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EXISTENCE AND FORMS OF ENTIRE SOLUTIONS TO SYSTEM OF NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. The main objective of this article is to explore the existence and forms of transcendental entire solutions of some systems of non-linear partial differential equations. We obtain two results and illustrate the results with several examples. This article improves the results in [5, 15]. In the last section we discuss the differences between the solutions involving homogeneous and non-homogeneous operators, and state an open question for the sake of future research.

1. Introduction

The development of the difference analogue of the Nevanlinna theory [4, 11] has greatly influenced the study of difference and difference-differential equations. Naturally, this topic has become a central focus for many researchers in the field. In 1966, Gross [6] studied the existence and form of transcendental entire solution of the equation $f(z)^m+g(z)^m=1$, and settled the problem for $m=2$ and pointed out that the equation does not possess any non-constant transcendental entire solution if $m>2$. This significant result opened new avenues for further exploration about the existence and form of transcendental entire solutions for variants of classical Fermat-type equations. In course of time, this line of research has gained momentum, leading to a number of interesting results by many researchers, thereby enriching the field.

Theorem 1.1 ([12]). *For any two positive integers m and n with $m \neq n$, the equation*
$$f'(z)^n + f(z+c)^m = 1,$$
has no transcendental entire solution with finite order.

Theorem 1.2 ([12]). *The finite order transcendental entire solution of*
$$f'(z)^2 + f(z+c)^2 = 1,$$
must satisfy $f(z)=\sin(z\pm Bi)$, where B is a constant and $c=2k\pi$ or $c=(2k+1)\pi$, k is an integer.

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In 2017, Gao [5] investigated the existence and form of transcendental entire solutions, for the following systems of Fermat-type equations:

$$\begin{aligned} f_1'(z)^{n_1} + f_2(z+c)^{m_1} &= Q_1(z),\\ f_2'(z)^{n_2} + f_1(z+c)^{m_2} &= Q_2(z), \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} f_1'(z)^2 + f_2(z+c)^2 &= Q_1(z),\\ f_2'(z)^2 + f_1(z+c)^2 &= Q_2(z), \end{aligned} \tag{1.2}$$

where $Q_j(z)$, $j = 1, 2$ are non-zero polynomials in $\mathbb{C}$. For systems (1.1) and (1.2), the following results were obtained:

Theorem 1.3 ([5]). *There does not exist any finite order transcendental entire solutions $(f_1(z), f_2(z))$ of (1.1) if any of the following conditions is satisfied:*
(i) $m_1m_2 > n_1n_2$;
(ii) $m_j > \frac{n_j}{n_j-1}$, $j = 1, 2$.

Theorem 1.4 ([5]). *Let $(f_1(z), f_2(z))$ be a finite order transcendental entire solution of (1.2) in $\mathbb{C}$. Then $Q_1(z) = c_{11}c_{12}$, $Q_2(z) = c_{21}c_{22}$ and*

$$f_1(z) = \frac{c_{11}e^{az+b_1} - c_{12}e^{-az-b_1}}{2a}, \quad f_2(z) = \frac{c_{21}e^{az+b_2} - c_{21}e^{-az-b_2}}{2a},$$

where $a^4 = 1$, b_1, b_2, $c_{kj}(\neq 0)$, $k, j = 1, 2$ are constants.

In 2018, Xu-Cao [15] investigated the existence and form of transcendental entire solutions of shift-differential equation in $\mathbb{C}^2$ to obtain the following result.

Theorem 1.5 ([15]). *Let $c = (c_1, c_2)$ be a non-zero constant in $\mathbb{C}^2$. Then the Fermat-type partial differential equation*

$$\Big(\frac{\partial f(z_1, z_2)}{\partial z_1}\Big)^n + f(z_1 + c_1, z_2 + c_2)^m = 1,$$

does not have a finite order transcendental entire solution whenever m, n are two distinct positive integer.

Theorem 1.6 ([15, 16]). *Let $c = (c_1, c_2)$ be a non-zero constant in $\mathbb{C}^2$, then each finite order transcendental entire solution of the Fermat-type partial differential equation*

$$\Big(\frac{\partial f(z_1, z_2)}{\partial z_1}\Big)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1,$$

has the form $f(z_1, z_2) = sin(Az_1 + Bz_2 + H(z_2))$, where A, B are constants in $\mathbb{C}$ satisfying $A^2 = 1$, $e^{i(Ac_1+Bc_2)} = 1$ and $H(z_2)$ is a polynomial in one variable z_2 such that $H(z_2) = H(z_2 + c_2)$. In special case whenever $c_2 \neq 0$, we have $f(z_1, z_2) = sin(Az_1 + Bz_2 + constant)$.

Motivated by the results several authors made contribution to this field; see [1, 14], [17]-[21] and the references therein.

2. Formulation of main problem and relevant examples

To proceed further, we introduce the following differential-operator

Definition 2.1. The partial differential operator P_{L_n} in $\mathbb{C}^n$ is defined as

$$P_{L_n} = \sum_{|J|=1}^{n} b_{j_1\dots j_n} \frac{\partial^{|J|}}{\partial z_1^{j_1}\dots\partial z_n^{j_n}},$$

where $b_{j_1\dots j_n} \neq 0$ are constants in $\mathbb{C}$ and $J = (j_1, j_2, \dots, j_n)$, $|J| = \sum_{t=1}^{n} j_t$. From now onwards, we use $\vec{z_n} = (z_1, z_2, \dots, z_n)$, $\vec{c_n} = (c_1, c_2, \dots, c_n)$ and $\vec{z_n} + \vec{c_n} = (z_1 + c_1, z_2 + c_2, \dots, z_n + c_n)$ and $\vec{0} = (0,\ 0, \dots,\ 0)$.

This article is based on exploring existence of finite order transcendental entire solutions in n ($n \geq 1$) dimensional complex field of the equations

$$\begin{aligned}(P_{L_n}(f_1(\vec{z_n})))^{l_1} + f_2(\vec{z_n} + \vec{c_n})^{k_1} &= Q_1(\vec{z_n})\\ (P_{L_n}(f_2(\vec{z_n})))^{l_2} + f_1(\vec{z_n} + \vec{c_n})^{k_2} &= Q_2(\vec{z_n});\end{aligned} \tag{2.1}$$

where $Q_j(\vec{z_n})$, $j = 1, 2$ are two non-zero polynomials in $\mathbb{C}^n$ and are of finite order transcendental entire solution for $n = 2$, i.e. in $\mathbb{C}^2$ of the equations

$$\begin{aligned}(P_{L_2}(f_1(\vec{z_2})))^2 + f_2(\vec{z_2} + \vec{c_2})^2 &= 1\\ (P_{L_2}(f_2(\vec{z_2})))^2 + f_1(\vec{z_2} + \vec{c_2})^2 &= 1.\end{aligned} \tag{2.2}$$

Theorem 2.2. *Let $\vec{c} = (c_1, c_2, \dots, c_n)$ be a non-zero constant in $\mathbb{C}^n$. Then* (2.1) *can not have a finite order transcendental entire solution $(f_1(\vec{z_n}), f_2(\vec{z_n}), \dots, f_n(\vec{z_n}))$ if the exponents satisfy one of the following two conditions:*

(i) $k_1 k_2 > l_1 l_2$;
(ii) $k_t > \frac{l_t}{l_t - 1}$ *for* $l_t \geq 2$, $t = 1, 2$.

The above theorem motivate us to explore the case $l_t = 1$, and $k_t = 1$; $t = 1,\ 2$. In this respect, the following example shows that the solution exists.

Example 2.3. Let $l_1 = 1$, $l_2 = 1$, $k_1 = 1$, $k_2 = 1$, $b_{10} = 1$, $b_{01} = 1$, $b_{11} = 1$, $b_{20} = 1$, $b_{02} = 1$, $Q_1(\vec{z_2}) = 1$, $Q_2(\vec{z_2}) = 1$. Then $f(z) = (f_1(\vec{z_2}), f_2(\vec{z_2}))$, where $f_j(\vec{z_2}) = e^{z_1+z_2} + 1$, $j = 1, 2$ is a solution of (2.1) when $e^{c_1+c_2} = -5$.

