obtained in Theorem 3.4, and let the sequences  $\varepsilon_n \in (0, 1)$  and  $(x_n) \subset \mathbb{R}^N$  be such that  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ . The sequences  $(\theta_n)$  and  $(\vartheta_n)$  defined by*

$$\theta_n(x) := w_{\varepsilon_n}(x + x_n), \quad \vartheta_n(x) := z_{\varepsilon_n}(x + x_n)$$

*belong to  $L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$  and have subsequences that converge uniformly over compact sets of  $\mathbb{R}^N$  to  $\theta, \vartheta \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ , respectively. Moreover, there exist constants  $C_1, C_2, C_3, C_4 > 0$  such that*

$$\theta(x) \leq C_1 \exp(-C_2|x|), \quad \vartheta(x) \leq C_3 \exp(-C_4|x|), \quad \forall x \in \mathbb{R}^N.$$

The proof of the above lemma follows from adapting the arguments used in the proof of [18, Lemma (4.1)], combined with [4, Corollary 4.3].

**Lemma 3.6.** *Suppose that  $W$  and  $V$  satisfy (A6), and that either  $W$  or  $V$  belongs to Class 1 or class 2. Furthermore, suppose that  $Q$  satisfies (A10)–(A15). Then*

$$m_\varepsilon := \max_{x \in \partial\Omega_\varepsilon} |(w_\varepsilon(x), z_\varepsilon(x))| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0^+,$$

where we define  $\Omega_\varepsilon := B_{R_\varepsilon/\varepsilon}$ .

*Proof.* Assume, by contradiction, that the lemma is not true. Then, there exist  $\delta > 0$  and a sequence  $\varepsilon_n \rightarrow 0^+$ , such that

$$m_{\varepsilon_n} \geq \delta, \quad \forall n \in \mathbb{N}.$$

Since  $w_{\varepsilon_n}, z_{\varepsilon_n} \in C^{1,\alpha}(\mathbb{R}^N)$ , there exist  $x_n \in \partial B_{R_{\varepsilon_n}/\varepsilon_n}$  such that

$$w_{\varepsilon_n}^2(x_n) + z_{\varepsilon_n}^2(x_n) \geq \delta^2, \quad \forall n \in \mathbb{N}. \tag{3.15}$$

We define the functions  $\theta_n, \vartheta_n : \mathbb{R}^N \rightarrow \mathbb{R}$  by

$$\theta_n(x) := w_{\varepsilon_n}(x + x_n), \quad \vartheta_n(x) := z_{\varepsilon_n}(x + x_n).$$

By Lemma 3.1, the sequence  $(w_{\varepsilon_n}, z_{\varepsilon_n})$  is bounded in  $X$ . Thus, by the invariance of  $\mathbb{R}^N$  by translation,  $(\theta_n, \vartheta_n)$  is also bounded in  $X$ . Moreover,  $(\theta_n, \vartheta_n)$  is a solution of the following system in  $\mathbb{R}^N$ :

$$\begin{aligned} -\Delta\theta_n + W(\varepsilon_n x + \varepsilon_n x_n)f(\theta_n)f'(\theta_n) &= H_w(\varepsilon_n x + \varepsilon_n x_n, f(\theta_n), f(\vartheta_n))f'(\theta_n), \\ -\Delta\vartheta_n + V(\varepsilon_n x + \varepsilon_n x_n)f(\vartheta_n)f'(\vartheta_n) &= H_z(\varepsilon_n x + \varepsilon_n x_n, f(\theta_n), f(\vartheta_n))f'(\vartheta_n), \\ \theta_n, \vartheta_n > 0, \quad \theta_n, \vartheta_n &\in H^1(\mathbb{R}^N). \end{aligned} \tag{3.16}$$

