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NORMALIZED SOLUTIONS OF FRACTIONAL KIRCHHOFF EQUATIONS IN THE DEFOCUSING CASE

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ABSTRACT. In this article, we focus on the normalized solutions to the fractional Kirchhoff equations with subcritical nonlinearities in the defocusing case. By applying distinct suppositions to the coefficients of nonlinearities, namely $q < p$, we prove the existence and nonexistence of normalized solutions. Also we obtain new results on the characterization of ground states of the fractional Kirchhoff equations.

1. INTRODUCTION

The aim of article is to study the fractional Kirchhoff equation

$$\Big(a + b\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 \mathrm{d}x\Big)(-\Delta)^s u = \lambda u + \mu|u|^{q-2}u + |u|^{p-2}u \quad \text{in } \mathbb{R}^3 \tag{1.1}$$

with a prescribed mass

$$\int_{\mathbb{R}^3} u^2 \mathrm{d}x = c^2. \tag{1.2}$$

In this article, except for additional statements, we assume that $a, b, c > 0$, $0 < s < 1$, $2 < q < p \leq 2_s^* = \frac{6}{3-2s}$ (2_s^* is the Sobolev critical exponent), $N = 3$, $\lambda \in \mathbb{R}$ is a Lagrange multiplier and $\mu < 0$ is a parameter. For equation (1.1), we name the focusing case when $\mu > 0$, and defocusing case when $\mu < 0$. In equation (1.1), $(-\Delta)^s$ with $s \in (0, 1)$), namely the fractional Laplacian, is generally specified as

$$\begin{aligned}(-\Delta)^s v(x) &= C_s \text{P.V.} \int_{\mathbb{R}^3} \frac{v(x) - v(y)}{|x-y|^{3+2s}} \mathrm{d}y \\ &= C_s \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^3 \setminus B_\epsilon(x)} \frac{v(x) - v(y)}{|x-y|^{3+2s}} \mathrm{d}y \\ &= -\frac{1}{2} C_s \int_{\mathbb{R}^3} \frac{v(x+y) + v(x-y) - 2v(x)}{|y|^{3+2s}} \mathrm{d}y\end{aligned} \tag{1.3}$$

for $v \in S(\mathbb{R}^3)$, where $S(\mathbb{R}^3)$ denotes the Schwartz space of rapidly decaying C^∞ function, $B_\epsilon(x)$ represents an open ball of radius ϵ centered at x, P.V. is the principle value, which is defined by the latter expression in (1.3),

$$C_s = \Big(\int_{\mathbb{R}^3} \frac{1 - cos(\xi_1)}{|\xi|^{3+2s}} \mathrm{d}\xi\Big)^{-1}.$$

For $u \in S(\mathbb{R}^3)$, the fractional Laplacian $(-\Delta)^s$ can be regulated by the Fourier transform $(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u)$, $\mathcal{F}$ denotes the usual Fourier transform.

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When setting $a=1$, $s=1$ and $b=0$ (that is to say, (1.1) turns into the Schrödinger equation), in 1997, Jeanjean [13] studied the existence of the normalized solutions to the Schrödinger equation in the case of the corresponding energy functional

$$F(u)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u(x)|^2\mathrm{d}x-\sum_{i=1}^{m}\frac{a_i}{\sigma_i+2}\int_{\mathbb{R}^N}|u(x)|^{\sigma_i+2}\mathrm{d}x$$

is unbounded from below on the L^2-constraint set

$$S(c)=\{u\in H^1(\mathbb{R}^N),\|u\|_{L^2(\mathbb{R}^N)}=c\}$$

initially. Lately, such type of problems have attracted extensive attention in the field of partial differential equations. For instance, Soave [24] gave the existence and some properties of ground states for the nonlinear Schrödinger equation with combined power nonlinearities

$$-\Delta u=\lambda u+\mu|u|^{q-2}u+|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \quad N\geq 1,$$

on the normalized manifold

$$\int_{\mathbb{R}^N}|u|^2\mathrm{d}x=a^2.$$

As for other results of the Schrödinger equation, we refer readers to [4, 11, 25, 27] and the references therein.

When considering the case $a=1$, $s\neq 1$ and $b=0$, i.e., for the fractional Schrödinger equations, see [18, 29] and the references therein for results about the normalized solutions to the fractional Schrödinger equations.

When $b>0$ and $s=1$, equation (1.1) becomes the classic Kirchhoff model; this type of problems has also been researched by many authors [1, 14, 21]. In fact, such model has relation to the stationary solutions of equation

$$u_{tt}-\Big(a+b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\Big)\Delta u=f(x,u). \tag{1.4}$$

This equation comes from the traditional D'Alembert wave equation which was given by Kirchhoff [14] in 1876 in the process of studying the changes in the length of the string during vibrations, where $f(x,u)$ denotes a general nonlinear term. Additionally, it deserves attention that in [1] equation (1.4) models some physical systems, where u explains a process which is related to the average of itself. A lot of papers about the Kirchhoff type equations emerged with the emergence of this ground breaking article [21]. For example, considering a Kirchhoff model, together with a critical Trudinger-Moser nonlinearity $f(x,u)$, a class of fractional Kirchhoff-type equation with Trudinger-Moser nonlinearity was discussed by Xiang, Rădulescu and Zhang [20]. Using appropriate assumptions on the potential function V and some energy estimates techniques, Chen and Huang [6] obtained the existence results of normalized solutions for a fractional Kirchhoff-type equation

$$\Big(a+b\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u|^2\mathrm{d}x\Big)(-\Delta)^s u+V(x)u=c|u|^{p-2}u+\mu u \text{ in } \mathbb{R}^N$$

with doubly critical exponents (when considering the case $N=4s$, the critical Sobolev exponent $2_s^*=\frac{2N}{N-2s}$ and the fractional Gagliardo-Nirenberg-Sobolev critical exponent $2_{GNS}^*=\frac{2N+8s}{N}$ are equal, and $\frac{2N}{N-2s}=\frac{2N+8s}{N}=4$). In addition, in 2024, the existence of the normalized solutions to the fractional Kirchhoff equation with subcritical nonlinearity

$$\Big(a+b\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u|^2\mathrm{d}x\Big)(-\Delta)^s u=\lambda u+\mu|u|^{q-2}u+|u|^{p-2}u \quad \text{in } \mathbb{R}^N$$

was studied in [8] in $\mathbb{R}^3$ with $s\in(3/4,1)$ and $\mu<0$.

If the constraint condition (1.2) is considered on this basis, some universal methods do not take effect. As a result, we need to establish extra claims to solve the technical obstacles. As is known (such as in [13]), (1.2) has definite physical motivations. Consequently, it has sparked a wave of research on the normalized solutions. More specifically, the practical application background of operator $(-\Delta)^s$ includes the following aspects such as fractional quantum mechanics [15], physics and chemistry [19], conformal geometry and minimal surfaces [5], obstacle problems [23]. Caffarelli and Silvestre [2] adopted the extension method which converted this nonlocal problem (due to the nonlocal feature of the operator $(-\Delta)^s$ on $\mathbb{R}^N$ $(N \geq 1)$) to a local one in higher dimensions. What makes this amusing, of course, is that this method can go for nonlinear equations with a fractional Laplacian, we refer readers to [7, 9] and the references therein.

About equation (1.1), there are usually two classes of treatments. On one hand, we can think of it as a fixed frequency problem, in other words, we look for solutions $u \in H^s(\mathbb{R}^3)$ by hunting for critical points of the action functional $M : H^s(\mathbb{R}^3) \to \mathbb{R}$:

$$\begin{aligned} M(u) := &\frac{a}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 \mathrm{d}x + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 \mathrm{d}x\Big)^2 \\ &- \frac{\lambda}{2}\int_{\mathbb{R}^3} |u|^2 \mathrm{d}x - \frac{\mu}{q}\int_{\mathbb{R}^3} |u|^q \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p \mathrm{d}x, \end{aligned} \tag{1.5}$$

where $\lambda \in \mathbb{R}$ is a fixed frequency, readers can see [12, 17] for more results.

On the other hand, we can also search for solutions to (1.1) with a prescribed L^2 norm. In this case, we see $\lambda \in \mathbb{R}$ as part of unknown quantity. As everyone knows, equation (1.1) has roots in the standing wave type solution $\psi(x,t) = e^{-i\lambda t}u(x)$, $\lambda \in \mathbb{R}$ to the time-dependent nonlinear fractional equation defined by:

$$i\frac{\partial \psi}{\partial t} = \Big(a + b\int_{\mathbb{R}^3} |(-\Delta)^{s/2}\psi|^2 \mathrm{d}x\Big)(-\Delta)^s\psi - f(|\psi|)\psi, \quad \text{in } \mathbb{R}^3, \tag{1.6}$$

where $s \in (0,1)$, i represents the imaginary unit, and $\psi = \psi(x,t) \colon \mathbb{R}^3 \times [0,+\infty) \to \mathbb{C}$. It is easy to see that ψ solves (1.6) if and only if the standing wave $u(x)$ satisfies (1.1) with $f(u) = \mu u^{q-2} + u^{p-2}$. After computations, we can see that solutions $\psi \in C([0,T); H^s(\mathbb{R}^3))$ to (1.6) has conservation of mass along time, therefore this method is extremely significative from the physical perspective.

Now mention some publications that consider normalized solutions to (1.1). Li, Luo and Yang [16] proved the existence and properties of solutions to (1.1) with $s = 1$ under normalized constraint $\int_{\mathbb{R}^3} |u|^2 \mathrm{d}x = c^2$ when $a, b > 0$ and $\mu > 0$, namely the focusing case. As far as we know, the defocusing case of problem (1.1) with the condition (1.2) was mainly studied by Soave [24] with $b = 0$. The situation $b > 0$, $\mu < 0$ and $s = 1$ was studied in [3]. The defocusing case of fractional Kirchhoff equation was part of Ding's result [8]. We further extended his results (see Theorem 2.8 and 2.10). In this paper, we take the case of $b > 0$, $\mu < 0$ and $s \in (0,1)$ into consideration.