For the sake of convenience and to proceed further we us use the following expressions

$$\begin{gathered}A_1(r,s) = -b_{10}s + b_{01}r + b_{11}(d_2 s - d_1 r) + 2b_{20}d_1 s - 2b_{02}d_2 r,\\ A_2(r,s) = b_{10}s - b_{01}r + b_{11}(d_2 s - d_1 r) + 2b_{20}d_1 s - 2b_{02}d_2 r,\\ B(r,s) = -b_{11}rs + b_{20}s^2 + b_{02}r^2,\\ D_1(r,s) = -b_{10}r - b_{01}s + b_{11}rs + b_{20}r^2 + b_{02}s^2,\\ D_2(r,s) = b_{10}r + b_{01}s + b_{11}rs + b_{20}r^2 + b_{02}s^2,\\ L(r,s) = d_1 r + d_2 s,\end{gathered}$$

where r, s are parameters and d_1, d_2 are two constants in $\mathbb{C}$.

Theorem 2.4. *Let $(c_1, c_2) \neq (0,0) \in \mathbb{C}^2$ be a constant and $(f_1(\vec{z_2}), f_2(\vec{z_2}))$ be a finite order transcendental entire solution of* (2.2) *in $\mathbb{C}^2$. Also let $B(c_1, c_2)$,*

$A_1(c_1,c_2)$ and $B(c_1,c_2)$, $A_2(c_1,c_2)$ be nonzero simultaneously. Then $(f_1(\vec{z_2})f_2(\vec{z_2}))$ takes one of the following form:

(A) When

$$D_1(d_1,d_2)D_2(d_1,d_2)=1, \quad e^{2L(\vec{c_2})}=\frac{D_2(d_1,d_2)}{D_1(d_1,d_2)},$$
$$e^{2(W_1+W_2)}=-1, \quad e^{W_1+W_2}=\frac{i}{D_1(d_1,d_2)}e^{-L(\vec{c_2})},$$

we have

$$f_1(\vec{z_2})=\frac{e^{-L(\vec{z_2})+L(\vec{c_2})+W_2}-e^{L(\vec{z_2})-L(\vec{c_2})-W_2}}{2i},$$
$$f_2(\vec{z_2})=\frac{e^{L(\vec{z_2})-L(\vec{c_2})+W_1}-e^{-L(\vec{z_2})+L(\vec{c_2})-W_1}}{2i},$$

where W_1, W_2 are two constants in $\mathbb{C}$.

(B) When

$$D_1(d_1,d_2)D_2(d_1,d_2)=1, \quad e^{2L(\vec{c_2})}=-\frac{D_2(d_1,d_2)}{D_1(d_1,d_2)}, e^{2(W_1-W_2)}=1,$$
$$e^{W_1-W_2}=-\frac{i}{D_1(d_1,d_2)}e^{-L(\vec{c_2})},$$

$$f_1(\vec{z_2})=\frac{e^{L(\vec{z_2})-L(\vec{c_2})+W_2}-e^{-L(\vec{z_2})+L(\vec{c_2})-W_2}}{2i},$$
$$f_2(\vec{z_2})=\frac{e^{L(\vec{z_2})-L(\vec{c_2})+W_1}-e^{-L(\vec{z_2})+L(\vec{c_2})-W_1}}{2i},$$

where W_1, W_2 are two constants in $\mathbb{C}$.

The following examples justify Theorem 2.4.

Example 2.5. *Let $b_{10}=-2i$, $b_{01}=i$, $b_{11}=-2i$, $b_{20}=i$, $b_{02}=i$, $d_1=1$, $d_2=1$, $c_1=\frac{\pi i}{4}$, $c_2=\frac{\pi i}{4}$, $W_1=\frac{-\pi i}{4}$, $W_2=\frac{-\pi i}{4}$. Then*

$$(f_1(\vec{z}),f_2(\vec{z_2}))=\Big(\frac{-ie^{-z_1-z_2+\frac{\pi i}{4}}+ie^{z_1+z_2-\frac{\pi i}{4}}}{2},-\frac{e^{-z_1-z_2+\frac{\pi i}{4}}+e^{z_1+z_2-\frac{\pi i}{4}}}{2}\Big)$$

is a solution of (2.2).

Example 2.6. Let $b_{10}=2i$, $b_{01}=i$, $b_{11}=2i$, $b_{20}=i$, $b_{02}=i$, $d_1=1$, $d_2=-1$, $c_1=\frac{\pi i}{4}$, $c_2=-\frac{\pi i}{4}$, $W_1=\frac{\pi i}{4}$, $W_2=\frac{\pi i}{4}$. Then

$$(f_1(\vec{z}),f_2(\vec{z_2}))=\Big(\frac{ie^{-z_1+z_2-\frac{\pi i}{4}}-ie^{z_1-z_2+\frac{\pi i}{4}}}{2},-\frac{e^{-z_1+z_2-\frac{\pi i}{4}}+e^{z_1-z_2+\frac{\pi i}{4}}}{2}\Big)$$

is a solution of (2.2).

Example 2.7. Let $b_{10}=2i$, $b_{01}=i$, $b_{11}=2i$, $b_{20}=i$, $b_{02}=i$, $d_1=1$, $d_2=-1$, $c_1=2\pi i$, $c_2=-2\pi i$, $W_1=0$, $W_2=0$. Then

$$(f_1(\vec{z_2}),f_2(\vec{z_2}))=\Big(\frac{ie^{-z_1+z_2}-ie^{z_1-z_2}}{2},\frac{ie^{-z_1+z_2}-ie^{z_1-z_2}}{2}\Big)$$

is a solution of (2.2).

Example 2.8. Let $b_{10} = 2i$, $b_{01} = i$, $b_{11} = 2i$, $b_{20} = i$, $b_{02} = i$, $d_1 = 1$, $d_2 = -1$, $c_1 = 1$, $c_2 = 1$, $W_1 = 0$, $W_2 = 0$. Then

$$(f_1(\vec{z}), f_2(\vec{z_2})) = \Big(\frac{ie^{-z_1+z_2} - ie^{z_1-z_2}}{2}, \frac{ie^{-z_1+z_2} - ie^{z_1-z_2}}{2}\Big)$$

is a solution of (2.2).

Example 2.9. Let $b_{10} = i$, $b_{01} = 2i$, $b_{11} = 2i$, $b_{20} = i$, $b_{02} = i$, $d_1 = 1$, $d_2 = -1$, $c_1 = \pi i$, $c_2 = -\pi i$, $W_1 = \frac{\pi i}{2}$, $W_2 = -\frac{\pi i}{2}$. Then

$$(f_1(\vec{z_2}), f_2(\vec{z_2})) = \Big(\frac{-ie^{-z_1+z_2} + ie^{z_1-z_2}}{2}, \frac{ie^{-2z_1+z_2} - ie^{z_1-z_2}}{2}\Big)$$

is a solution of (2.2).

Example 2.10. Let $b_{10} = -2i$, $b_{01} = -i$, $b_{11} = 2i$, $b_{20} = i$, $b_{02} = i$, $d_1 = 1$, $d_2 = -1$, $c_1 = 1$, $c_2 = 1$, $W_1 = 2\pi i$, $W_2 = \pi i$. Then

$$(f_1(\vec{z}), f_2(\vec{z_2})) = \Big(\frac{-ie^{-z_1+z_2} + ie^{z_1-z_2}}{2}, \frac{ie^{-z_1+z_2} - ie^{z_1-z_2}}{2}\Big)$$

is a solution of (2.2).

Example 2.11. Let $b_{10} = 1$, $b_{01} = 1$, $b_{11} = 1$, $b_{20} = 1$, $b_{02} = 1$, $d_1 = 1$, $d_2 = -1$, $c_1 = c_2 = 1$, $W_1 = \frac{\pi i}{4}$, $W_2 = \frac{\pi i}{4}$. Then

$$(f_1(\vec{z}), f_2(\vec{z_2})) = \Big(\frac{e^{-z_1+z_2-\frac{\pi i}{4}} + e^{z_1-z_2+\frac{\pi i}{4}}}{2}, \frac{ie^{-z_1+z_2-\frac{\pi i}{4}} - ie^{z_1-z_2+\frac{\pi i}{4}}}{2}\Big)$$

is a solution of (2.2).