Note that, up to a subsequence,  $(\theta_n, \vartheta_n) \rightharpoonup (\theta, \vartheta)$  in  $X$ , for some  $(\theta, \vartheta) \in X$ . By Lemma 3.5,  $(\theta_n)$  and  $(\vartheta_n)$  converge uniformly over compact sets of  $\mathbb{R}^N$  to  $\theta$  and  $\vartheta$ , respectively. Moreover,  $\theta, \vartheta \in C(\mathbb{R}^N)$ . Thus, from this fact and the condition above, it follows that:

$$\theta^2(0) + \vartheta^2(0) \geq \delta^2.$$

Hence,

$$\theta \not\equiv 0 \quad \text{or} \quad \vartheta \not\equiv 0. \tag{3.17}$$

Since  $(W(\varepsilon_n x_n))$  and  $(V(\varepsilon_n x_n))$  are bounded, there exist  $\alpha_W, \alpha_V > 0$  such that

$$W(\varepsilon_n x_n) \rightarrow \alpha_W, \quad V(\varepsilon_n x_n) \rightarrow \alpha_V. \quad (3.18)$$

It follows from (3.16) that

$$\begin{aligned} & \int_{\mathbb{R}^N} \nabla \theta_n \nabla \phi + \int_{\mathbb{R}^N} W(\varepsilon_n x + \varepsilon_n x_n) f(\theta_n) f'(\theta_n) \phi \\ &= \int_{\mathbb{R}^N} H_w(\varepsilon_n x + \varepsilon_n x_n, f(\theta_n), f(\vartheta_n)) f'(\theta_n) \phi + o_n(1), \end{aligned} \quad (3.19)$$

and similarly,

$$\begin{aligned} & \int_{\mathbb{R}^N} \nabla \vartheta_n \nabla \varphi + \int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n x_n) f(\vartheta_n) f'(\vartheta_n) \varphi \\ &= \int_{\mathbb{R}^N} H_z(\varepsilon_n x + \varepsilon_n x_n, f(\theta_n), f(\vartheta_n)) f'(\vartheta_n) \varphi + o_n(1). \end{aligned} \quad (3.20)$$

Using (3.19) and (3.20), passing to the limit, and the density of  $C_0^\infty(\mathbb{R}^N)$  in  $H^1(\mathbb{R}^N)$ , we obtain

$$\int_{\mathbb{R}^N} [\nabla \theta \nabla \phi + \alpha_W f(\theta) f'(\theta) \phi] = \int_{\mathbb{R}^N} g_1(x, f(\theta), f(\vartheta)) f'(\theta) \phi, \quad (3.21)$$

$$\int_{\mathbb{R}^N} [\nabla \vartheta \nabla \varphi + \alpha_V f(\vartheta) f'(\vartheta) \varphi] = \int_{\mathbb{R}^N} g_2(x, f(\theta), f(\vartheta)) f'(\vartheta) \varphi, \quad (3.22)$$

for all  $(\phi, \varphi) \in X$ , where

$$\begin{aligned} g_1(x, f(\theta), f(\vartheta)) &:= \tilde{I}(x) Q_u(f(\theta), f(\vartheta)) + (1 - \tilde{I}(x)) \widehat{Q}_w(f(\theta), f(\vartheta)), \\ g_2(x, f(\theta), f(\vartheta)) &:= \tilde{I}(x) Q_v(f(\theta), f(\vartheta)) + (1 - \tilde{I}(x)) \widehat{Q}_z(f(\theta), f(\vartheta)), \end{aligned}$$

for some function  $\tilde{I} \in L^\infty(\mathbb{R}^N)$ .