Before presenting the main results of our paper, let us first recall that the fractional Sobolev space $H^s(\mathbb{R}^3)$ can be defined as follows:

$$H^s(\mathbb{R}^3) = \big\{u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x < +\infty\big\}$$

with the norm

$$\|u\|^2 = \int_{\mathbb{R}^3} (|(-\Delta)^{s/2}u|^2 + |u|^2)\mathrm{d}x,$$

where

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x = \iint_{\mathbb{R}^6} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y.$$

Also we define

$$H_r^s(\mathbb{R}^3) = \big\{u \in H^s(\mathbb{R}^3) : u(x) = u(|x|),\ x \in \mathbb{R}^3\big\}.$$

In this paper, we denote by $|\cdot|_p$ the usual norm in the $L^p(\mathbb{R}^3)$ space, $S_c^r = S_c \cap H_r^s$ and u^* the symmetric decreasing rearrangement of the modulus of $u \in H^s(\mathbb{R}^3)$. The functional $E_\mu : S_c \to \mathbb{R}$ is regulated as

$$E_\mu(u) = \frac{a}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 \mathrm{d}x + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 \mathrm{d}x\Big)^2 - \frac{\mu}{q}\int_{\mathbb{R}^3} |u|^q \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p \mathrm{d}x,$$

where S_c is the constraint space

$$S_c = \big\{u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x = c^2\big\}.$$

It is easy to check that $E_\mu \in C^1(\mathbb{R}^3, \mathbb{R})$. Thus, the weak solutions of (1.1) under the constraint (1.2) can be obtained as critical points of the functional E_μ.

We can easily prove that, if $u \in H^s(\mathbb{R}^3)$ is a weak solution of (1.1), then we have the Pohožaev identity

$$P_\mu(u) := a|(-\Delta)^{s/2}u|_2^2 + b|(-\Delta)^{s/2}u|_2^4 - \mu\delta_{s,q}|u|_q^q - \delta_{s,p}|u|_p^p = 0,$$

where $\delta_{s,p} = \frac{3(p-2)}{2sp}$, $\delta_{s,q} = \frac{3(q-2)}{2sq}$. Therefore, the critical points of E_μ is certainly contained in the Pohožaev set

$$\mathcal{P}_{c,\mu} = \big\{u \in S_c \colon P_\mu(u) = 0\big\}$$

(see lemma 2.3 for a proof).

By simple calculations, we can show that $\delta_{s,p} \in (0,1)$ (when $2 < p < 2_s^*$) and

$$q\delta_{s,q} \le 4 < p\delta_{s,p}, \quad \text{if } 2 < q \le 2 + \frac{8s}{3} < p < \frac{6}{3-2s},$$

where $2 + \frac{8s}{3}$ is the mass critical exponent for the Kirchhoff constrained minimization problem, namely, $2+\frac{8s}{N}$ is the threshold exponent for many dynamic problems, see [28] for more information.

In this article, we will be concerned with ground state solutions, wihci are defined as follows.

Definition 1.1. We say that $\widetilde{u}$ is a ground state of (1.1) on S_c if it is a solution to (1.1) having minimal energy among all the solutions which belong to S_c:

$$dE_{\mu|S_c}(\widetilde{u}) = 0 \text{ and } E_\mu(\widetilde{u}) = \inf\{E_\mu(u) : dE_{\mu|S_c}(u) = 0, \text{ and } u \in S_c\}.$$

And the set of ground states will be denoted by $Z_{c,\mu}$.

To go over the obstacles about the convergence of the Palais Smale (hereinafter referred to as PS) sequence of E_μ, we build the sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfying

$$P_\mu(u_n) \to 0,$$

when $n \to \infty$. Thanks to the normalized condition (1.2), we define the dilations

$$(\omega * u)(x) = e^{3\omega/2}u(e^\omega x), \qquad \text{a.e. in } \mathbb{R}^3$$

which retain the L^2 norm, more precisely,

$$\int_{\mathbb{R}^3} (\omega * u)^2 \mathrm{d}x = \int_{\mathbb{R}^3} u^2 \mathrm{d}x,$$

and it is a continuous map from $\mathbb{R} \times H^s(\mathbb{R}^3)$ into $H^s(\mathbb{R}^3)$. Furthermore, we introduce the following fiber map

$$J_u^\mu(\omega) := E_\mu(\omega * u) = \frac{ae^{2s\omega}}{2}|(-\Delta)^{s/2}u|_2^2 + \frac{be^{4s\omega}}{4}|(-\Delta)^{s/2}u|_2^4 - \mu\frac{e^{q\delta_{s,q}s\omega}}{q}|u|_q^q - \frac{e^{p\delta_{s,p}s\omega}}{p}|u|_p^p,$$

where $\delta_{s,q} = \frac{3(q-2)}{2sq}$ and $\delta_{s,p} = \frac{3(p-2)}{2sp}$. By using the functional J_u^μ, we cast a function into the Pohožaev set. Soave [24] and Li, Luo and Yang [16] have also applied such idea.

The main results of this paper are organized as follows:

- If $\mu < 0, 2 < q < p = 2 + \frac{8s}{3} = \bar{p}$, we prove that (1.1) under the condition (1.2) does not have solution.
- If $2 < q \le 2 + \frac{8s}{3} < p < \frac{6}{3-2s}$ are given constants and $\mu < 0$ satisifies an additional assumption, we prove that there exists $\lambda < 0$ such that (1.1) under the condition (1.2) has a solution. The solution is radially symmetric, and is a ground state on S_c.
- If $2 < q \le 2 + \frac{8s}{3} < p < \frac{6}{3-2s}$ are given constants and $\mu < 0$ satisfies an additional assumption, we also give a characterization to the set of ground states.

For overcoming some technical difficulties, the main proof of our results involves the techniques used by Soave [24], Li, Luo and Yang [16]. This paper is organized as follows. In section 2 we give the notation and assumptions and we enunciate our main results. In section 3 we prove some results concerning the subcritical case (namely, the main results in this paper).

2. Preliminaries and main results

Lemma 2.1 ([7]). *Let $s \in (0,1)$ and $p \in [1,+\infty)$ be such that $sp < N$. Then, there exists a positive constant $S_s = S_s(N,p,s)$ such that, for any measurable and compactly supported function $u : \mathbb{R}^N \to \mathbb{R}$, we have*

$$S_s|u|^2_{2^*_s} \le \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y, \tag{2.1}$$

where $2^*_s = \frac{2N}{N-2s}$ is the so-called fractional critical exponent. Moreover, (2.1) becomes equality if and only if $\widetilde{u} = \mathcal{K}(\bar{\mu}^2 + |x-x_0|^2)^{-\frac{N-2s}{2}}$ with $\mathcal{K} \in \mathbb{R}\backslash\{0\}$, $\bar{\mu} > 0$, $x_0 \in \mathbb{R}^N$ fixed constants, S_s is the best Sobolev embedding constant.

It is known that if $p \in (2, 2^*_s)$, then there exists an optimal constant $C(s,p)$ such that

$$|u|_p \le C(s,p)|(-\Delta)^{s/2}u|_2^{\delta_{s,p}}|u|_2^{1-\delta_{s,p}}, \tag{2.2}$$

holds for all $u \in H^s(\mathbb{R}^N)$. (2.2) is called of the fractional Gagliardo-Nirenberg inequality.

Lemma 2.2 ([18]). *Let $u \in H^s(\mathbb{R}^N)$, $N \ge 2$ satisfy the equation*

$$(-\Delta)^s u = g(u),$$

then

$$\frac{N-2s}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \mathrm{d}x = N\int_{\mathbb{R}^N} G(u)\mathrm{d}x,$$

where $G(u) = \int_0^u g(t)\mathrm{d}t$.

Lemma 2.3. *Let $p, q \in (2, \frac{2N}{N-2s}]$ and $\lambda, \mu \in \mathbb{R}$. If $u \in H^s(\mathbb{R}^N)$ is a weak solution of the equation in the N-dimensional space corresponding to equation (1.1), then it satisfies the Pohožaev identity*

$$P_\mu(u) := a|(-\Delta)^{s/2}u|_2^2 + b|(-\Delta)^{s/2}u|_2^4 - \mu\delta_{s,q}|u|_q^q - \delta_{s,p}|u|_p^p = 0, \tag{2.3}$$

where $\delta_{s,q} = \frac{N(q-2)}{2sq}$ and $\delta_{s,p} = \frac{N(p-2)}{2sp}$.