Example 2.12. Let $b_{10} = 1$, $b_{01} = 2$, $b_{11} = 2$, $b_{20} = 1$, $b_{02} = 1$, $d_1 = 2$, $d_2 = -1$, $c_1 = 2\pi i$, $c_2 = 2\pi i$, $W_1 = \frac{\pi i}{4}$, $W_2 = \frac{\pi i}{4}$. Then

$$(f_1(\vec{z}), f_2(\vec{z_2})) = \Big(\frac{e^{-2z_1+z_2-\frac{\pi i}{4}} + e^{2z_1-z_2+\frac{\pi i}{4}}}{2}, \frac{ie^{-2z_1+z_2-\frac{\pi i}{4}} - ie^{2z_1-z_2+\frac{\pi i}{4}}}{2}\Big)$$

is a solution of (2.2).

Example 2.13. Let $b_{10} = 1$, $b_{01} = 2$, $b_{11} = 2$, $b_{20} = 1$, $b_{02} = 1$, $d_1 = 2i$, $d_2 = -i$, $c_1 = i$, $c_2 = 2i$, $W_1 = -\frac{\pi i}{4}$, $W_2 = -\frac{\pi i}{4}$. Then

$$(f_1(\vec{z}), f_2(\vec{z_2})) = \Big(-\frac{e^{-2iz_1+iz_2+\frac{\pi i}{4}} + e^{2iz_1-iz_2-\frac{\pi i}{4}}}{2}, \frac{ie^{-2iz_1+iz_2+\frac{\pi i}{4}} - ie^{2iz_1-iz_2-\frac{\pi i}{4}}}{2}\Big)$$

is a solution of (2.2).

Example 2.14. Let $b_{10} = 1$, $b_{01} = 2$, $b_{11} = 2$, $b_{20} = 1$, $b_{02} = 1$, $d_1 = 2i$, $d_2 = -i$, $c_1 = 2\pi$, $c_2 = 2\pi$, $W_1 = -\frac{\pi i}{4}$, $W_2 = -\frac{\pi i}{4}$, Then

$$(f_1(\vec{z}), f_2(\vec{z_2})) = \Big(-\frac{e^{-2iz_1+iz_2+\frac{\pi i}{4}} + e^{2iz_1-iz_2-\frac{\pi i}{4}}}{2}, \frac{ie^{-2iz_1+iz_2+\frac{\pi i}{4}} - ie^{2iz_1-iz_2-\frac{\pi i}{4}}}{2}\Big)$$

is a solution of (2.2).

Example 2.15. Let $b_{10} = 1$, $b_{01} = 2$, $b_{11} = 2$, $b_{20} = 1$, $b_{02} = 1$, $d_1 = 2$, $d_2 = -1$, $c_1 = \frac{\pi i}{2}$, $c_2 = -\frac{\pi i}{2}$, $W_1 = \frac{\pi i}{2}$, $W_2 = \frac{\pi i}{2}$. Then

$$(f_1(\vec{z}), f_2(\vec{z_2})) = \Big(\frac{e^{-2z_1+z_2} + e^{2z_1-z_2}}{2}, \frac{e^{-2z_1+z_2} + e^{2z_1-z_2}}{2}\Big)$$

is a solution of (2.2).

Example 2.16. Let $b_{10} = b_{01} = -\frac{6}{5}$, $b_{11} = \frac{1}{5}$, $b_{20} = \frac{8}{5}$, $b_{02} = \frac{4}{5}$, $d_1 = 1$, $d_2 = 1$, $c_1 = 1$, $c_2 = \log(\frac{1}{5}) - 1$, $W_1 = W_2 = \frac{\pi i}{4}$. Then

$$(f_1(\vec{z}), f_2(\vec{z_2})) = \Big(\frac{\frac{1}{5}e^{-z_1-z_2+\frac{\pi i}{4}} - 5e^{z_1+z_2-\frac{\pi i}{4}}}{2i}, \frac{5e^{z_1+z_2+\frac{\pi i}{4}} - \frac{1}{5}e^{-z_1-z_2-\frac{\pi i}{4}}}{2i}\Big)$$

is a solution of (2.2).

Corollary 2.17. *Let $(f_1(\vec{z_2}), f_2(\vec{z_2}))$ be a transcendental entire function of order properly greater than one with $B(c_1, c_2)$, $A_1(c_1, c_2)$ and $A_2(c_1, c_2)$ are not zero simultaneously. Then $(f_1(\vec{z_2}), f_2(\vec{z_2}))$ can not be a solution of (2.2).*

3. Lemmas

We assume that the readers are familiar with the basic notations of the Nevanlinna theory such as $N(r, f)$, $N(r, \frac{1}{f})$, $m(r, f)$, $T(r, f)$ in complex variable [7]. For several complex variables we refer to [10] and the references therein. By $S(r, f)$ we will mean any quantity satisfying $S(r, f) = o(T(r, f))$, $r \to \infty$, outside possibly an exceptional set of finite logarithmic measure. Based on the notations, the following lemmas will play important role in proving our theorems.

Lemma 3.1 ([13])**.** *For each entire function F in $\mathbb{C}^n$, $F(\vec{0}) \neq \vec{0}$ and put $\rho(n_F) = \rho < \infty$. Then there exists a canonical function f_F and a function $g_F \in \mathbb{C}^n$ such that $F(z) = f_{F(z)} e^{g_F(\vec{z}_n)}$. For special case $n = 1$, f_F is the canonical product of Weierstrass. Here $\rho(n_f)$ denotes the order of the counting function of zeros of F.*

Lemma 3.2 ([22, 2])**.** *Let $f(z)$ be a non-constant meromorphic function in $\mathbb{C}^n$ and let $I = (i_1, \dots, i_n)$ be a multi index with length $|I| = \sum_{j=1}^n i_j$. Assume that $T(r_0, f) \geq e$ for some r_0. Then*

$$m\Big(r, \frac{\partial^I f}{f}\Big) = S(r, f),$$

holds for all $r \geq r_0$, outside a set $E \subset (0, +\infty)$ of finite logarithmic measure $\int_E \frac{dt}{t} < \infty$, where $\partial^I f = \frac{\partial^I f}{\partial z_1^{i_1} \dots \partial z_n^{i_n}}$.

Lemma 3.3 ([9])**.** *Let $f_j (\not\equiv 0)$, $j = 1, 2, 3$ be meromorphic function in $\mathbb{C}^n$ such that f_1 is not constant, $f_1 + f_2 + f_3 = 1$ and*

$$\sum_{j=1}^{3} \Big\{N_2\big(r, \frac{1}{f_j}\big) + 2\overline{N}(r, f_j)\Big\} < \lambda T(r, f_1) + o(log^+ T(r, f_1)),$$

for all r outside possibly a set with finite logarithmic measure, where $\lambda < 1$ is a positive number, then either $f_2 \equiv 1$ or $f_3 \equiv 1$.

Lemma 3.4 ([3])**.** *Let $f(\vec{z}_n)$ be a non-constant meromorphic function with finite order in $\mathbb{C}^n$ such that $f(\vec{0}) \neq 0, \infty$ and let $\epsilon > 0$. Then, for $\vec{c}_n \in \mathbb{C}^n$,*

$$m\Big(r, \frac{f(\vec{z}_n)}{f(\vec{z}_n + \vec{c}_n)}\Big) + m\Big(r, \frac{f(\vec{z}_n + \vec{c}_n)}{f(\vec{z}_n)}\Big) = S(r, f),$$

holds for all $r \geq r_0$, outside a set $E \subset (0, +\infty)$ of finite logarithmic measure $\int_E \frac{dt}{t} < \infty$.