Noting that  $(\theta, \vartheta) \in W_{\text{loc}}^{2,2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  and

$$\nabla(f f')(w) = (f')^2(w) \nabla w + f(w) f''(w) \nabla w, \quad f''(w) = -2f(w) [f'(w)]^4,$$

for all  $w \in H^1(\mathbb{R}^N)$ , by property (2) of Lemma 2.1, we have that  $(f f')(w) \in H^1(\mathbb{R}^N)$  for all  $w \in H^1(\mathbb{R}^N)$ . Thus, there exist  $\phi_j, \varphi_j \in C_0^\infty(\mathbb{R}^N)$  such that

$$\|\phi_j - (f f')(\theta)\| \leq \frac{1}{j}, \quad \|\varphi_j - (f f')(\vartheta)\| \leq \frac{1}{j}, \quad \forall j \in \mathbb{N}. \quad (3.23)$$

We assert that (3.17) and (3.21) imply that  $\theta \not\equiv 0$  and  $\vartheta \not\equiv 0$ . Indeed, suppose, by contradiction, that  $\theta \not\equiv 0$  and  $\vartheta \equiv 0$ . Since  $f(0) = 0$ , by (2.4), (3.15), and (3.19),

$$\int_{\mathbb{R}^N} |\nabla \theta|^2 + \alpha_W \int_{\mathbb{R}^N} f(\theta) f'(\theta) \theta = \int_{\mathbb{R}^N} g_1(x, f(\theta), 0) f'(\theta) \theta = 0.$$

Using that  $\theta \geq 0$ , we conclude that  $\theta \equiv 0$ , and thus,  $(\theta, \vartheta) = (0, 0)$ , which contradicts (3.17). A similar conclusion is obtained if we consider  $\theta \equiv 0$  and  $\vartheta \not\equiv 0$ . Thus, the assertion is true.

Now, suppose  $W$  satisfies the (PS) condition, that is,  $W$  belongs to Class 1. Choosing  $\frac{\partial \phi_j}{\partial x_i}$  as a test function in (3.19), we have

$$\begin{aligned} & \int_{\mathbb{R}^N} \nabla \theta_n \nabla \left( \frac{\partial \phi_j}{\partial x_i} \right) + \int_{\mathbb{R}^N} W(\varepsilon_n x + \varepsilon_n x_n) f(\theta_n) f'(\theta_n) \frac{\partial \phi_j}{\partial x_i} \\ & - \int_{\mathbb{R}^N} H_u(\varepsilon_n x + \varepsilon_n x_n, f(\theta_n), f(\vartheta_n)) f'(\theta_n) \frac{\partial \phi_j}{\partial x_i} = o_n(1). \end{aligned}$$

Thus, exploring the fact that  $\theta \neq 0$  and arguing, we can conclude that

$$\nabla W(\varepsilon_n x_n) \rightarrow 0, \quad W(\varepsilon_n x_n) \rightarrow \alpha_W.$$

Thus,  $(\varepsilon_n x_n)$  is a  $(PS)_{\alpha_W}$  sequence for  $W$ . Therefore, from (A8),  $(\varepsilon_n x_n)$  should have a convergent subsequence, but

$$|\varepsilon_n x_n| = R_{\varepsilon_n} = \frac{1}{\varepsilon_n} \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Therefore, the lemma is true for  $W$  in Class 1. The case where  $V$  belongs to Class 1 is analogous. □

**Lemma 3.7.** *Suppose that  $W$  and  $V$  satisfy (A6) and that either  $W$  or  $V$  belongs to Class 2. Furthermore, suppose that  $Q$  satisfies (A10)-(A15). Then*

$$m_\varepsilon := \max_{x \in \partial\Omega_\varepsilon} |(u_\varepsilon(x), v_\varepsilon(x))| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0^+,$$

where we define  $\Omega_\varepsilon := \frac{1}{\varepsilon} \Lambda$  with  $\Lambda$  given in hypothesis (A9).