Proof. Set $A = a + b\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2\mathrm{d}x$. According to the lemma 2.2, we have

$$(-\Delta)^s u = \frac{1}{A}(\lambda u + \mu|u|^{q-2}u + |u|^{p-2}u). \tag{2.4}$$

Let $f(u) = \frac{1}{A}(\lambda u + \mu|u|^{q-2}u + |u|^{p-2}u)$, then $F(u) = \frac{1}{A}(\frac{\lambda}{2}|u|^2 + \frac{\mu}{q}|u|^q + \frac{1}{p}|u|^p)$. According to the assumption that u is a weak solution to the equation in the N-dimensional space corresponding to equation (1.1) (in other words, multiplying the above equation by u and integrating), one has that

$$\int_{\mathbb{R}^N} A(-\Delta)^s u \cdot u\mathrm{d}x = (\lambda|u|_2^2 + \mu|u|_q^q + |u|_p^p). \tag{2.5}$$

The above equality implies that

$$A|(-\Delta)^{s/2}u|_2^2 = (\lambda|u|_2^2 + \mu|u|_q^q + |u|_p^p). \tag{2.6}$$

Multiplying by $\frac{N}{N-2s}$, it holds that

$$\frac{N}{N-2s}A|(-\Delta)^{s/2}u|_2^2 = \frac{N}{N-2s}\lambda|u|_2^2 + \frac{N}{N-2s}\mu|u|_q^q + \frac{N}{N-2s}|u|_p^p. \tag{2.7}$$

By lemma 2.2, we have that

$$|(-\Delta)^{s/2}u|_2^2 = \frac{2N}{A(N-2s)}\Big(\frac{\lambda}{2}|u|_2^2 + \frac{\mu}{q}|u|_q^q + \frac{1}{p}|u|_p^p\Big). \tag{2.8}$$

Combining the above two equations, we obtain that

$$\Big(1 - \frac{N}{N-2s}\Big)A|(-\Delta)^{s/2}u|_2^2 = \mu\Big(\frac{2N}{q(N-2s)} - \frac{N}{N-2s}\Big)|u|_q^q + \Big(\frac{2N}{p(N-2s)} - \frac{N}{N-2s}\Big)|u|_p^p.$$

From the above equality, one deduces that

$$A|(-\Delta)^{s/2}u|_2^2 = \mu\delta_{s,q}|u|_q^q + \delta_{s,p}|u|_p^p.$$

Thus, the fractional Pohožaev identity holds, namely,

$$a|(-\Delta)^{s/2}u|_2^2 + b|(-\Delta)^{s/2}u|_2^4 = \mu\delta_{s,q}|u|_q^q + \delta_{s,p}|u|_p^p. \qquad \square$$

For convenience, we decompose the set $\mathcal{P}_{c,\mu}$ into three disjoint sets as follows:

$$\mathcal{P}_{c,\mu} = \mathcal{P}_{c,\mu}^+ \cup \mathcal{P}_{c,\mu}^0 \cup \mathcal{P}_{c,\mu}^-,$$

where

$$\begin{aligned}
\mathcal{P}_{c,\mu}^+ &= \{u \in \mathcal{P}_{c,\mu}, 2a|(-\Delta)^{s/2}u|_2^2 + 4b|(-\Delta)^{s/2}u|_2^4 - \mu q\delta_{s,q}^2|u|_q^q - p\delta_{s,p}^2|u|_p^p > 0\} \\
&= \big\{u \in \mathcal{P}_{c,\mu}, (J_u^\mu)''(0) > 0\big\}, \\
\mathcal{P}_{c,\mu}^- &= \{u \in \mathcal{P}_{c,\mu}, 2a|(-\Delta)^{s/2}u|_2^2 + 4b|(-\Delta)^{s/2}u|_2^4 - \mu q\delta_{s,q}^2|u|_q^q - p\delta_{s,p}^2|u|_p^p < 0\} \\
&= \big\{u \in \mathcal{P}_{c,\mu}, (J_u^\mu)''(0) < 0\big\}, \\
\mathcal{P}_{c,\mu}^0 &= \{u \in \mathcal{P}_{c,\mu}, 2a|(-\Delta)^{s/2}u|_2^2 + 4b|(-\Delta)^{s/2}u|_2^4 - \mu q\delta_{s,q}^2|u|_q^q - p\delta_{s,p}^2|u|_p^p = 0\} \\
&= \big\{u \in \mathcal{P}_{c,\mu}, (J_u^\mu)''(0) = 0\big\},
\end{aligned}$$

where

$$(J_u^\mu)''(0) = \Big(2a|(-\Delta)^{s/2}u|_2^2 + 4b|(-\Delta)^{s/2}u|_2^4 - \mu q\delta_{s,q}^2|u|_q^q - p\delta_{s,p}^2|u|_p^p\Big)s^2.$$

Next we state a lemma that is a type of minimax principle. But first, we state a related definition.

Definition 2.4. Let X be a topological space and B be a closed subset of X. We say that a class F of compact subsets of X is a homotopy-stable family with extended boundary B if for any set A in F and any $\eta \in C([0,1]\times X; X)$ satisfying $\eta(t,x) = x$ for all $(t,x) \in (\{0\}\times X) \cup ([0,1]\times B)$, we have that $\eta(\{1\}\times A)$ is in F.

Lemma 2.5 ([10, Theorem 5.2])**.** *Let Φ be a C^1 functional on a complete connected C^1-Finsler manifold X and consider a homotopy-stable family F with an extended closed boundary B. Set $m = m(\Phi, F) = \inf_{A\in F}\max_{x\in A}\Phi(x)$ and let F be a closed subset of X satisfying*

1. *$A \cap F\backslash B \neq \emptyset$ for each $A \in F$.*
2. *$\sup\Phi(B) \le m \le \inf\Phi(F)$. Then, for any sequence of sets $\{A_n\}_n$ in F such that $\lim_n \sup_{A_n}\Phi = m$, there exists a sequence $\{x_n\}_n$ in $X\backslash B$ such that*

$$\lim_{n\to\infty}\Phi(x_n) = m, \quad \lim_{n\to\infty}\|d\Phi(x_n)\| = 0,$$
$$\lim_{n\to\infty}\operatorname{dist}(x_n, F) = 0, \quad \lim_{n\to\infty}\operatorname{dist}(x_n, A_n) = 0.$$

Lemma 2.6 ([18])**.** *Let $N \ge 2$, then $H_r^s(\mathbb{R}^N)$ is compactly embedding into $L^p(\mathbb{R}^N)$ for $p \in (2, 2_s^*)$.*

Lemma 2.7 ([18])**.** *Let $s \in (0,1)$. For any $u \in H^s(\mathbb{R}^N)$, the following inequality holds*

$$\iint_{\mathbb{R}^{2N}} \frac{(u^*(x)-u^*(y))^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \le \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y.$$

Next, we state the main results of this paper.

Theorem 2.8 (Subcritical case)**.** *Let $N = 3$ and $2 < q < p = 2 + \frac{8s}{3} = \bar{p}$. If $b \ge \frac{4}{\bar{p}}C(s,\bar{p})^{\bar{p}}c^{\bar{p}-4}$, then there is no solution to problem (1.1)-(1.2) for any $\mu < 0$.*

Now, we define the constant

$$C_0 := \Big(\frac{a}{\delta_{s,p}C(s,p)^p c^{p(1-\delta_{s,p})}}\Big)^{\frac{1}{p\delta_{s,p}-2}}.$$

Theorem 2.9 (Subcritical case)**.** *Let $N = 3$ and $2 < q \le 2 + \frac{8s}{3} < p < 2_s^* = \frac{6}{3-2s}$ be given constants. If $\mu < 0$ satisfies*

$$(1 - \frac{1}{\delta_{s,p}})(a + bC_0^2)C_0^{2-q\delta_{s,q}} + \mu\Big(\frac{\delta_{s,q}}{\delta_{s,p}} - 1\Big)C(s,q)^q c^{q(1-\delta_{s,q})} := \epsilon_0 < 0, \tag{2.9}$$

then $E_{\mu|S_c}$ has a critical point $\widetilde{u}$ at a positive level $m(c,\mu) = \inf_{u\in\mathcal{P}_{c,\mu}} E_\mu(u) > 0$ satisfying: $\widetilde{u}$ is radially symmetric, it solves (1.1) for some $\widetilde{\lambda} < 0$ and it is a ground state of (1.1) on S_c, where

$C(s,q)$ is an optimal constant such that the fractional Gagliardo-Nirenberg inequality holds (see inequality (2.2)).

The following theorem characterizes the ground states.

Theorem 2.10 (Subcritical case)**.** *Under the assumptions of Theorem 2.9, if $u \in Z_{c,\mu}$ (see Definition 1.1 for the definition of $Z_{c,\mu}$), then $e^{i\theta}|u| \in Z_{c,\mu}$ for each $\theta \in \mathbb{R}$. Moreover, if u is a ground state, then the associated Lagrange multiplier λ is negative.*

3. Subcritical case

In this part we prove Theorems 2.8, 2.9 and 2.10. First we list some lemmas. Since some of these lemmas appear in the references, we omit their proof here.

Lemma 3.1 (Jeanjean [13])**.** *For $u \in S_c$ and $s \in \mathbb{R}$, the map $\phi \mapsto s * \phi$ from T_uS_c to $T_{s*u}S_c$ is a linear isomorphism with inverse $\psi \mapsto (-s) * \psi$, where $T_uS_c = \{\phi \in S_c : \int_{\mathbb{R}^N} u\phi \mathrm{d}x = 0\}$.*

Next, we will discuss the convergence of a class of special PS sequences satisfying appropriate additional assumptions. The idea used in the proof was first introduced by Jeanjean [13]. Then Soave [24] applied this idea to study the normalized solutions to the nonlinear Schrödinger equation with mixed nonlinearities.

Lemma 3.2 (Compactness of PS sequences)**.** *Let $2 < q \le 2 + \frac{8s}{3} < p < \frac{6}{3-2s}$ be given constants. We suppose that $\{u_n\}_{n\in\mathbb{N}} \subset S_c$ is a PS sequence for $E_{\mu|S_c}$ at level $c \neq 0$ and it holds*

- (i) *$P_\mu(u_n) \to 0$ as $n \to \infty$.*
- (ii) *$\mu < 0$ and (2.9) holds.*

Then, going to a subsequence, $u_n \to u$ strongly in $H^s(\mathbb{R}^3)$, and $u \in S_c$ is a radial solution to (1.1) for some $\lambda < 0$.