Lemma 3.5 ([9, Lemma 3.1]). *Suppose that* $a_0(\vec{z}_m), a_1(\vec{z}_m), \dots, a_n(\vec{z}_m)$, $n \geq 1$, *are meromorphic in* $\mathbb{C}^m$ *and* $g_0(\vec{z}_m),\ g_2(\vec{z}_m),\ \dots, g_n(\vec{z}_m)$ *are entire in* $\mathbb{C}^m$. $g_j(\vec{z}_m) - g_k(\vec{z}_m)$ *are non-constant for* $0 \leq j < k \leq n$. *If*

$$\sum_{j=0}^{n} a_j(\vec{z}_m) e^{g_j(\vec{z}_m)} \equiv 0$$

and $T(r, a_j) = o\left(T(r)\right)$, $j = 0, 1, 2 \dots, n$,

$$T(r) = \min_{0 \leq j < k \leq n} T(r,\ e^{g_k - g_j}),$$

then $a_j \equiv 0$.

4. Proof of the main results

Proof of Theorem 2.2. Let $(f_1(\vec{z_n}), f_2(\vec{z_n}), \dots, f_n(\vec{z_n}))$ be a finite order transcendental entire solution of (2.1) in $\mathbb{C}^n$. We consider the following 2 cases:

Case 1: Let $k_1 k_2 > l_1 l_2$. Using Lemma 3.4, we have that

$$m\Big(r, \frac{f_j(\vec{z_n})}{f_j(\vec{z_n} + \vec{c_n})}\Big) = S(r, f_j), \tag{4.1}$$

holds for all $r > 0$ outside a possible set $E_j \subset [1, +\infty)$, $j = 1, 2, \dots, n$ of finite logarithmic measure $\int_{E_j} \frac{dt}{t} < \infty$. Clearly, we have the following

$$\begin{aligned} T(r, f_j(\vec{z_n})) &= m(r, f_j(\vec{z_n})), \\ &\leq m\Big(r, \frac{f_j(\vec{z_n})}{f_j(\vec{z_n} + \vec{c_n})} \times f_j(\vec{z_n} + \vec{c_n})\Big), \\ &\leq m\Big(r, \frac{f_j(\vec{z_n})}{f_j(\vec{z_n} + \vec{c_n})}\Big) + m\left(r, f_j(\vec{z_n} + \vec{c_n})\right) + \log 2, \\ &= m(r, f_j(\vec{z_n} + \vec{c_n})) + \log 2 + S(r, f_j), \\ &= T(r, f_j(\vec{z_n} + \vec{c_n})) + \log 2 + S(r, f_j), \end{aligned} \tag{4.2}$$

for all $r \notin E_1 \cup E_2$. Applying Valliron Mohon'ko theorem in several complex variables [8] we have

$$\begin{aligned} k_1 T(r, f_2(\vec{z_n})) &\leq k_1 T(r, f_2(\vec{z_n} + \vec{c_n})) + S(r, f_2) \\ &\leq T(r, f_2(\vec{z_n} + \vec{c_n}))^{k_1}) + S(r, f_2), \\ &= T\left(r, (P_{L_n}(f_1(\vec{z_n})))^{l_1} - Q_1(\vec{z_n})\right) + S(r, f_2), \\ &= l_1 T\left(r, P_{L_n}(f_1(\vec{z_n}))\right) + S(r, f_1) + S(r, f_2), \\ &= l_1 m\left(r, P_{L_n}(f_1(\vec{z_n}))\right) + S(r, f_1) + S(r, f_2), \\ &\leq l_1 \Big[m\Big(r, \frac{P_{L_n}(f_1(\vec{z_n}))}{f_1(\vec{z_n})}\Big) + m(r, f_1(\vec{z_n})) + \log 2\Big] \\ &\quad + S(r, f_1) + S(r, f_2), \\ &= l_1 T(r, f_1(\vec{z_n}) + S(r, f_1) + S(r, f_2), \end{aligned} \tag{4.3}$$

i.e. from (4.3) wee obtain

$$(k_1 + o(1)) T(r, f_2(\vec{z_n})) \leq (l_1 + o(1)) T(r, f_1(\vec{z_n})), \quad r \notin E_1. \tag{4.4}$$

Similarly,we obtain

$$(k_2+o(1))T(r,f_2(\vec{z_n}))\le (l_2+o(1))T(r,f_1(\vec{z_n})),\quad r\notin E_2. \tag{4.5}$$

From (4.4) and (4.5) clearly we have a contradiction.

Case 2: Let $k_t>\frac{l_t}{l_t-1}$, $l_t\ge 2$, $t=1,2$. Using the Nevanlinna second main theorem, from (2.1) we obtain

$$\begin{aligned}
&(l_1-1)T\left(r,P_{L_n}(f_1(\vec{z_n}))\right)\\
&\le \overline{N}\Big(r,P_{L_n}(f_1(\vec{z_n}))\Big)+\overline{N}\Big(r,\frac{1}{(P_{L_n}(f_1(\vec{z_n})))^{l_1}-Q_1(\vec{z_n})}\Big)+S\left(r,P_{L_n}(f_1)\right),\\
&\le \overline{N}\Big(r,\frac{1}{f_2(\vec{z_n}+\vec{c_n})}\Big)+S(r,f_1),\\
&\le T(r,f_2(\vec{z_n}+\vec{c_n}))+S(r,f_1),\\
&\le T(r,f_2(\vec{z_n}))+S(r,f_1)+S(r,f_2).
\end{aligned} \tag{4.6}$$

Proceeding with the similar arguments, from the second equation we obtain

$$(l_2-1)T\left(r,P_{L_n}f_2((\vec{z_n}))\le T(r,f_1(\vec{z_n}))+S(r,f_1)+S(r,f_2). \tag{4.7}$$

From the first equation of (2.1) and using Valliron Mohon'ko theorem in several complex variables [8] we obtain

$$\begin{aligned}
k_1T(r,f_2(\vec{z_n}+\vec{c_n}))&=T(r,(P_{L_n}(f_1(\vec{z_n})))^{l_1}-Q_1(\vec{z_n}))+S(r,f_1)\\
&\le l_1T\left(r,P_{L_n}(f_1(\vec{z_n}))\right)+S(r,f_1)+S(r,f_1).
\end{aligned} \tag{4.8}$$

Proceeding, in the similar way from the second equation of (2.1) we obtain

$$k_2T(r,f_1(\vec{z_n}+\vec{c_n}))\le l_2T\left(r,P_{L_n}(f_2(\vec{z_n}))+S(r,f_1)+S(r,f_2). \tag{4.9}$$

From (4.6)-(4.9) we obtain

$$\begin{gathered}
\Big(k_1-\frac{l_1}{l_1-1}+o(1)\Big)T(r,f_2(\vec{z_n}))\le S(r,f_1),\\
\Big(k_2-\frac{l_2}{l_2-1}+o(1)\Big)T(r,f_1(\vec{z_n}))\le S(r,f_2).
\end{gathered}$$

Since $(f_1(\vec{z_n}),f_2(\vec{z_n}),\dots,f_n(\vec{z_n}))$ is a transcendental entire function, we obtain

$$\Big(k_1-\frac{l_1}{l_1-1}+o(1)\Big)\Big(k_2-\frac{l_2}{l_2-1}+o(1)\Big)\le 0.$$

Since $k_t>\frac{l_t}{l_t-1}$, $t=1,2$, we have a contradiction. The proof of Theorem 2.2 is complete □

The following expression is used several times to prove the next theorem.

$$\begin{aligned}
M_{m,u}(p)=b_{10}\frac{\partial p_m(\vec{z_2})}{\partial z_1}+b_{01}\frac{\partial p_m(\vec{z_2})}{\partial z_2}+b_{11}\Big\{\frac{\partial^2 p_m(\vec{z_2})}{\partial z_1\partial z_2}\\
+(-1)^{u-1}\frac{\partial p_m(\vec{z_2})}{\partial z_1}\frac{\partial p_m(\vec{z_2})}{\partial z_2}\Big\}+b_{20}\Big\{\frac{\partial^2 p_m(\vec{z_2})}{\partial z_1^2}\\
+(-1)^{u-1}\Big(\frac{\partial p_m(\vec{z_2})}{\partial z_1}\Big)^2\Big\}+b_{02}\Big\{\frac{\partial^2 p_m(\vec{z_2})}{\partial z_2^2}+(-1)^{u-1}\Big(\frac{\partial p_m(\vec{z_2})}{\partial z_2}\Big)^2\Big\},
\end{aligned}$$

for $m,u=1,2$.

