EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 6732
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Some New Boundedness Results for Variable Marcinkiewicz Fractional Integral Operator on Herz-Morrey-Hardy Spaces

Babar Sultan[1], Amjad Hussain[1,*], Mehvish Sultan[2], Ioan-Lucian Popa[3,4,*]

[1] *Department of Mathematics, Quaid-I-Azam University, Islamabad 45320, Pakistan*
[2] *Department of Mathematics, Capital University of Science and Technology, Islamabad, Pakistan*
[3] *Department of Computing, Mathematics and Electronics, "1 Decembrie 1918" University of Alba Iulia, 510009 Alba Iulia, Romania*
[4] *Faculty of Mathematics and Computer Science, Transilvania University of Brasov, Iuliu Maniu Street 50, 500091 Brasov, Romania*

Abstract. In this paper, we define the idea of Herz-Morrey-Hardy spaces by using variable Herz-Morrey spaces and Hardy spaces. Then we give the atomic characterization of these spaces by using the grand maximal function. Then our main objective is to prove the boundedness of higher order commutators of variable Marcinkiewicz fractional integral operator on Herz-Morrey-Hardy spaces where the exponents defining these spaces are variable. These results also hold for variable Herz-Hardy spaces. The higher order commutators of variable Marcinkiewicz fractional integral operator is the generalization of Marcinkiewicz integral operators, variable Marcinkiewicz fractional integral operator and commutators on Marcinkiewicz fractional integral operators, so these proofs generalize some previous results.

2020 Mathematics Subject Classifications: 42B20, 47B38

Key Words and Phrases: BMO spaces, Marcinkiewicz fractional integral, Herz-Morrey-Hardy spaces

1. Introduction and preliminaries

Let E be an open set in $\mathbb{R}^n$, consider a measurable function $p(\cdot) : E \to [1, \infty)$. The conjugate exponent denoted by $p'(\cdot)$, is defined as $p'(\cdot) = p(\cdot)/(p(\cdot) - 1)$.

The set $\mathcal{P}(E)$ comprises all functions $p(\cdot) : E \to [1, \infty)$. We suppose that

$$1 \leq p^-(E) \leq p(x) \leq p^+(E) < \infty, \tag{1.1}$$

*Corresponding author.
*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6732

Email addresses: babarsultan40@yahoo.com (B. Sultan), a.hussain@qau.edu.pk (A. Hussain), mehvishsultanbaz@gmail.com (M. Sultan), lucian.popa@uab.ro (I.-L. Popa)

such that

$$p^{-} = \operatorname{ess\,inf}\big\{p(x) : x \in E\big\} > 1,$$
$$p^{+} = \operatorname{ess\,sup}\big\{p(x) : x \in E\big\} < \infty.$$

We use the notation $L^{p(\cdot)}(E)$ to represent the space of all measurable functions f defined on E, such that, for a certain $\eta > 0$, $\int_E \left(\frac{|f(x)|}{\eta}\right)^{p(y)} dy < \infty$. Its norm is given as

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\left\{\eta > 0 : \int_E \left(\frac{|f(y)|}{\eta}\right)^{p(y)} dy \leq 1\right\}.$$

Herz spaces with variable exponents have emerged as a generalization of Lebesgue spaces with variable exponents. Boundedness of sublinear operators on Herz spaces with variable exponents, $\dot{K}^{\alpha,q}_{p(\cdot)}$ and $K^{\alpha,q}_{p(\cdot)}$, was shown by Izuki in [1] in 2010. Boundedness results for a wide class of classical operators on Herz spaces were later developed by Almeida and Drihem in 2012 [2]. These spaces were represented as $\dot{K}^{\alpha(\cdot)q}_{p(\cdot)}$ and $K^{\alpha(\cdot),q}_{p(\cdot)}$. Grand variable Herz spaces are the generalization of Herz spaces, for boundedness results in these spaces see [3–8]. For more results in variable exponent function spaces see [9–27]. In [28], the authors introduced Herz-Morrey-Hardy spaces with variable exponents and established the characterization of these spaces in terms of atom. The authors were able to determine the boundedness of certain singular integral operators on these spaces by applying the characterization. In this paper, we define the idea of variable Herz-Morrey-Hardy spaces and using the characterization we obtain boundedness results for some new operator in these spaces.

Many classical function spaces, alongside Hardy-type spaces linked with operators, exhibit atomic and molecular decompositions. These decompositions simplify the action of linear operators on these spaces significantly; see [29–31].