*Proof.* Assume, for the sake of contradiction, that the statement does not hold. Then, there exists a constant  $\delta > 0$  and a sequence  $\varepsilon_n \rightarrow 0^+$  such that

$$\max_{x \in \partial\Omega_{\varepsilon_n}} |(u_{\varepsilon_n}(x), v_{\varepsilon_n}(x))| \geq \delta, \quad \forall n \in \mathbb{N}.$$

This implies that for each  $n$ , there exists a point  $x_n \in \partial\Omega_{\varepsilon_n}$  satisfying

$$|(u_{\varepsilon_n}(x_n), v_{\varepsilon_n}(x_n))| \geq \delta.$$

We define the translated functions

$$\theta_n(x) := u_{\varepsilon_n}(x + x_n), \quad \vartheta_n(x) := v_{\varepsilon_n}(x + x_n).$$

Since  $(u_{\varepsilon_n}, v_{\varepsilon_n})$  is a solution of the auxiliary system (2.10), the pair  $(\theta_n, \vartheta_n)$  satisfies the system

$$\begin{aligned} -\Delta\theta_n + W(\varepsilon_n x + \varepsilon_n x_n)f(\theta_n)f'(\theta_n) &= H_w(\varepsilon_n x + \varepsilon_n x_n, f(\theta_n), f(\vartheta_n))f'(\theta_n), \\ -\Delta\vartheta_n + V(\varepsilon_n x + \varepsilon_n x_n)f(\vartheta_n)f'(\vartheta_n) &= H_z(\varepsilon_n x + \varepsilon_n x_n, f(\theta_n), f(\vartheta_n))f'(\vartheta_n). \end{aligned}$$

From the boundedness of  $(u_{\varepsilon_n}, v_{\varepsilon_n})$  in  $H^1(\mathbb{R}^N)$ , the sequences  $(\theta_n, \vartheta_n)$  are also bounded in  $H^1(\mathbb{R}^N)$ . Specifically, there exists a constant  $C > 0$  such that

$$\|\theta_n\|_{H^1(\mathbb{R}^N)} \leq C, \quad \|\vartheta_n\|_{H^1(\mathbb{R}^N)} \leq C, \quad \forall n \in \mathbb{N}.$$

By the Sobolev embedding theorem, up to a subsequence,  $(\theta_n, \vartheta_n)$  converges weakly in  $H^1(\mathbb{R}^N)$  and strongly in  $L^p_{\text{loc}}(\mathbb{R}^N)$ , for  $p \in [2, 2^*]$ , to some  $(\theta, \vartheta) \in H^1(\mathbb{R}^N)$ .

The strong convergence in  $L^p_{\text{loc}}(\mathbb{R}^N)$  implies that for each compact set  $K \subset \mathbb{R}^N$ , we have

$$\lim_{n \rightarrow \infty} \int_K |\theta_n(x) - \theta(x)|^p dx = 0, \quad \lim_{n \rightarrow \infty} \int_K |\vartheta_n(x) - \vartheta(x)|^p dx = 0.$$

In particular, for  $K = B_1(0)$  (the unit ball centered at the origin), we have

$$\lim_{n \rightarrow \infty} \int_{B_1(0)} |\theta_n(x) - \theta(x)|^2 dx = 0, \quad \lim_{n \rightarrow \infty} \int_{B_1(0)} |\vartheta_n(x) - \vartheta(x)|^2 dx = 0.$$

From the assumption  $|(u_{\varepsilon_n}(x_n), v_{\varepsilon_n}(x_n))| \geq \delta$ , we have

$$\theta_n(0) = u_{\varepsilon_n}(x_n), \quad \vartheta_n(0) = v_{\varepsilon_n}(x_n).$$

Since  $(\theta_n, \vartheta_n)$  converges strongly in  $L^p_{loc}(\mathbb{R}^N)$ , it follows that

$$\theta(0)^2 + \vartheta(0)^2 = \lim_{n \rightarrow \infty} (\theta_n(0)^2 + \vartheta_n(0)^2) \geq \delta^2.$$

This implies that at least one of  $\theta$  or  $\vartheta$  is not identically zero. Without loss of generality, assume  $\theta \not\equiv 0$ .