Proof of Theorem 2.4. Let $(f_1(\vec{z_2}), f_2(\vec{z_2}))$ be a pair of finite order transcendental entire solution of (2.2) in $\mathbb{C}^2$. Clearly, system (2.2) can be re-written as follows

$$\begin{aligned}
&\{P_{L_2}(f_1(\vec{z_2}))+if_2(\vec{z_2}+\vec{c_2})\}\{P_{L_2}(f_1(\vec{z_2}))-if_2(\vec{z_2}+\vec{c_2})\}=1,\\
&\{P_{L_2}(f_2(\vec{z_2})+if_1(\vec{z_2}+\vec{c_2})\}\{P_{L_2}(f_2(\vec{z_2}))-if_1(\vec{z_{2_2}}+\vec{c_2})\}=1.
\end{aligned}\tag{4.10}$$

Now using Lemma 3.1, from (4.10) we obtain

$$\begin{aligned}
P_{L_2}(f_1(\vec{z_2}))+if_2(\vec{z_2}+\vec{c_2})&=e^{p_1(\vec{z_2})},\\
P_{L_2}(f_1(\vec{z_2}))-if_2(\vec{z_2}+\vec{c_2})&=e^{-p_1(\vec{z_2})},\\
P_{L_2}(f_2(\vec{z_2}))+if_1(\vec{z_2}+\vec{c_2})&=e^{p_2(\vec{z_2})},\\
P_{L_2}(f_2(\vec{z_2})-if_1(\vec{z_2}+\vec{c_2})&=e^{-p_2(\vec{z_2})},
\end{aligned}\tag{4.11}$$

where $p_1(\vec{z_2})$, $p_2(\vec{z_2})$ are two non-constant polynomials in $\mathbb{C}^2$. By an easy computation from (4.11), we obtain

$$\begin{aligned}
P_{L_2}(f_1(\vec{z_2}))&=\frac{e^{p_1(\vec{z_2})}+e^{-p_1(\vec{z_2})}}{2},\\
f_2(\vec{z_2}+\vec{c_2})&=\frac{e^{p_1(\vec{z_2})}-e^{-p_1(\vec{z_2})}}{2i},\\
P_{L_2}(f_2(\vec{z_2}))&=\frac{e^{p_2(\vec{z_2})}+e^{-p_2(\vec{z_2})}}{2},\\
f_1(\vec{z_2}+\vec{c_2})&=\frac{e^{p_2(\vec{z_2})}-e^{-p_2(\vec{z_2})}}{2i}.
\end{aligned}\tag{4.12}$$

Combining the first and the last equations, and the second and the third equations of (4.12) we obtain respectively

$$-iM_{2,1}e^{p_1(\vec{z_2}+\vec{c_2})+p_2(\vec{z_2})}-iM_{2,2}e^{p_1(\vec{z_2}+\vec{c_2})-p_2(\vec{z_2})}-e^{2p_1(\vec{z_2}+\vec{c_2})}=1,\tag{4.13}$$

and

$$-iM_{1,1}e^{p_2(\vec{z_2}+\vec{c_2})+p_1(\vec{z_2})}-iM_{1,2}e^{p_2(\vec{z_2}+\vec{c_2})-p_1(\vec{z_2})}-e^{2p_2(\vec{z_2}+\vec{c_2})}=1.\tag{4.14}$$

Now taking into consideration equation (4.13), we discuss the following possibilities:

(i) Let $M_{2,1}\equiv 0$, $M_{2,2}\equiv 0$. Then we have $-e^{2p_1(\vec{z_2}+\vec{c_2})}=1$, which shows that $p_1(\vec{z_2})$ is a constant polynomial, a contradiction.

(ii) Let $M_{2,1}\equiv 0$ and $M_{2,2}\not\equiv 0$. Then we have

$$-iM_{2,2}e^{p_1(\vec{z_2}+\vec{c_2})-p_2(\vec{z_2})}-e^{2p_1(\vec{z_2}+\vec{c_2})}=1.\tag{4.15}$$

Since $p_1(\vec{z_2})$ is a non-constant polynomial, (4.15) implies that $p_1(\vec{z_2}+\vec{c_2})-p_2(\vec{z_2})$ is also non-constant. We claim that $-p_2(\vec{z_2})-p_1(\vec{z_2}+\vec{c_2})$ is also non-constant. On the contrary, let $-p_2(\vec{z_2})-p_1(\vec{z_2}+\vec{c_2})=A_1'$, where A_1' is a constant in $\mathbb{C}$.

Then from (4.15) we obtain

$$\begin{gathered}
-iM_{2,2}e^{A_1'+2p_1(\vec{z_2}+\vec{c_2})}-e^{2p_1(\vec{z_2}+\vec{c_2})}=1,\\
\text{i.e. } (iM_{2,2}e^{A_1'}+1)e^{2p_1(\vec{z_2}+\vec{c_2})}=-1.
\end{gathered}$$

Then we have $p_1(\vec{z_2})$ is a constant polynomial, a contradiction. Clearly, we can rewrite (4.15) as

$$-iM_{2,2}e^{-p_2(\vec{z_2})}-e^{p_1(\vec{z_2}+\vec{c_2})}-e^{-p_1(\vec{z_2}+\vec{c_2})}=0.\tag{4.16}$$

Now applying Lemma 3.5 in (4.16) we have $M_{2,2}\equiv 0$, a contradiction.

(iii) Let $M_{2,1} \not\equiv 0$ and $M_{2,2} \equiv 0$. Then proceeding in the similar way as done in case (ii) we obtain a contradiction.

So we must have $M_{2,1} \not\equiv 0$ and $M_{2,2} \not\equiv 0$. Using similar arguments from (4.14) we obtain $M_{1,1} \not\equiv 0$ and $M_{1,2} \not\equiv 0$. Hence using Lemma 3.3, in (4.13) and (4.14) we obtain

$$\begin{gathered}
-iM_{2,1}e^{p_1(\vec{z_2}+\vec{c_2})+p_2(\vec{z_2})} \equiv 1 \text{ or } -iM_{2,2}e^{p_1(\vec{z_2}+\vec{c_2})-p_2(\vec{z_2})} \equiv 1; \\
-iM_{1,1}e^{p_2(\vec{z_2}+\vec{c_2})+p_1(\vec{z_2})} \equiv 1 \text{ or } -iM_{1,2}e^{p_2(\vec{z_2}+\vec{c_2})-p_1(\vec{z_2})} \equiv 1,
\end{gathered}$$

respectively.

Now we consider the following four cases:

Case 1:

$$\begin{gathered}
-iM_{2,1}e^{p_1(\vec{z_2}+\vec{c_2})+p_2(\vec{z_2})} \equiv 1, \\
-iM_{1,1}e^{p_2(\vec{z_2}+\vec{c_2})+p_1(\vec{z_2})} \equiv 1.
\end{gathered}$$