Proof. In this lemma, we argue directly. Since $P_\mu(u_n) \to 0$ as $n \to \infty$, we have that

$$a|(-\Delta)^{s/2}u_n|_2^2 + b|(-\Delta)^{s/2}u_n|_2^4 - \mu\delta_{s,q}|u_n|_q^q - \delta_{s,p}|u_n|_p^p = o(1), \tag{3.1}$$

as $n \to \infty$. Therefore,

$$\begin{aligned}
c+1 &\ge E_\mu(u_n) \\
&= \frac{a}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 \mathrm{d}x + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 \mathrm{d}x\Big)^2 - \frac{\mu}{q}\int_{\mathbb{R}^3} |u_n|^q \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3} |u_n|^p \mathrm{d}x \\
&= a\Big(\frac{1}{2} - \frac{1}{p\delta_{s,p}}\Big)\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 \mathrm{d}x + b\Big(\frac{1}{4} - \frac{1}{p\delta_{s,p}}\Big)\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 \mathrm{d}x\Big)^2 \\
&\quad - \frac{\mu}{q}\Big(1 - \frac{q\delta_{s,q}}{p\delta_{s,p}}\Big)\int_{\mathbb{R}^3} |u_n|^q \mathrm{d}x + o(1) \\
&\ge a\Big(\frac{1}{2} - \frac{1}{p\delta_{s,p}}\Big)\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 \mathrm{d}x + b\Big(\frac{1}{4} - \frac{1}{p\delta_{s,p}}\Big)\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 \mathrm{d}x\Big)^2 + o(1),
\end{aligned}$$

as $n \to \infty$, where we used the fact that $E_\mu(u_n) \to c$, the equivalent deformation of (3.1) and $-\frac{\mu}{q}\big(1 - \frac{q\delta_{s,q}}{p\delta_{s,p}}\big) > 0$ (since $\mu < 0, 0 < q\delta_{s,q} < p\delta_{s,p}$). From the above inequality and the fact that $|u_n|_2^2 = c^2$, we deduce that $\{u_n\}$ is a bounded sequence in $H^s(\mathbb{R}^3)$. Besides, Hilbert space $H^s(\mathbb{R}^3)$ is a reflexive Banach space. In the reflexive Banach space $H^s(\mathbb{R}^3)$, bounded sequence $\{u_n\}$ has weakly convergent subsequence $\{u_n\}$ (for the sake of brevity, the subsequence of $\{u_n\}$ is still represented by $\{u_n\}$). According to lemma 2.6, $H_r^s(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ compactly for $p \in (2, 2_s^*)$, there exists $u \in H_r^s(\mathbb{R}^3)$ such that

$$u_n \rightharpoonup u \text{ in } H_r^s(\mathbb{R}^3), \quad u_n \to u \text{ in } L^p(\mathbb{R}^3), \quad u_n(x) \to u(x) \text{ a.e. in } \mathbb{R}^3, \tag{3.2}$$

as $n \to \infty$.

Since $\{u_n\}$ is a bounded PS sequence of $E_{\mu|S_c}$, by applying the Lagrange multipliers rule, we conclude that there exists $\lambda_n \in \mathbb{R}$ such that

$$\begin{aligned} &a\int_{\mathbb{R}^3}(-\Delta)^{s/2}u_n(-\Delta)^{s/2}\phi \mathrm{d}x + b|(-\Delta)^{s/2}u_n|_2^2\int_{\mathbb{R}^3}(-\Delta)^{s/2}u_n(-\Delta)^{s/2}\phi \mathrm{d}x \\ &- \mu\int_{\mathbb{R}^3}|u_n|^{q-2}u_n\phi \mathrm{d}x - \int_{\mathbb{R}^3}|u_n|^{p-2}u_n\phi \mathrm{d}x - \lambda_n\int_{\mathbb{R}^3}u_n\phi \mathrm{d}x \\ &= o(1)\|\phi\|_{H^s}, \end{aligned} \tag{3.3}$$

for all $\phi \in H^s(\mathbb{R}^3)$. Letting $\phi = u_n$, we have that

$$a|(-\Delta)^{s/2}u_n|_2^2 + b|(-\Delta)^{s/2}u_n|_2^4 - \mu|u_n|_q^q - |u_n|_p^p - \lambda_n|u_n|_2^2 = o(1)\|u_n\|_{H^s}. \tag{3.4}$$

Therefore,

$$\lambda_n = \frac{1}{c^2}\Big(a|(-\Delta)^{s/2}u_n|_2^2 + b|(-\Delta)^{s/2}u_n|_2^4 - \mu|u_n|_q^q - |u_n|_p^p\Big) + o(1)\|u_n\|_{H^s}. \tag{3.5}$$

As $\{u_n\}$ is a bounded sequence in $H^s(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)\cap L^q(\mathbb{R}^3)$, from the above equation we obtain that $\{\lambda_n\}$ is a bounded sequence. Thus, going if necessary to a subsequence, there exists $\lambda \in \mathbb{R}$ such that

$$\lambda_n \to \lambda \tag{3.6}$$

as $n \to \infty$. In the remaining of this proof, we prove that $\lambda < 0$. From $P_\mu(u_n) \to 0$ as $n \to \infty$, we obtain that

$$a|(-\Delta)^{s/2}u_n|_2^2 + b|(-\Delta)^{s/2}u_n|_2^4 = \mu\delta_{s,q}|u_n|_q^q + \delta_{s,p}|u_n|_p^p + o(1) \le \delta_{s,p}|u_n|_p^p + o(1). \tag{3.7}$$

Employing the fractional Gagliardo-Nirenberg inequality (2.2), we derive that

$$\begin{aligned} a|(-\Delta)^{s/2}u_n|_2^2 &\le a|(-\Delta)^{s/2}u_n|_2^2 + b|(-\Delta)^{s/2}u_n|_2^4 \\ &\le \delta_{s,p}|u_n|_p^p + o(1) \\ &\le \delta_{s,p}C(s,p)^p|(-\Delta)^{s/2}u_n|_2^{p\delta_{s,p}}|u_n|_2^{p(1-\delta_{s,p})} + o(1). \end{aligned} \tag{3.8}$$

It is easy to obtain that $u \not\equiv 0$: assuming by contradiction that $u \equiv 0$, then we obtain that $\lim_{n\to\infty}|u_n|_q^q = \lim_{n\to\infty}|u_n|_p^p = 0$. Using that $P_\mu(u_n) \to 0$ we deduce that $E_\mu(u_n) \to 0$, while this contradicts the assumption that $E_\mu(u_n) \to c \not\equiv 0$. Thus we have $u \not\equiv 0$. Since $u_n \in S_c$ and the weak lower semi-continuity of the norm, it follows that $|u|_2 \le c$, then we have that

$$C_0 = \Big(\frac{a}{\delta_{s,p}C(s,p)^p c^{p(1-\delta_{s,p})}}\Big)^{\frac{1}{p\delta_{s,p}-2}} \le B, \tag{3.9}$$

where $B = \lim_{n\to\infty}|(-\Delta)^{s/2}u_n|_2$. That is to say, for n large enough, we obtain that

$$|(-\Delta)^{s/2}u_n|_2 \ge C_0. \tag{3.10}$$

Inserting (3.1) into (3.5), we have that

$$\lambda_n = \frac{1}{c^2}\Big[\Big(1 - \frac{1}{\delta_{s,p}}\Big)\Big(a + b|(-\Delta)^{s/2}u_n|_2^2\Big)|(-\Delta)^{s/2}u_n|_2^2 + \mu\Big(\frac{\delta_{s,q}}{\delta_{s,p}} - 1\Big)|u_n|_q^q\Big] + o(1). \tag{3.11}$$

According to the fractional Gagliardo-Nirenberg inequality (2.2), we obtain that

$$|u_n|_q^q \le C(s,q)^q|(-\Delta)^{s/2}u_n|_2^{q\delta_{s,q}}|u_n|_2^{q(1-\delta_{s,q})}. \tag{3.12}$$

By $u_n \in S_c$, we have that

$$|u_n|_q^q \le C(s,q)^q|(-\Delta)^{s/2}u_n|_2^{q\delta_{s,q}}c^{q(1-\delta_{s,q})}. \tag{3.13}$$

Further, by $1 - \frac{1}{\delta_{s,p}} < 0$ (since $0 < \delta_{s,p} < 1$), combining (3.9), (3.11) with (3.13), we can deduce that

$$\lambda_n \le \frac{1}{c^2}|(-\Delta)^{s/2}u_n|_2^{q\delta_{s,q}}\Big[\Big(1-\frac{1}{\delta_{s,p}}\Big)(a+bC_0^2)C_0^{2-q\delta_{s,q}} + \mu\Big(\frac{\delta_{s,q}}{\delta_{s,p}}-1\Big)C(s,q)^q c^{q(1-\delta_{s,q})}\Big] + o(1). \tag{3.14}$$

By (3.10) and (2.9), we derive that

$$\lambda_n \le \frac{1}{c^2} C_0^{q\delta_{s,q}} \Big[\Big(1-\frac{1}{\delta_{s,p}}\Big)(a+bC_0^2)C_0^{2-q\delta_{s,q}} + \mu\Big(\frac{\delta_{s,q}}{\delta_{s,p}}-1\Big)C(s,q)^q c^{q(1-\delta_{s,q})}\Big] + o(1). \tag{3.15}$$

Therefore, considering condition (2.9), we obtain that

$$\lambda_n \le \frac{1}{c^2} C_0^{q\delta_{s,q}}\epsilon_0 + o(1), \tag{3.16}$$

for n adequately large. Taking the limit of the above formula as $n \to \infty$, we have that

$$\lambda \le \frac{1}{c^2} C_0^{q\delta_{s,q}}\epsilon_0 < 0. \tag{3.17}$$

Thus $\lambda < 0$, and the claim is proved.