Using the convergence of  $(\theta_n, \vartheta_n)$  and the continuity of  $W, V$ , and  $H$ , we can pass to the limit in the weak formulation of the system. This yields

$$\begin{aligned} -\Delta\theta + \alpha_W f(\theta)f'(\theta) &= g_1(x, f(\theta), f(\vartheta))f'(\theta), \\ -\Delta\vartheta + \alpha_V f(\vartheta)f'(\vartheta) &= g_2(x, f(\theta), f(\vartheta))f'(\vartheta), \end{aligned}$$

where

$$\alpha_W = \lim_{n \rightarrow \infty} W(\varepsilon_n x_n), \quad \alpha_V = \lim_{n \rightarrow \infty} V(\varepsilon_n x_n),$$

and  $g_1, g_2$  are the limits of  $H_w$  and  $H_z$ , respectively.

Since  $W$  or  $V$  belongs to Class 2, we have (A9), which implies that  $\nabla V(x) \neq 0$  for all  $x \in \partial\Lambda$ . Now, observe that the sequence  $(\varepsilon_n x_n)$  lies on  $\partial\Lambda_{\varepsilon_n}$ , and as  $\varepsilon_n \rightarrow 0$ ,  $\varepsilon_n x_n \rightarrow x_0 \in \partial\Lambda$ , and by the continuity of  $\nabla V$ , we have  $\nabla V(x_0) = 0$ . However, this contradicts (A9), which requires that  $\nabla V(x_0) \neq 0$  for all  $x_0 \in \partial\Lambda$ .

The contradiction implies that our initial assumption must be false. Therefore, we conclude that

$$\max_{x \in \partial\Omega_\varepsilon} |(u_\varepsilon(x), v_\varepsilon(x))| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0^+. \quad \square$$

#### 4. PROOF OF THEOREM 1.2

*Proof.* Suppose, by contradiction, that there exists  $y_\varepsilon \in \mathbb{R}^N \setminus \Omega_\varepsilon$  such that

$$w_\varepsilon(y_\varepsilon) \geq f^{-1}\left(\frac{a}{2}\right).$$

Combining the previous lemma with the fact that  $|(w_\varepsilon(x), z_\varepsilon(x))| \rightarrow 0$  as  $|x| \rightarrow +\infty$ , see Lemma 3.6, we conclude that there exists a maximum point  $x_\varepsilon \in \mathbb{R}^N \setminus \Omega_\varepsilon$  for  $w_\varepsilon$ .

Since  $(w_\varepsilon, z_\varepsilon) \in C^{2,\alpha}_{loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  is a solution of (2.10), we have

$$W(\varepsilon x_\varepsilon) f(w_\varepsilon(x_\varepsilon)) f'(w_\varepsilon(x_\varepsilon)) = H_w(\varepsilon x_\varepsilon, f(w_\varepsilon(x_\varepsilon)), f(z_\varepsilon(x_\varepsilon))) f'(w_\varepsilon(x_\varepsilon)).$$

Using that  $f$  is an increasing function and  $f' > 0$  in  $(0, \infty)$ , we obtain

$$W_0 \frac{a}{2} \leq H_w(\varepsilon x_\varepsilon, f(w_\varepsilon(x_\varepsilon)), f(z_\varepsilon(x_\varepsilon))),$$

which contradicts hypothesis (A18).

Thus, we conclude that

$$w_\varepsilon(x) < f^{-1}\left(\frac{a}{2}\right) \quad \text{in } \mathbb{R}^N \setminus \Omega_\varepsilon.$$

Similarly, we have

$$z_\varepsilon(x) < f^{-1}\left(\frac{a}{2}\right) \quad \text{in } \mathbb{R}^N \setminus \Omega_\varepsilon.$$

Hence,

$$|(f(w_\varepsilon), f(z_\varepsilon))| < a \quad \text{in } \mathbb{R}^N \setminus \Omega_\varepsilon.$$

Therefore, by the definition of  $H$ ,  $(w_\varepsilon, z_\varepsilon)$  is also a solution of (2.3). □

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