Clearly we have $p_1(\vec{z_2}+\vec{c_2}) + p_2(\vec{z_2}) \equiv \eta_1$, $p_2(\vec{z_2}+\vec{c_2}) + p_1(z) \equiv \eta_2$, where η_1, η_2 are two constants in $\mathbb{C}$. Then we have $p_1(\vec{z_2}) = L(\vec{z_2}) + H(s) + W_1$, $p_2(\vec{z_2}) = -L(\vec{z_2}) - H(s) + W_2$, where W_1, W_2 are two constants in $\mathbb{C}$, $H(s)$ is a polynomial in $s = c_2z_1 - c_1z_2$. Now combining with (4.13) and (4.14) we obtain

$$\begin{aligned}
& b_{10}\left(-d_1 - H'(s)c_2\right) + b_{01}\left(-d_2 + H'(s)c_1\right) \\
& + b_{11}\{H''(s)c_1c_2 + \left(-d_1 - H'(s)c_2\right)\left(-d_2 + H'(s)c_1\right)\} \\
& + b_{20}\{-H''(s)c_2^2 + \left(-d_1 - H'(s)c_2\right)^2\} \\
& + b_{02}\{-H''(s)c_1^2 + \left(-d_2 + H'(s)c_1\right)^2\}e^{L(\vec{c_2})+W_1+W_2} \equiv i, \\
& b_{10}\left(d_1 + H'(s)c_2\right) + b_{01}\left(d_2 - H'(s)c_1\right) \\
& + b_{11}\{-H''(s)c_1c_2 + \left(d_1 + H'(s)c_2\right)\left(d_2 - H'(s)c_1\right)\} + b_{20}\{H''(s)c_2^2 \\
& + \left(d_1 + H'(s)c_2\right)^2\} + b_{02}\{H''(s)c_1^2 + \left(d_2 - H'(s)c_1\right)^2\}e^{-L(\vec{c_2})+W_1+W_2} \equiv i, \\
& b_{10}\left(-d_1 - H'(s)c_2\right) + b_{01}\left(-d_2 + H'(s)c_1\right) \\
& + b_{11}\{H''(s)c_1c_2 - \left(-d_1 - H'(s)c_2\right)\left(-d_2 + H'(s)c_1\right)\} \\
& + b_{20}\{-H''(s)c_2^2 - \left(-d_1 - H'(s)c_2\right)^2\} \\
& + b_{02}\{-H''(s)c_1^2 - \left(-d_2 + H'(s)c_1\right)^2\}e^{-L(\vec{c_2})-W_1-W_2} \equiv i, \\
& b_{10}\left(d_1 + H'(s)c_2\right) + b_{01}\left(d_2 - H'(s)c_1\right) \\
& + b_{11}\{-H''(s)c_1c_2 - \left(d_1 + H'(s)c_2\right)\left(d_2 - H'(s)c_1\right)\} \\
& + b_{20}\{H''(s)c_2^2 - \left(d_1 + H'(s)c_2\right)^2\} \\
& + b_{02}\{H''(s)c_1^2 - \left(d_2 - H'(s)c_1\right)^2\}e^{L(\vec{c_2})-W_1-W_2} \equiv i.
\end{aligned} \tag{4.17}$$

We note that coefficient of $H'(s)$ of the first, second, third, and fourth equations are $A_1(c_1, c_2)$, $A_2(c_1, c_2)$, $-A_2(c_1, c_2)$, and $-A_1(c_1, c_2)$ respectively. Also, coefficients of $H'(s)^2$ of the first, second, third, and fourth equations are $B(c_1, c_2)$, $B(c_1, c_2)$, $-B(c_1, c_2)$, and $-B(c_1, c_2)$ respectively. Further, the coefficients of $H''(s)$ of the first, second, third, and fourth equations are $-B(c_1, c_2)$, $B(c_1, c_2)$, $-B(c_1, c_2)$, and $B(c_1, c_2)$ respectively.

Then (4.17) reduces to

$$\begin{aligned} &[D_1(d_1,d_2)+A_1(c_1,c_2)H'(s)+B(c_1,c_2)\{H'(s)^2-H''(s)\}]\\ &\times e^{L(\vec{c_2})+W_1+W_2}\equiv i,\\ &[D_2(d_1,d_2)+A_2(c_1,c_2)H'(s)+B(c_1,c_2)\{H'(s)^2+H''(s)\}]\\ &\times e^{-L(\vec{c_2})+W_1+W_2}\equiv i,\\ &[-D_2(d_1,d_2)-A_2(c_1,c_2)H'(s)-B(c_1,c_2)\{H'(s)^2+H''(s)\}]\\ &\times e^{-L(\vec{c_2})-W_1-W_2}\equiv i,\\ &[-D_1(d_1,d_2)-A_1(c_1,\ c_2)H'(s)-B(c_1,c_2)\{H'(s)^2-H''(s)\}]\\ &\times e^{L(\vec{c_2})-W_1-W_2}\equiv i. \end{aligned} \tag{4.18}$$

Taking into consideration the first and fourth equations of (4.18), we have the following:

(a) $A_1(c_1,c_2)\neq 0$, $B(c_1,c_2)\neq 0$, then degree of $H(s)\leq 1$.
(b) $A_1(c_1,c_2)=0$, $B(c_1,c_2)\neq 0$, then degree of $H(s)\leq 1$.
(c) $A_1(c_1,c_2)\neq 0$, $B(c_1,c_2)=0$, then degree of $H(s)\leq 1$.
(d) $A_1(c_1,c_2)=0$, $B(c_1,c_2)=0$, then degree of $H(s)$ can be any finite number.

Now using the first assumption of Theorem 2.4, i.e. $B(c_1,c_2)$ and $A_1(c_1,c_2)$ are not zero simultaneously, we obtain degree of $H(s)\leq 1$. Since under $H(s)\leq 1$; $p_1(\vec{z_2})$, $p_2(\vec{z_2})$ both become linear polynomials, without loss of generality we can consider $H(s)\equiv 0$. Then from first and fourth equations of (4.18) we must have

$$\begin{aligned} D_1(d_1,d_2)e^{L(\vec{c_2})+W_1+W_2}&=i,\\ -D_1(d_1,d_2)e^{L(\vec{c_2})-W_1-W_2}&=i. \end{aligned} \tag{4.19}$$

Let us consider the second and third equations of (4.18). We have the following 4 possibilities:

(e) $A_2(c_1,c_2)\neq 0$, $B(c_1,c_2)\neq 0$, degree of $H(s)\leq 1$.
(f) $A_2(c_1,c_2)=0$, $B(c_1,c_2)\neq 0$, degree of $H(s)\leq 1$.
(g) $A_2(c_1,c_2)\neq 0$, $B(c_1,c_2)=0$, degree of $H(s)\leq 1$.
(h) $A_2(c_1,c_2)=0$, $B(c_1,c_2)=0$, degree of $H(s)$ is arbitrary finite number.

Using the assumption of Theorem 2.4, which is $B(c_1,c_2)$ and $A_2(c_1,c_2)$ are not zero simultaneously, we must have $\deg(H(s)\leq 1$. Since $p_1(\vec{z_2})$, $p_2(\vec{z_2})$ becomes a linear polynomial, without any loss of generality we consider $H(s)\equiv 0$. Then from second and third equations of (4.18) we obtain

$$\begin{aligned} D_2(d_1,d_2)e^{-L(\vec{c_2})+W_1+W_2}&=i,\\ -D_2(d_1,d_2)e^{-L(\vec{c_2})-W_1-W_2}&=i. \end{aligned} \tag{4.20}$$

Considering all conditions such that degree of $H(s)\leq 1$ i.e. $B(c_1,c_2)$, $A_1(c_1,c_2)$ and $B(c_1,c_2)$, $A_2(c_1,c_2)$ are not zero simultaneously, from (4.19) and (4.20) we have

$$\begin{gathered} D_1(d_1,d_2)D_2(d_1,d_2)=1,\quad e^{2L(\vec{c_2})}=\frac{D_2(d_1,d_2)}{D_1(d_1,d_2)},\\ e^{2(W_1+W_2)}=-1,\quad e^{W_1+W_2}=\frac{i}{D_1(d_1,d_2)}e^{-L(\vec{c_2})}. \end{gathered}$$

The form of the solution is

$$f_1(\vec{z_2}) = \frac{e^{-L(\vec{z_2})+L(\vec{c_2})+W_2} - e^{L(\vec{z_2})-L(\vec{c_2})-W_2}}{2i},$$
$$f_2(\vec{z_2}) = \frac{e^{L(\vec{z_2})-L(\vec{c_2})+W_1} - e^{-L(\vec{z_2})+L(\vec{c_2})-W_1}}{2i}.$$

Case 2: Let

$$-iM_{2,1}e^{p_1(\vec{z_2}+\vec{c_2})+p_2(\vec{z_2})} \equiv 1,$$
$$-iM_{1,2}e^{p_2(\vec{z_2}+\vec{c_2})-p_1(\vec{z_2})} \equiv 1.$$

Clearly we have $p_1(\vec{z_2}+\vec{c_2})+p_2(\vec{z_2}) \equiv \eta_1$, $p_2(\vec{z_2}+\vec{c_2})-p_1(\vec{z_2}) \equiv \eta_2$, where η_1, η_2 are two constants in $\mathbb{C}$. Then by easy computation we obtain $p_1(\vec{z_2}+2\vec{c_2})+p_1(\vec{z_2}) \equiv \eta_1-\eta_2$, which contradicts that $p_1(\vec{z_2})$ is a non-constant polynomial.