Finally, we prove that $u_n \to u$ strongly in $H^s(\mathbb{R}^3)$. By taking the limit of (3.3), one has

$$\begin{aligned} &a\int_{\mathbb{R}^3} (-\Delta)^{s/2}u(-\Delta)^{s/2}\phi \mathrm{d}x + bB^2\int_{\mathbb{R}^3} (-\Delta)^{s/2}u(-\Delta)^{s/2}\phi \mathrm{d}x - \mu\int_{\mathbb{R}^3} |u|^{q-2}u\phi \mathrm{d}x \\ &- \int_{\mathbb{R}^3} |u|^{p-2}u\phi \mathrm{d}x - \lambda\int_{\mathbb{R}^3} u\phi \mathrm{d}x = 0, \end{aligned} \tag{3.18}$$

for all $\phi \in H^s(\mathbb{R}^3)$, that is, u satisfies

$$(a+B^2b)(-\Delta)^s u = \lambda u + \mu|u|^{q-2}u + |u|^{p-2}u,$$

namely, u solves (1.1) for some $\lambda < 0$. Testing (3.18), (3.3) with $\phi = u_n - u$, we can see that

$$(a+B^2b)\int_{\mathbb{R}^3} |(-\Delta)^{s/2}(u_n-u)|^2 \mathrm{d}x - \lambda\int_{\mathbb{R}^3} |u_n-u|^2 \mathrm{d}x \to 0,$$

as $n \to \infty$. Since $\lambda < 0$, we deduce that $\{u_n\}$ converges strongly to u in $H^s(\mathbb{R}^3)$. □

Lemma 3.3. *Let $\mu < 0$, and $2 < q \le 2 + \frac{8s}{3} < p < 2_s^*$ be given constants. Then $\mathcal{P}_{c,\mu}^0 = \emptyset$ and $\mathcal{P}_{c,\mu}$ is a smooth manifold of codimension 2 in $H^s(\mathbb{R}^3)$.*

Proof. Suppose by contradiction that this is not the case, namely we set $\mathcal{P}_{c,\mu}^0 \ne \emptyset$, then from the definition of $\mathcal{P}_{c,\mu}^0$, we can derive that there exists $u \in S_c$ such that

$$P_\mu(u) = 0 \quad \text{and} \quad (J_u^\mu)''(0) = 0.$$

Therefore,

$$a|(-\Delta)^{s/2}u|_2^2 + b|(-\Delta)^{s/2}u|_2^4 = \mu\delta_{s,q}|u|_q^q + \delta_{s,p}|u|_p^p, \tag{3.19}$$

$$2a|(-\Delta)^{s/2}u|_2^2 + 4b|(-\Delta)^{s/2}u|_2^4 = \mu q\delta_{s,q}^2|u|_q^q + p\delta_{s,p}^2|u|_p^p. \tag{3.20}$$

Combining (3.19) with (3.20), one has

$$(p\delta_{s,p}-2)a|(-\Delta)^{s/2}u|_2^2 + (p\delta_{s,p}-4)b|(-\Delta)^{s/2}u|_2^4 = \mu\delta_{s,q}(p\delta_{s,p}-q\delta_{s,q})|u|_q^q \le 0, \tag{3.21}$$

where $p\delta_{s,p} > 4 \ge q\delta_{s,q}$ by $2 < q \le 2 + \frac{8s}{3} < p < 2_s^*$, $\delta_{s,q} > 0$ by $q > 2$. Then, it follows that

$$|(-\Delta)^{s/2}u|_2 = 0. \tag{3.22}$$

Further, from (3.19), (3.21) and the fractional Gagliardo-Nirenberg inequality (2.2), we obtain that

$$|u|_q = 0 \quad \text{and} \quad |u|_p = 0.$$

Thus, we deduce that $u \equiv 0$, which is in contradiction with $u \in S_c$. So we obtain $\mathcal{P}_{c,\mu}^0 = \emptyset$. To proof that $\mathcal{P}_{c,\mu}$ is a smooth manifold of codimension 2 in $H^s(\mathbb{R}^3)$ is very similar to the one of [24, lemma 5.2], therefore we omit it here. □

Since $\mathcal{P}_{c,\mu}^0 = \emptyset$ by lemma 3.3, we observe that $\mathcal{P}_{c,\mu}$ is a natural constraint in the following sense.

Lemma 3.4. *Let $\mu < 0$ and $2 < q \le 2 + \frac{8s}{3} < p < 2_s^*$ be given constants. If $u \in \mathcal{P}_{c,\mu}$ is a critical point for $E_{\mu|\mathcal{P}_{c,\mu}}$, then u is a critical point for $E_{\mu|S_c}$.*

Proof. From lemma 3.3, we see that $\mathcal{P}_{c,\mu}$ is a smooth manifold of codimension 2 in $H^s(\mathbb{R}^3)$ and $\mathcal{P}^0_{c,\mu}=\emptyset$. If $u\in\mathcal{P}_{c,\mu}$ is a critical point for $E_{\mu|\mathcal{P}_{c,\mu}}$, then by the Lagrange multipliers rule, one gets that there exist $\lambda,\nu\in\mathbb{R}$ such that

$$\langle E'_\mu(u),\phi\rangle-\lambda\int_{\mathbb{R}^3}u\phi\mathrm{d}x-\nu\langle P'_\mu(u),\phi\rangle=0$$

for any $\phi\in H^s(\mathbb{R}^3)$, that is, u solves

$$\begin{aligned}&\Big[(1-2\nu)a+(1-4\nu)b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2\mathrm{d}x\Big](-\Delta)^s u\\&=\lambda u+\mu(1-\nu q\delta_{s,q})|u|^{q-2}u+(1-\nu p\delta_{s,p})|u|^{p-2}u\quad\text{in }\mathbb{R}^3.\end{aligned}\tag{3.23}$$

By lemma 2.2, we obtain that

$$\begin{aligned}&\frac{3-2s}{2}(1-2\nu)a\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2\mathrm{d}x+\frac{3-2s}{2}(1-4\nu)b\Big(\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2\mathrm{d}x\Big)^2\\&-\frac{3}{2}\lambda\int_{\mathbb{R}^3}u^2\mathrm{d}x+\frac{3\mu(\nu q\delta_{s,q}-1)}{q}\int_{\mathbb{R}^3}u^q\mathrm{d}x+\frac{3(\nu p\delta_{s,p}-1)}{p}\int_{\mathbb{R}^3}u^p\mathrm{d}x=0\quad\text{in }\mathbb{R}^3.\end{aligned}\tag{3.24}$$

Multiplying (3.23) by u and integrating, then combining it with (3.24), we obtain that

$$\begin{aligned}&(1-2\nu)a\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2\mathrm{d}x+(1-4\nu)b\Big(\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2\mathrm{d}x\Big)^2\\&+\mu\delta_{s,q}(\nu q\delta_{s,q}-1)\int_{\mathbb{R}^3}u^q\mathrm{d}x+\delta_{s,p}(\nu p\delta_{s,p}-1)\int_{\mathbb{R}^3}u^p\mathrm{d}x=0.\end{aligned}\tag{3.25}$$

By (3.25) and $P_\mu(u)=0$, we have that

$$\nu\Big(2a|(-\Delta)^{s/2}u|_2^2+4b|(-\Delta)^{s/2}u|_2^4-\mu q\delta_{s,q}^2|u|_q^q-p\delta_{s,p}^2|u|_p^p\Big)=0,$$

which implies that $\nu=0$ since $u\notin\mathcal{P}^0_{c,\mu}$: from lemma 3.3, we see that $\mathcal{P}_{c,\mu}$ is a smooth manifold of codimension 2 in $H^s(\mathbb{R}^3)$ and $\mathcal{P}^0_{c,\mu}=\emptyset$, namely,

$$\Big(2a|(-\Delta)^{s/2}u|_2^2+4b|(-\Delta)^{s/2}u|_2^4-\mu q\delta_{s,q}^2|u|_q^q-p\delta_{s,p}^2|u|_p^p\Big)\neq0.$$ □

Lemma 3.5. *For every $u\in S_c$, there exists a unique $t_u\in\mathbb{R}$ such that $t_u*u\in\mathcal{P}_{c,\mu}$. Moreover, t_u is the unique critical point of J^μ_u and it is a strict maximum point at the positive level. Moreover,*

- (i) $\mathcal{P}_{c,\mu}=\mathcal{P}^-_{c,\mu}$.
- (ii) *J^μ_u is strictly decreasing and concave on $(t_u,+\infty)$ and $t_u<0$ implies that $P_\mu(u)<0$.*
- (iii) *The function $u\in S_c\mapsto t_u$ is of class C^1.*
- (iv) *If $P_\mu(u)<0$, then $t_u<0$.*

Proof. For every $u\in S_c$, according to the definition of $J^\mu_u(\tau)$, we obtain

$$\lim_{\tau\to-\infty}J^\mu_u(\tau)=0^+\quad\text{and}\quad\lim_{\tau\to+\infty}J^\mu_u(\tau)=-\infty.$$

Thus, J^μ_u has at least one global maximum point t_u at positive level. Besides, this is the one and only critical point of J^μ_u. For each $u\in S_c$, we define

$$h(t)=\frac{at^{2s}}{2}|(-\Delta)^{s/2}u|_2^2+\frac{bt^{4s}}{4}|(-\Delta)^{s/2}u|_2^4-\frac{t^{p\delta_{s,p}s}}{p}|u|_p^p-\mu\frac{t^{q\delta_{s,q}s}}{q}|u|_q^q.$$

Indeed, $J^\mu_u(\tau)=h(e^\tau)$, then we obtain $J^{\mu\prime}_u(\tau)=h'(e^\tau)e^\tau$. Thus, it is sufficient to study the function h. The derivative of h can be written as $h'(t)=t^{4s-1}\eta(t)$, where

$$\eta(t)=ast^{-2s}|(-\Delta)^{s/2}u|_2^2+bs|(-\Delta)^{s/2}u|_2^4-\delta_{s,p}st^{(p\delta_{s,p}-4)s}|u|_p^p-\mu s\delta_{s,q}t^{(q\delta_{s,q}-4)s}|u|_q^q,$$

and η satisfies

$$\lim_{t\to0^+}\eta(t)=+\infty,\quad\lim_{t\to+\infty}\eta(t)=-\infty,\quad\eta'(t)<0,\quad\text{for all }t>0.$$

From the above analysis for the function η, we obtain it has a unique zero point $\widetilde{t}$ in $(0,+\infty)$. By $h'(t)=t^{4s-1}\eta(t)$, it holds that the function h has a unique critical point $\widetilde{t}$ and $t_u=\ln\widetilde{t}$. By

$$\lim_{t\to 0^+} h(t)=0^+, \quad \text{and} \quad \lim_{t\to+\infty} h(t)=-\infty,$$

we deduce that $h(\widetilde{t})>0$.