Let $\mathbb{S}^{n-1}$ is denoting the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ with the normalized Lebesgue measure. Let $\Phi \in L^r(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero such that

$$\int_{\mathbb{S}^{n-1}} \Phi(y') d\Phi(y') = 0, \tag{1.2}$$

where $y' = y/|y|$ and y is not zero. The Marcinkiewicz integral is define as

$$\mu_\Phi(g)(z_1) = \left(\int_0^\infty |R_{\Phi,s}(g)(z_1)|^2 \frac{ds}{s^3}\right)^{\frac{1}{2}},$$

where

$$R_{\Phi,s}(g)(z_1) = \int_{|z_1 - z_2| \leq s} \frac{\Phi(z_1 - z_2)}{|z_1 - z_2|^{n-\beta(z_1)-1}} g(z_2) dz_2.$$

Let $h \in BMO(\mathbb{R}^n)$, then the commutators on variable Marcinkiewicz fractional integral operator are given as

$$[h, \mu_\Phi]^m_\beta (g)(z_1) = \left(\int_0^\infty \left| \int_{|z_1 - z_2| \le s} \frac{\Phi(z_1 - z_2)[h(z_1) - h(z_2)]^m}{|z_1 - z_2|^{n-1-\beta(z_1)}} g(z_2) dz_2 \right|^2 \frac{ds}{s^3} \right)^{\frac{1}{2}}.$$

Assume $g \in L^1_{\text{loc}}(\mathbb{R}^n)$, the definition of the Hardy-Littlewood maximal operator is expressed as

$$Mg(z_1) = \sup_{r>0} \frac{1}{|B_r(z_1)|} \int_{B_r(z_1)} |g(z_2)| \, dz_2,$$

where $B_r(z_1) = \{z_2 \in \mathbb{R}^n : |z_1 - z_2| < r\}$.

The set $\mathcal{B}(\mathbb{R}^n)$ is comprised of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ that fulfill the requirement that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Now we will define the well known log-condition

$$|p(h_1) - p(h_2)| \le \frac{C(p)}{-\ln|h_1 - h_2|}, \quad |h_1 - h_2| \le \frac{1}{2}, \quad h_1, h_2 \in E, \tag{1.3}$$

where $C(p) > 0$. And the decay condition: there exists a number $p_\infty \in (1, \infty)$, such that

$$|p(h) - p_\infty| \le \frac{C}{\ln(e + |h|)}, \tag{1.4}$$

and also decay condition

$$|p(h) - p_0| \le \frac{C}{\ln|h|}, |h| \le \frac{1}{2}, \tag{1.5}$$

holds for some $p_0 \in (1, \infty)$. We use these notations in this article:

(i) The set $\mathcal{P}(E)$ consists of all measurable functions $p(\cdot)$ satisfying $p^- > 1$ and $p^+ < \infty$.

(ii) $\mathcal{P}^{\log} = \mathcal{P}^{\log}(E)$ consists of all functions $p \in \mathcal{P}(E)$ satisfying (1.1) and (1.3).

(iii) $\mathcal{P}_\infty(E)$ and $\mathcal{P}_{0,\infty}(E)$ are the subsets of $\mathcal{P}(E)$ and values of these subsets lies in $[1, \infty)$ which satisfy the condition (1.4) and both conditions (1.4) and (1.5) respectively.

(iv)
$$\chi_i = \chi_{F_i}, \; F_i = B_i \setminus B_{i-1}, \; B_i = B(0, 2^i) = \{x \in \mathbb{R}^n : |x| < 2^i\}$$
for all $i \in \mathbb{Z}$.

(v) Let E be a measurable subset in $\mathbb{R}^n$, then $|E|$ denotes the Lebesgue measure and χ_E is the characteristic function.

(vi) Let $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ then $|\beta|$ is defined as $|\beta| = \beta_1, \beta_2 + \cdots + \beta_n$.

(vii) The symbol $\mathbb{N}_0$ denotes the set of all nonnegative integers. For $m \in \mathbb{N}_0$, we denote $\tilde{\chi}_m := \chi_{F_m}$ if $m \geq 1$ and $\tilde{\chi}_0 := \chi_{B_0}$.

(viii) C is a positive constant.

(ix) By $a \lesssim b$, we mean $a \leq Cb$.

Lemma 1. *[32] Let $D > 1$ and $p \in \mathcal{P}_{0,\infty}(\mathbb{R}^n)$. Then*

$$\frac{1}{r_0} o^{\frac{n}{p(0)}} \leq \|\chi_{F_{o,D_o}}\|_{p(\cdot)} \leq r_0 o^{\frac{n}{p(0)}}, \text{ for } 0 < o \leq 1 \tag{1.6}$$

and

$$\frac{1}{r_\infty} o^{\frac{n}{p_\infty}} \leq \|\chi_{F_{o,D_o}}\|_{p(\cdot)} \leq r_\infty o^{\frac{n}{p_\infty}}, \text{ for } o \geq 1, \tag{1.7}$$

respectively, where $r_0 \geq 1$ and $r_\infty \geq 1$ and depending on D but independent of o.