Case 3: Let

$$-iM_{2,2}e^{p_1(\vec{z_2}+\vec{c_2})-p_2(\vec{z_2})} \equiv 1,$$
$$-iM_{1,1}e^{p_2(\vec{z_2}+\vec{c_2})+p_1(\vec{z_2})} \equiv 1.$$

Then by using similar arguments as in Case 2, we obtain a contradiction.

Case 4: Let

$$-iM_{2,2}e^{p_1(\vec{z_2}+\vec{c_2})-p_2(\vec{z_2})} \equiv 1,$$
$$-iM_{1,2}e^{p_2(\vec{z_2}+\vec{c_2})-p_1(\vec{z_2})} \equiv 1.$$

Then clearly we have $p_1(\vec{z_2}+\vec{c_2})-p_2(\vec{z_2}) \equiv \eta_1$, $p_2(\vec{z_2}+\vec{c_2})-p_1(\vec{z_2}) \equiv \eta_2$, where η_1, η_2 be two constants in $\mathbb{C}$. Let us take $p_1(\vec{z_2}) = L(\vec{z_2})+H(s)+W_1$, $p_2(\vec{z_2}) = L(\vec{z_2})+H(s)+W_2$, where W_1, W_2 are constants in $\mathbb{C}$, $H(s)$ is a polynomial in $s=c_2z_1-c_1z_2$.

Then combining this with (4.13) and (4.14), we obtain

$$\begin{aligned}
&b_{10}\,(d_1+H'(s)c_2)+b_{01}\,(d_2-H'(s)c_1)\\
&\ +b_{11}\{-H''(s)c_1c_2-(d_1+H'(s)c_2)\,(d_2-H'(s)c_1)\}\\
&\ +b_{20}\{H''(s)c_2^2-(d_1+H'(s)c_2)^2\}\\
&\ +b_{02}\{H''(s)c_1^2-(d_2-H'(s)c_1)^2\}e^{L(\vec{c_2})+W_1-W_2} \equiv i,\\
&b_{10}\,(d_1+H'(s)c_2)+b_{01}\,(d_2-H'(s)c_1)\\
&\ +b_{11}\{-H''(s)c_1c_2-(d_1+H'(s)c_2)\,(d_2-H'(s)c_1)\}\\
&\ +b_{20}\{H''(s)c_2^2-(d_1+H'(s)c_2)^2\}\\
&\ +b_{02}\{H''(s)c_1^2-(d_2-H'(s)c_1)^2\}e^{L(\vec{c_2})-W_1+W_2} \equiv i,\\
&b_{10}\,(d_1+H'(s)c_2)+b_{01}\,(d_2-H'(s)c_1)\\
&\ +b_{11}\{-H''(s)c_1c_2+(d_1+H'(s)c_2)\,(d_2-H'(s)c_1)\}\\
&\ +b_{20}\{H''(s)c_2^2+(d_1+H'(s)c_2)^2\}\\
&\ +b_{02}\{H''(s)c_1^2+(d_2-H'(s)c_1)^2\}e^{-L(\vec{c_2})-W_1+W_2} \equiv i,
\end{aligned} \tag{4.21}$$

$$
\begin{aligned}
&b_{10}\left(d_1+H'(s)c_2\right)+b_{01}\left(d_2-H'(s)c_1\right)\\
&+b_{11}\{-H''(s)c_1c_2+\left(d_1+H'(s)c_2\right)\left(d_2-H'(s)c_1\right)\}\\
&+b_{20}\{H''(s)c_2^2+\left(d_1+H'(s)c_2\right)^2\}\\
&+b_{02}\{H''(s)c_1^2+\left(d_2-H'(s)c_1\right)^2\}e^{-L(\vec{c_2})+W_1-W_2}\equiv i.
\end{aligned}
$$

Proceeding with the similar methods as done in Case 1 we conclude that $H(s)\equiv 0$. Then from (4.21) we obtain

$$
\begin{aligned}
-D_1(d_1,d_2)e^{L(\vec{c_2})+W_1-W_2}&=i,\\
-D_1(d_1,d_2)e^{L(\vec{c_2})-W_1+W_2}&=i,\\
D_2(d_1,d_2)e^{-L(\vec{c_2})-W_1+W_2}&=i,\\
D_2(d_1,d_2)e^{-L(\vec{c_2})+W_1-W_2}&=i.
\end{aligned}
$$

Clearly we have

$$
\begin{gathered}
D_1(d_1,d_2)D_2(d_1,d_2)=1,\quad e^{2L(\vec{c_2})}=-\frac{D_2(d_1,d_2)}{D_1(d_1,d_2)},\\
e^{2(W_1-W_2)}=1,\quad e^{W_1-W_2}=-\frac{i}{D_1(d_1,d_2)}e^{-L(\vec{c_2})}.
\end{gathered}
$$

In this case the form of the solution is

$$
\begin{aligned}
f_1(\vec{z_2})&=\frac{e^{L(\vec{z_2})-L(\vec{c_2})+W_2}-e^{-L(\vec{z_2})+L(\vec{c_2})-W_2}}{2i},\\
f_2(\vec{z_2})&=\frac{e^{L(\vec{z_2})-L(\vec{c_2})+W_1}-e^{-L(\vec{z_2})+L(\vec{c_2})-W_1}}{2i}.
\end{aligned}
$$

□

5. Discussion related to Theorem 2.4 and an open question

From the expressions of $D_1(d_1,d_2)$, $D_2(d_1,d_2)$ we see that they are related in a certain way. More elaborately, when we consider only the second degree homogeneous differential operator, then $D_1(d_1,d_2)=D_2(d_1,d_2)$ and when we consider only the first order differential operator, then $D_1(d_1,d_2)=-D_2(d_1,d_2)$. Now we discuss the following cases:

Case 1: Let $D_1(d_1,d_2)=D_2(d_1,d_2)=D(d_1,d_2)$. Then from Case 1 and Case 4 in Theorem 2.4, we obtain $D^2(d_1,d_2)=1$, that is $D(d_1,d_2)=\pm 1$.

Case 2: Let $D_1(d_1,d_2)=-D_2(d_1,d_2)=D(d_1,d_2)$. Then from Case 1 and Case 4 in Theorem 2.4, we obtain $D^2(d_1,d_2)=-1$, that is $D(d_1,d_2)=\pm i$.

Combining Case 1 and Case 2 we clearly see that $D_1(d_1,d_2)$ and $D_2(d_1,d_2)$ can take the values $\{1,-1,i,-i\}$ with $D_1(d_1,d_2)D_2(d_1,d_2)=1$. In particular, we can write the solution of equation (2.2) as the follows: Let

$$
f_1(\vec{z})=\frac{S_{11}e^{-L(\vec{z_2})-W_1}+S_{12}e^{L(\vec{z_2})+W_1}}{2},\quad f_2(\vec{z})=\frac{S_{21}e^{-L(\vec{z_2})-W_1}+S_{22}e^{L(\vec{z_2})+W_1}}{2},
$$

where W_1, W_2 and S_{11}, S_{12}, S_{21}, S_{22} are constants in $\mathbb{C}$.