From the above arguments and $sP_\mu(t_u*u)=(J_u^\mu)'(t_u)$, we deduce that for each $u\in S_c$, there exists a unique $t_u\in\mathbb{R}$ such that $P_\mu(t_u*u)=0$, namely, $t_u*u\in\mathcal{P}_{c,\mu}$. Let $u\in\mathcal{P}_{c,\mu}$, then we obtain $t_u=0$ and as t_u is a maximum point of J_u^μ, we obtain that $(J_u^\mu)''(0)\le 0$. Since $\mathcal{P}^0_{c,\mu}=\emptyset$, we conclude that $(J_u^\mu)''(0)<0$. Thus, $\mathcal{P}_{c,\mu}=\mathcal{P}^-_{c,\mu}$. From the calculus above, we can also deduce that J_u^μ is strictly decreasing and concave on $(t_u,+\infty)$.

Since $(J_u^\mu)'(t)<0$ if and only if $t>t_u$, we conclude that $P_\mu(u)=\frac{1}{s}(J_u^\mu)'(0)<0$ if and only if $t_u<0$.

Item (iii) holds when applying the implicit function theorem to the function $\Phi(\tau,u)=(J_u^\mu)'(\tau)$. We use that $\Phi(t_u,u)=(J_u^\mu)'(t_u)=0$, that $\partial_\tau\Phi(t_u,u)=(J_u^\mu)''(t_u)<0$, and the fact that it is not possible to pass with continuity from $\mathcal{P}^+_{c,\mu}$ to $\mathcal{P}^-_{c,\mu}$ (since $\mathcal{P}^0_{c,\mu}=\emptyset$), therefore we obtain $u\mapsto t_u$ is C^1. □

Lemma 3.6. *It holds that*

$$m(c,\mu)=\inf_{u\in\mathcal{P}_{c,\mu}} E_\mu(u)>0.$$

Proof. Setting $u\in\mathcal{P}_{c,\mu}$, by fractional Gagliardo-Nirenberg inequality and $\mu<0$ we obtain that

$$a|(-\Delta)^{s/2}u|_2^2\le\delta_{s,p}|u|_p^p\le\delta_{s,p}C(s,p)^p|-\Delta^{s/2}u|_2^{p\delta_{s,p}}|u|_2^{p(1-\delta_{s,p})}. \tag{3.26}$$

From the definition of $\mathcal{P}_{c,\mu}$, we obtain $u\in S_c$, namely, $|u|_2=c$. Therefore,

$$|(-\Delta)^{s/2}u|_2\ge\Big(\frac{a}{\delta_{s,p}C(s,p)^pc^{p(1-\delta_{s,p})}}\Big)^{\frac{1}{p\delta_{s,p}-2}}\Rightarrow\inf_{\mathcal{P}_{c,\mu}}|(-\Delta)^{s/2}u|_2>0. \tag{3.27}$$

For every $u\in\mathcal{P}_{c,\mu}$, we obtain that

$$\begin{aligned}E_\mu(u)&=\Big(\frac12-\frac{1}{p\delta_{s,p}}\Big)a|(-\Delta)^{s/2}u|_2^2+\Big(\frac14-\frac{1}{p\delta_{s,p}}\Big)b|(-\Delta)^{s/2}u|_2^4-\frac{\mu}{q}\Big(1-\frac{q\delta_{s,q}}{p\delta_{s,p}}\Big)|u|_q^q\\&\ge\Big(\frac12-\frac{1}{p\delta_{s,p}}\Big)a|(-\Delta)^{s/2}u|_2^2.\end{aligned}$$

Then $p\delta_{s,p}>2$ and (3.27) imply that

$$m(c,\mu)=\inf_{\mathcal{P}_{c,\mu}}E_\mu(u)>0.$$

□

Lemma 3.7. *For $2<q\le 2+\frac{8s}{3}<p<2_s^*$ and $\mu<0$, there exists $k>0$ small enough such that*

$$0<\sup_{\bar{A}_k}E_\mu(u)<m(c,\mu)\quad\textit{and}\quad u\in\bar{A}_k\Rightarrow E_\mu(u),P_\mu(u)>0,$$

where $\bar{A}_k=\{u\in S_c:|(-\Delta)^{s/2}u|_2^2\le k\}$.

Proof. For $u\in\bar{A}_k$ with k small enough, thanks to the fractional Gagliardo-Nirenberg inequality and the fact that $p\delta_{s,p}>4$ (because of $p>2+\frac{8s}{3}$), we have that

$$\begin{aligned}E_\mu(u)&=\frac a2|(-\Delta)^{\frac s2}u|_2^2+\frac b4|(-\Delta)^{\frac s2}u|_2^4-\frac{\mu}{q}|u|_q^q-\frac1p|u|_p^p\\&\ge\frac a2|(-\Delta)^{\frac s2}u|_2^2-\frac1pC(s,p)^pc^{p(1-\delta_{s,p})}|(-\Delta)^{\frac s2}u|_2^{p\delta_{s,p}}>0\end{aligned}$$

and that

$$\begin{aligned}P_\mu(u)&=a|(-\Delta)^{\frac s2}u|_2^2+b|(-\Delta)^{\frac s2}u|_2^4-\mu\delta_{s,q}|u|_q^q-\delta_{s,p}|u|_p^p\\&\ge a|(-\Delta)^{\frac s2}u|_2^2-C(s,p)^pc^{p(1-\delta_{s,p})}|(-\Delta)^{\frac s2}u|_2^{p\delta_{s,p}}\delta_{s,p}>0.\end{aligned}$$

Then, if $u \in \bar{A}_k$, we have that $\sup_{\bar{A}_k} E_\mu(u) \geq E_\mu(u) > 0$ and $P_\mu(u) > 0$. Now, replacing k with a smaller quantity, recalling that $m(c,\mu) > 0$ by lemma 3.6 and applying fractional Gagliardo-Nirenberg inequality, we conclude that

$$E_\mu(u) \leq \frac{a}{2}|(-\Delta)^{\frac{s}{2}}u|_2^2 + \frac{b}{4}|(-\Delta)^{\frac{s}{2}}u|_2^4 + \frac{|\mu|}{q}C(s,q)^q c^{q(1-\delta_{s,q})}|(-\Delta)^{\frac{s}{2}}u|_2^{q\delta_{s,q}} < m(c,\mu).$$

Thus $\sup_{\bar{A}_k} E_\mu(u) < m(c,\mu)$. □

Now, we define $E_\mu^c = \{u \in S_c : E_\mu(u) \leq c\}$ and the minimax class

$$\widetilde{\tau} := \{\gamma = (\alpha,\beta) \in C([0,1], \mathbb{R} \times S_c^r) : \gamma(0) \in (0, \bar{A}_k) \text{ and } \gamma(1) \in (0, E_\mu^0)\},$$

where $S_c^r = S_c \cap H_r^s$. We define the minimax level as follows:

$$\sigma(c,\mu) = \inf_{\gamma \in \widetilde{\tau}} \max_{(\tau,u) \in \gamma([0,1])} \widetilde{E}_\mu(\tau,u),$$

where

$$\begin{aligned}\widetilde{E}_\mu(\tau,u) &= E_\mu(\tau * u) = J_u^\mu(\tau)\\ &= \frac{ae^{2s\tau}}{2}|(-\Delta)^{s/2}u|_2^2 + \frac{be^{4s\tau}}{4}|(-\Delta)^{s/2}u|_2^4 - \mu\frac{e^{q\delta_{s,q}s\tau}}{q}|u|_q^q - \frac{e^{p\delta_{s,p}s\tau}}{p}|u|_p^p.\end{aligned}$$

Proof of Theorem 2.8. Assume by contradiction that if there exists a solution u to (1.1)-(1.2), then by the Pohožaev identity $P_\mu(u) = 0$, we obtain that

$$a|(-\Delta)^{s/2}u|_2^2 + b|(-\Delta)^{s/2}u|_2^4 = \mu\delta_{s,q}|u|_q^q + \frac{4}{\bar{p}}|u|_{\bar{p}}^{\bar{p}},$$

where $\delta_{s,\bar{p}} = \frac{4}{\bar{p}}$ when $p = 2 + \frac{8s}{3} = \bar{p}$. Since $b \geq \frac{4}{\bar{p}}C(s,\bar{p})^{\bar{p}}c^{\bar{p}-4}$, by the fractional Gagliardo-Nirenberg inequality, we have that

$$\begin{aligned}&a|(-\Delta)^{s/2}u|_2^2 + b|(-\Delta)^{s/2}u|_2^4 - \frac{4}{\bar{p}}|u|_{\bar{p}}^{\bar{p}}\\ &\geq a|(-\Delta)^{s/2}u|_2^2 + b|(-\Delta)^{s/2}u|_2^4 - \frac{4}{\bar{p}}C(s,p)^{\bar{p}}c^{\bar{p}-4}|(-\Delta)^{s/2}u|_2^4 \geq 0,\end{aligned}$$

and hence we deduce that

$$0 > \mu\delta_{s,q}|u|_q^q = a|(-\Delta)^{s/2}u|_2^2 + b|(-\Delta)^{s/2}u|_2^4 - \frac{4}{\bar{p}}|u|_{\bar{p}}^{\bar{p}} \geq 0,$$

which is a contradiction. □

Proof of Theorem 2.9. Through simple analysis we can see that it is suffice to prove that there exists a PS sequence such that conditions (i) and (ii) of lemma 3.2 hold. By

$$\lim_{\tau\to-\infty} J_u^\mu(\tau) = 0^+ \quad \text{and} \quad \lim_{\tau\to+\infty} J_u^\mu(\tau) = -\infty,$$

it follows that there exist τ_1 and $\tau_2 \in \mathbb{R}$ satisfying

$$\begin{gathered}J_u^\mu(\tau) \leq k \quad \text{for all } \tau < \tau_1 \ (k > 0),\\ J_u^\mu(\tau) \leq 0 \quad \text{for all } \tau > \tau_2.\end{gathered}$$

Next, we define $\gamma_u : [0,1] \to \mathbb{R} \times S_c^r$ by

$$\gamma_u(p) = \big(0, ((1-p)\tau_1 + p\tau_2) * u\big), \tag{3.28}$$

which is a path in $\widetilde{\tau}$, therefore $\sigma(c,\mu)$ is a real number.