Now under the conclusion (A) in Theorem 2.4, we have the following:

(i) $D_1(d_1,d_2)=i$, $D_2(d_1,d_2)=-i$ and $e^{L(\vec{c_2})}=i$, $e^{W_1+W_2}=-i$, then $S_{11}=-i$, $S_{12}=i$, $S_{21}=-1$, $S_{22}=-1$ or $e^{L(\vec{c_2})}=-i$, $e^{W_1+W_2}=i$, then $S_{11}=-i$, $S_{12}=i$, $S_{21}=1$, $S_{22}=1$; or

(ii) $D_1(d_1,d_2) = -i$, $D_2(d_1,d_2) = i$ and $e^{L(\vec{c_2})} = i$, $e^{W_1+W_2} = i$, then $S_{11} = i$, $S_{12} = -i$, $S_{21} = -1$, $S_{22} = -1$ or $e^{L(\vec{c_2})} = -i$, $e^{W_1+W_2} = -i$, then $S_{11} = i$, $S_{12} = -i$, $S_{21} = 1$, $S_{22} = 1$; or

(iii) $D_1(d_1,d_2) = 1$, $D_2(d_1,d_2) = 1$ and $e^{L(\vec{c_2})} = 1$, $e^{W_1+W_2} = i$, then $S_{11} = 1$, $S_{12} = 1$, then $S_{21} = i$, $S_{22} = -i$; $e^{L(\vec{c_2})} = -1$, $e^{W_1+W_2} = -i$, then $S_{11} = 1$, $S_{12} = 1$, then $S_{21} = i$, $S_{22} = -i$; or

(iv) $D_1(d_1,d_2) = -1$, $D_2(d_1,d_2) = -1$ and $e^{L(\vec{c_2})} = 1$, $e^{W_1+W_2} = -i$, then $S_{11} = -1$, $S_{12} = -1$, $S_{21} = i$, $S_{22} = -i$ or $e^{L(\vec{c_2})} = -1$, $e^{W_1+W_2} = i$, then $S_{11} = -1$, $S_{12} = -1$, $S_{21} = -i$, $S_{22} = i$.

Similarly, under conclusion (B) in Theorem 2.4, we have the following

(i) $D_1(d_1,d_2) = -i$, $D_2(d_1,d_2) = i$, and $e^{L(\vec{c_2})} = 1$, $e^{W_1-W_2} = 1$, then $S_{11} = i$, $S_{12} = -i$, $S_{21} = i$, $S_{22} = -i$; or $e^{L(\vec{c_2})} = -1$, $e^{W_1-W_2} = -1$, then $S_{11} = i$, $S_{12} = -i$, $S_{21} = -i$, $S_{22} = i$; o

(ii) $D_1(d_1,d_2) = i$, $D_2(d_1,d_2) = -i$ and $e^{L(\vec{c_2})} = 1$, $e^{W_1-W_2} = -1$, then $S_{11} = -i$, $S_{12} = i$, $S_{21} = i$, $S_{22} = -i$; or $e^{L(\vec{c_2})} = -1$, $e^{W_1-W_2} = 1$, then $S_{11} = -i$, $S_{12} = i$, $S_{21} = -i$, $S_{22} = i$; or

(iii) $D_1(d_1,d_2) = -1$, $D_2(d_1,d_2) = -1$ and $e^{L(\vec{c_2})} = i$, $e^{W_1-W_2} = 1$, then $S_{11} = -1$, $S_{12} = -1$, $S_{21} = -1$, $S_{22} = -1$, or $e^{L(\vec{c_2})} = -i$, $e^{W_1-W_2} = -1$, then $S_{11} = -1$, $S_{12} = -1$, $S_{21} = 1$, $S_{22} = 1$; or

(iv) $D_1(d_1,d_2) = 1$, $D_2(d_1,d_2) = 1$, and $e^{L(\vec{c_2})} = i$, $e^{W_1-W_2} = -1$, then $S_{11} = 1$, $S_{12} = 1$, $S_{21} = -1$, $S_{22} = -1$; $e^{L(\vec{c_2})} = -i$, $e^{W_1-W_2} = 1$, then $S_{11} = 1$, $S_{12} = 1$, $S_{21} = 1$, $S_{22} = 1$.

In view of (2.1) and (2.2) the following question is inevitable:

> What will be the possible form of transcendental entire solution of the following system of equation in $\mathbb{C}^n$
> $$\begin{gathered}(P_{L_2}(f_1(\vec{z_2})))^2 + f_2(\vec{z_2}+\vec{c_2})^2 = Q_1(\vec{z_2}),\\ (P_{L_2}(f_2(\vec{z_2}))^2 + f_1(\vec{z_2}+\vec{c_2})^2 = Q_2(\vec{z_2});\end{gathered}$$
> where $Q_j(\vec{z_2})$, $j = 1,2$ are two non-zero polynomials in $\mathbb{C}^n$?

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References

[1] A. Banerjee, J. Sarkar; *On the solutions of Fermat-type quadratic trinomial equations in* $\mathbb{C}^2$ *generated by first order linear c-shift and partial differential operators*, Advances. Pure. App. Math., **15**(1) (2024), 44-69.

[2] A. Biancofiore, W. Stoll; *Another proof of the lemma of the logarithmic derivative in several complex variables*, Princeton: Princeton University Press, (1981).

[3] T. B. Cao, R. J. Korhonen; *A new version of the second main theorem for meromorphic mappings intersecting hyper-planes in several complex variables*, J. Math. Anal. Appl., **444**(2) (2016), 1114-1132.

[4] Y. M. Chiang, S. J. Feng; *On the Nevanlinna characteristic of* $f(z+\eta)$ *and difference equations in the complex plane*, Ramanujan J., **16**(1) (2008), 105-129.

[5] L. Y. Gao; *Entire solutions of two types of systems of complex differential-difference equations*, Acta. Math. Scientia, **37B**(1) (2017), 187-194.

[6] F. Gross; *On the equation* $f^n + g^n = 1$, Bull. Am. Math. Soc., **72** (1966), 86-88.

[7] W. K. Hayman; *Meromorphic Functions*, Clarendon Press, Oxford, (1964).

[8] P. C. Hu; *Malmquist type theorem and factorization of meromorphic solutions of partial differential equations*, Complex Var., **27** (1995), 269-285.
[9] P. C. Hu, P. Li, C. C. Yang; *Unicity of meromorphic mappings*, Adv. Com. Anal. Appl., Kluwer Academic Dordrecht, The Netherlands, Boston, MA, USA, London, UK, (2003), Volume 1.
[10] I. Laine; *Nevanlinna theory and complex differential equations*, De Gruyter, Berlin, 1993.
[11] I. Lanie, J. Rieppo, H. Silvennoinen; *Remarks on complex difference equations*, Comput. Methods. Funct. Theo., **5**(1) (2005), 77-88.
[12] K. Liu, T. B. Cao; *Entire solutions of Fermat-type differential difference equations*, Arch. Math., **99** (2012), 147-155.
[13] W. Stoll; *Holomorphic functions of finite order in several complex variables*, American Mathematical Society, Providence, 1974.
[14] H. Y. Xu, S. Y. Liu, Q. P. Li; *Entire solutions for several systems of non-linear differential-difference equations of Fermat-type*, J. Math. Anal. Appl., **483** (2020), Art no. 123641.
[15] L. Xu, T. B. Cao; *Solutions of complex Fermat-type partial difference and differential-difference equations*, Mediterr. J. Math., **15**(227) (2018), 1-11.
[16] L. Xu, T. B. Cao; *Correction to : Solutions of complex Fermat-type partial difference and differential-difference equations*, Mediterr. J. Math., **17**(1)(8)(2020), 1-4.
[17] H. Y. Xu, L. Xu; *Transcendental entire solutions for several quadratic binomial and trinomial PDEs with constant coefficients*, Anal. Math. Phys. **12**(2022).
[18] H. Y. Xu, Y. Y. Jiang; *Results on entire and meromorphic solutions for several systems of quadratic trinomial functional equations with two complex variables*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matematicas, **116**(8)(2022), 1-19.
[19] H. Y. Xu, Y. H. Xu, X. L. Liu; *On solutions for several systems of complex nonlinear partial differential equations with two variables*, Anal. Math. Phys. **13**(47)(2023), 1-24.
[20] H. Y. Xu, G. Haldar; *Entire solutions to Fermat-type difference and partial differential-difference equations in* $\mathbb{C}^n$, Electronic J. Diff. Equa., **2024** (26) (2024), 1-21.
[21] H. Y. Xu, K. Liu, Z. X. Xuan; *Results on solutions of several product type nonlinear partial differential equations in* $\mathbb{C}^3$, J. Math. Anal. Appl. In press (2025), no. 128885, 1-20.
[22] Z. Ye; *On Nevanlinna's second main theorem in projective space*, Invent. Math., **122**(1) (1995), 475-507.

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