Now, we aim to prove the claim that for all $\gamma \in \widetilde{\tau}$, there exists $\tau_\gamma \in (0,1)$ such that

$$\alpha(\tau_\gamma) * \beta(\tau_\gamma) \in \mathcal{P}_{c,\mu}^-. \tag{3.29}$$

Actually, $\gamma(0) = (0, \beta(0)) \in (0, \bar{A}_k)$ and $sP_\mu(t * u) = (J_u^\mu)'(t)$. According to lemma 3.7, we obtain $P_\mu(\beta(0)) = \frac{1}{s}(J_u^\mu)'(0) > 0$, further, by lemma 3.5, we obtain $t_{0*\beta(0)} = t_{\beta(0)} > 0$.

In addition, by $E_\mu(\beta(1)) = \widetilde{E}_\mu(\gamma(1)) \leq 0$, we deduce that $t_{\alpha(1)*\beta(1)} = t_{\beta(1)} < 0$. Indeed, $J_{\beta(1)}(\tau) > 0$ for each $\tau \in (-\infty, t_{\beta(1)}]$, and $J_{\beta(1)}(0) = E_\mu(\beta(1)) \leq 0$, it is necessary that $t_{\beta(1)} < 0$.

Furthermore, lemma 3.5 implies that the function $u \in S_c \mapsto t_u \in \mathbb{R}$ is continuous. Then there exists $\tau_\gamma \in (0,1)$ such that $t_{\alpha(\tau_\gamma)*\beta(\tau_\gamma)} = 0$. This amounts to $\alpha(\tau_\gamma)*\beta(\tau_\gamma) = t_{\alpha(\tau_\gamma)*\beta(\tau_\gamma)} * \big(\alpha(\tau_\gamma)*\beta(\tau_\gamma)\big) \in \mathcal{P}_{c,\mu} = \mathcal{P}^-_{c,\mu}$. By (3.29), it follows that

$$\max_{\gamma([0,1])} \widetilde{E}_\mu \geq \widetilde{E}_\mu(\gamma(\tau_\gamma)) = E_\mu(\alpha(\tau_\gamma)*\beta(\tau_\gamma)) \geq \inf_{\mathcal{P}^-_{c,\mu}\cap S^r_c} E_\mu.$$

Then we obtain that

$$\sigma(c,\mu) \geq \inf_{\mathcal{P}^-_{c,\mu}\cap S^r_c} E_\mu. \tag{3.30}$$

Besides, taking $u \in \mathcal{P}^-_{c,\mu} \cap S^r_c$ and γ_u the corresponding path defined in (3.28), we derive that

$$E_\mu(u) = \widetilde{E}_\mu(0,u) = \max_{\gamma_u([0,1])} \widetilde{E}_\mu \geq \sigma(c,\mu),$$

then we obtain that

$$\inf_{\mathcal{P}^-_{c,\mu}\cap S^r_c} E_\mu \geq \sigma(c,\mu). \tag{3.31}$$

Formulas (3.30) and (3.31) imply that

$$\inf_{\mathcal{P}^-_{c,\mu}\cap S^r_c} E_\mu = \sigma(c,\mu). \tag{3.32}$$

Next, we prove a claim that:

$$\inf_{\mathcal{P}_{c,\mu}\cap S^r_c} E_\mu = \inf_{\mathcal{P}_{c,\mu}} E_\mu. \tag{3.33}$$

This is equivalent to verifying that $\inf_{\mathcal{P}_{c,\mu}} E_\mu \geq \inf_{\mathcal{P}_{c,\mu}\cap S^r_c} E_\mu$. Suppose by contradiction that there exists $u \in \mathcal{P}_{c,\mu} \setminus S^r_c$ with $E_\mu(u) < \inf_{\mathcal{P}_{c,\mu}\cap S^r_c} E_\mu$. Then we set $v := |u|^*$, the symmetric decreasing rearrangement of the modulus of u, which belongs to S^r_c. By lemma 2.7, we obtain $E_\mu(v) \leq E_\mu(u)$ and $P_\mu(v) \leq P_\mu(u)$. From $u \in \mathcal{P}_{c,\mu} \setminus S^r_c$, we have $P_\mu(u) = 0$. If $P_\mu(v) = 0$, we immediately derive a contradiction, hence we assume that $P_\mu(v) < 0$. In this case, from lemma 3.5, we know that $t_v < 0$. But then we obtain a contradiction in the following way

$$\begin{aligned}
E_\mu(u) &< E_\mu(t_v * v)\\
&= \frac{ae^{2st_v}}{2}|(-\Delta)^{s/2}v|_2^2 + \frac{be^{4st_v}}{4}|(-\Delta)^{s/2}v|_2^4 - \frac{\mu}{q}e^{q\delta_{s,q}st_v}|v|_q^q\\
&\quad - \frac{1}{p\delta_{s,p}}\Big[ae^{2st_v}|(-\Delta)^{s/2}v|_2^2 + be^{4st_v}|(-\Delta)^{s/2}v|_2^4 - \mu\delta_{s,q}e^{q\delta_{s,q}st_v}|v|_q^q\Big]\\
&= a(\frac{1}{2} - \frac{1}{p\delta_{s,p}})e^{2st_v}|(-\Delta)^{s/2}v|_2^2 + b(\frac{1}{4} - \frac{1}{p\delta_{s,p}})e^{4st_v}|(-\Delta)^{s/2}v|_2^4 - \frac{\mu}{q}(1 - \frac{q\delta_{s,q}}{p\delta_{s,p}})e^{q\delta_{s,q}st_v}|v|_q^q\\
&\leq a(\frac{1}{2} - \frac{1}{p\delta_{s,p}})|(-\Delta)^{s/2}u|_2^2 + b(\frac{1}{4} - \frac{1}{p\delta_{s,p}})|(-\Delta)^{s/2}u|_2^4 - \frac{\mu}{q}(1 - \frac{q\delta_{s,q}}{p\delta_{s,p}})|u|_q^q\\
&= E_\mu(u),
\end{aligned}$$

where we used that $t_v * v$ and u lie in $\mathcal{P}_{c,\mu}$. This proves that $\inf_{\mathcal{P}_{c,\mu}\cap S^r_c} E_\mu = \inf_{\mathcal{P}_{c,\mu}} E_\mu$. By the above claim, (3.33), and lemma 3.6, we obtain that

$$m(c,\mu) = \sigma(c,\mu). \tag{3.34}$$

We also obtain that

$$m(c,\mu) = \sigma(c,\mu) > \sup_{(\bar{A}_k\cup E^0_\mu)\cap S^r_c} E_\mu = \sup_{((0,\bar{A}_k)\cup(0,E^0_\mu))\cap(\mathbb{R}\times S^r_c)} \widetilde{E}_\mu. \tag{3.35}$$

The rest of the proof is the same as in Soave [24], existence of a second critical point of mountain pass type for $E_{\mu|S_c}$, thus we omit it here (namely, we use lemma 2.5 to obtain a PS sequence $\{u_n\}$ for $E_\mu|_{S^r_c}$ at level $\sigma(c,\mu) > 0$ and $\operatorname{dist}(u_n, \mathcal{P}_{c,\mu}) \to 0$, i.e., $P_\mu(u_n) \to 0$).

To verify that u is a ground state, we show that u achieves $\inf_{\mathcal{P}_{c,\mu}} E_\mu = m(c,\mu)$. From the above proof, we know that $\sigma(c,\mu) = \inf_{\gamma\in\widetilde{\tau}} \max_{(\tau,u)\in\gamma([0,1])} \widetilde{E}_\mu(\tau,u) = E_\mu(u) = \inf_{\mathcal{P}_{c,\mu}\cap S^r_c} E_\mu$, hence we have to show that $\inf_{\mathcal{P}_{c,\mu}} E_\mu = \inf_{\mathcal{P}_{c,\mu}\cap S^r_c} E_\mu$. By claim (3.33), this equality holds, hence u is a ground state. Therefore, Theorem 2.9 is proved. □

Proof of Theorem 2.10. We start by describing the structure of $Z_{c,\mu}$ of ground states. If $u \in Z_{c,\mu}$, then $u \in \mathcal{P}_{c,\mu}$ and $E_\mu(u) = m(c,\mu) = \inf_{\mathcal{P}_{c,\mu}} E_\mu$. We claim that

$$u \in Z_{c,\mu} \Rightarrow |u| \in Z_{c,\mu}, |(-\Delta)^{s/2}|u||_2 = |(-\Delta)^{s/2}u|_2. \tag{3.36}$$

To prove the claim, we observe that $E_\mu(|u|) \le E_\mu(u)$ and $P_\mu(|u|) \le P_\mu(u) = 0$. Then by lemma 3.5, there exists $t_{|u|} \le 0$ with $t_{|u|} * |u| \in \mathcal{P}_{c,\mu}$ and by definition of $t_{|u|}$, one has

$$\begin{aligned}
m(c,\mu) \le E_\mu(t_{|u|} * |u|) &= a\Big(\frac{1}{2}\frac{1}{p\delta_{s,p}}\Big)e^{2st_{|u|}}|(-\Delta)^{s/2}|u||_2^2 \\
&\quad + b\left(\frac{1}{4} - \frac{1}{p\delta_{s,p}}\right)e^{4st_{|u|}}|(-\Delta)^{s/2}|u||_2^4 - \frac{\mu}{q}\Big(1 - \frac{q\delta_{s,q}}{p\delta_{s,p}}\Big)e^{q\delta_{s,q}st_{|u|}}|u|_q^q \\
&\le a\Big(\frac{1}{2} - \frac{1}{p\delta_{s,p}}\Big)e^{2st_{|u|}}|(-\Delta)^{s/2}u|_2^2 + b\Big(\frac{1}{4} - \frac{1}{p\delta_{s,p}}\Big)e^{4st_{|u|}}|(-\Delta)^{s/2}u|_2^4 \\
&\quad - \frac{\mu}{q}\Big(1 - \frac{q\delta_{s,q}}{p\delta_{s,p}}\Big)e^{q\delta_{s,q}st_{|u|}}|u|_q^q \\
&\le \Big[a\Big(\frac{1}{2} - \frac{1}{p\delta_{s,p}}\Big)|(-\Delta)^{s/2}u|_2^2 + b\Big(\frac{1}{4} - \frac{1}{p\delta_{s,p}}\Big)|(-\Delta)^{s/2}u|_2^4 - \frac{\mu}{q}\Big(1 - \frac{q\delta_{s,q}}{p\delta_{s,p}}\Big)|u|_q^q\Big]e^{kst_{|u|}} \\
&= e^{kst_{|u|}}E_\mu(u) \\
&= e^{kst_{|u|}}m(c,\mu),
\end{aligned}$$

where $k = \min\{2, q\delta_{s,q}\}$, and we used the fact that $u, t_{|u|} * |u| \in \mathcal{P}_{c,\mu}$, and $E_\mu(u) = m(c,\mu)$. By $t_{|u|} \le 0$, we deduce that necessarily $t_{|u|} = 0$, that is $P_\mu(|u|) = 0$, and since also $P_\mu(u) = 0$, it holds that

$$|u| \in \mathcal{P}_{c,\mu}, |(-\Delta)^{s/2}|u||_2 = |(-\Delta)^{s/2}u|_2 \quad \text{and} \quad E_\mu(|u|) = m(c,\mu).$$

This proves claim (3.36). After proving that $|u|$ minimizes E_μ on $\mathcal{P}_{c,\mu}$, we obtain that $|u|$ is a non-negative solution to (1.1) for some $\lambda \in \mathbb{R}$, by lemma 3.4. By regularity and the strong maximum principle, it is a C^2 positive solution. Using also that $|(-\Delta)^{s/2}|u||_2 = |(-\Delta)^{s/2}u|_2$, and $u \in Z_{c,\mu}$ (then we obtain $|u| \in Z_{c,\mu}$, namely, $E_\mu(|u|) = m(c,\mu)$), we will prove that $e^{i\theta}|u| \in Z_{c,\mu}$ for any $\theta \in \mathbb{R}$.

By the definition of $E_\mu(u)$ and that the modulus of $e^{i\theta} = 1$ for any $\theta \in \mathbb{R}$, we obtain that

$$\begin{aligned}
E_\mu(e^{i\theta}|u|) &= \frac{a}{2}|(-\Delta)^{s/2}e^{i\theta}|u||_2^2 + \frac{b}{4}|(-\Delta)^{s/2}e^{i\theta}|u||_2^4 - \frac{\mu}{q}|e^{i\theta}|u||_q^q - \frac{1}{p}|e^{i\theta}|u||_p^p \\
&= \frac{a}{2}|(-\Delta)^{s/2}|u||_2^2 + \frac{b}{4}|(-\Delta)^{s/2}|u||_2^4 - \frac{\mu}{q}|u|_q^q - \frac{1}{p}|u|_p^p \\
&= E_\mu(|u|) \\
&= m(c,\mu).
\end{aligned}$$

In the remaining of this proof, we prove that if $u \in Z_{c,\mu}$, then the associated Lagrange multiplier λ is negative. Recalling that $u \in \mathcal{P}_{c,\mu}$, then $P_\mu(u) = 0$, and we have that

$$a|(-\Delta)^{s/2}u|_2^2 + b|(-\Delta)^{s/2}u|_2^4 = \mu\delta_{s,q}|u|_q^q + \delta_{s,p}|u|_p^p \le \delta_{s,p}|u|_p^p.$$

Then, by the fractional Gagliardo-Nirenberg inequality, we derive that

$$\begin{aligned}
a|(-\Delta)^{s/2}u|_2^2 &\le a|(-\Delta)^{s/2}u|_2^2 + b|(-\Delta)^{s/2}u|_2^4 \\
&\le \delta_{s,p}|u|_p^p \\
&\le \delta_{s,p}C(s,p)^p|(-\Delta)^{s/2}u|_2^{p\delta_{s,p}}|u|_2^{p(1-\delta_{s,p})}.
\end{aligned}$$

Next, we prove the claim that $u \not\equiv 0$. Since otherwise $E_\mu(u) = 0 = m(c,\mu)$, in contradiction with lemma 3.6. By the definition of $\mathcal{P}_{c,\mu}$, we obtain $|u|_2 = c$, then we can deduce that

$$|(-\Delta)^{s/2}u|_2 \ge \Big(\frac{a}{\delta_{s,p}C(s,p)^p c^{p(1-\delta_{s,p})}}\Big)^{\frac{1}{p\delta_{s,p}-2}}. \tag{3.37}$$

Now, since u is a weak radial and positive solution to

$$(a+b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2\mathrm{d}x)(-\Delta)^s u=\lambda u+\mu|u|^{q-2}u+|u|^{p-2}u \quad \text{in } \mathbb{R}^3. \tag{3.38}$$

By the Pohožaev identity, we infer that $P_\mu(u)=0$, i.e.,

$$\delta_{s,p}|u|_p^p=a|(-\Delta)^{s/2}u|_2^2+b|(-\Delta)^{s/2}u|_2^4-\mu\delta_{s,q}|u|_q^q. \tag{3.39}$$

Testing (3.38) with u and using (3.39), we obtain that

$$\lambda|u|_2^2=a\Big(1-\frac{1}{\delta_{s,p}}\Big)|(-\Delta)^{s/2}u|_2^2+b\Big(1-\frac{1}{\delta_{s,p}}\Big)|(-\Delta)^{s/2}u|_2^4+\mu\Big(\frac{\delta_{s,q}}{\delta_{s,p}}-1\Big)|u|_q^q,$$

where $1-\frac{1}{\delta_{s,p}}<0$ since $0<\delta_{s,p}<1$, while $\mu\Big(\frac{\delta_{s,q}}{\delta_{s,p}}-1\Big)>0$ since $\mu<0$. Using again the fractional Gagliardo-Nirenberg inequality and estimate (3.37), we infer that

$$\begin{aligned}
\lambda|u|_2^2 &\le a\Big(1-\frac{1}{\delta_{s,p}}\Big)|(-\Delta)^{s/2}u|_2^2+b\Big(1-\frac{1}{\delta_{s,p}}\Big)|(-\Delta)^{s/2}u|_2^4\\
&\quad+\mu\Big(\frac{\delta_{s,q}}{\delta_{s,p}}-1\Big)C(s,q)^q|(-\Delta)^{s/2}u|_2^{q\delta_{s,q}}|u|_2^{q(1-\delta_{s,q})}\\
&\le |(-\Delta)^{s/2}u|_2^{q\delta_{s,q}}\Big[a\Big(1-\frac{1}{\delta_{s,p}}\Big)|(-\Delta)^{s/2}u|_2^{2-q\delta_{s,q}}+b\Big(1-\frac{1}{\delta_{s,p}}\Big)|(-\Delta)^{s/2}u|_2^{4-q\delta_{s,q}}\\
&\quad+\mu\Big(\frac{\delta_{s,q}}{\delta_{s,p}}-1\Big)C(s,q)^q c^{q(1-\delta_{s,q})}\Big]\\
&\le |(-\Delta)^{s/2}u|_2^{q\delta_{s,q}}\Big[a\Big(1-\frac{1}{\delta_{s,p}}\Big)\Big(\frac{a}{\delta_{s,p}C(s,p)^p c^{p(1-\delta_{s,p})}}\Big)^{\frac{2-q\delta_{s,q}}{p\delta_{s,p}-2}}\\
&\quad+b\Big(1-\frac{a}{\delta_{s,p}}\Big)\Big(\frac{1}{\delta_{s,p}C(s,p)^p c^{p(1-\delta_{s,p})}}\Big)^{\frac{4-q\delta_{s,q}}{p\delta_{s,p}-2}}\\
&\quad+|\mu|\Big(1-\frac{\delta_{s,q}}{\delta_{s,p}}\Big)C(s,q)^q c^{q(1-\delta_{s,q})}\Big].
\end{aligned}$$

It is not difficult to check that the right hand side is strictly negative when (2.9) holds, finally implying that $\lambda<0$, as desired, hence the proof is complete. □

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