Lemma 2. *[33]*
Let $p(\cdot)$ be a function within the class $\mathcal{B}(\mathbb{R}^n)$. For any ball B in $\mathbb{R}^n$, there exists a positive constant C such that the inequality

$$\frac{1}{|B|} \|\chi_B\|_{p(\cdot)} \|\chi_B\|_{p'(\cdot)} \leq C,$$

holds.

Lemma 3. *[33]*
Assuming that $p(\cdot)$ is a function in the class $\mathcal{B}(\mathbb{R}^n)$, there exists a positive constant C such that, for every ball B in $\mathbb{R}^n$ and every measurable subset S within B, the following inequalities hold:

$$\frac{\|\chi_B\|_{p(\cdot)}}{\|\chi_S\|_{p(\cdot)}} \leq C \frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{p(\cdot)}}{\|\chi_B\|_{p(\cdot)}} \leq C \left(\frac{|S|}{|B|} \right)^{\delta_1}, \quad \frac{\|\chi_S\|_{p'(\cdot)}}{\|\chi_B\|_{p'(\cdot)}} \leq C \left(\frac{|S|}{|B|} \right)^{\delta_2},$$

where δ_1 and δ_2 are constants satisfying $0 < \delta_1, \delta_2 < 1$.

Lemma 4. *[34] Let $f \in L^{p(\cdot)}(E)$, $g \in L^{q(\cdot)}(E)$ where $E \subseteq \mathbb{R}^n$, and $1 \leq p_-(E) \leq p_+(E) \leq \infty$. Then*

$$\|fg\|_{r(\cdot)} \leq \|f\|_{p(\cdot)} \|g\|_{q(\cdot)}$$

where $\frac{1}{r(z)} = \frac{1}{p(z)} + \frac{1}{q(z)}$.

Definition 5 (BMO space)**.** Let h is a locally integrable function then a BMO function is consist of those functions whose mean oscillation given by $\frac{1}{|B|} \int_B |h(i) - h_B| di$ is bounded. A Mathematically,

$$\|h\|_{BMO} = \sup_B \frac{1}{|B|} \int\limits_B |h(i) - h_B| di < \infty.$$

Lemma 6. *[33] Let $j, i \in \mathbb{Z}$ for $i < \ell$, and $h \in BMO(\mathbb{R}^n)$, then*

$$\frac{1}{C}\|h\|_{BMO}^n \leq \sup_{B:ball} \frac{1}{\|\chi_B\|_{p(\cdot)}} \|(h - h_B)^n \chi_B\|_{p(\cdot)} \tag{1.8}$$

$$\leq C\|h\|_{BMO}^n, \tag{1.9}$$

$$||(h - h_{B_i})^n \chi_{B_k}||_{p(\cdot)} \leq C(k-i)^n ||b||_{BMO}^n ||\chi_{B_k}||_{p(\cdot)}. \tag{1.10}$$

Lemma 7 ([35]). *If $a > 0$, $s \in [1, \infty]$, $0 < d \leq s$ and $-m + (m-1)ds < u < \infty$, then*

$$\left(\int_{|z_2| \leq a|z_1|} |z_2|^u |\Phi(z_1 - z_2)|^d dz_2 \right)^{1/d} \leq |z_1|^{(u+m)/d} \|\Phi\|_{L^s(\mathbb{S}^{m-1})}.$$

Definition 8. Let $0 < p \leq \infty$, $q \in \mathcal{P}(\mathbb{R}^n)$, $\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The inhomogeneous Herz space $K_{p,q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ consists of all $f \in L_{\text{loc}}^{q(\cdot)}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{K_{p,q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)} := \|f\chi_{B_0}\|_{L^{q(\cdot)}} + \left(\sum_{k \geq 1} \left\| 2^{\alpha(\cdot)} f\chi_k \right\|_{q(\cdot)}^p \right)^{1/p} < \infty.$$

The homogeneous Herz space $\dot{K}_{p,q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ consists of all $f \in L_{\text{loc}}^{q(\cdot)}(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{\dot{K}_{p,q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)} := \left(\sum_{k \in \mathbb{Z}} \left\| 2^{\alpha(\cdot)} f\chi_k \right\|_{q(\cdot)}^p \right)^{1/p} < \infty.$$

Next we give the definition of variable Herz-Morrey spaces.

Definition 9. Let $p : \mathbb{R}^n \to [1, \infty)$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $u \in [1, \infty)$, and $0 \leq \Gamma < \infty$. The norm of variable Herz-Morrey spaces are defined as:

$$M\dot{K}_{\Gamma,p(\cdot)}^{\alpha(\cdot),u}(\mathbb{R}^n) = \left\{ g \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{M\dot{K}_{\Gamma,p(\cdot)}^{\alpha(\cdot),u}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|g\|_{M\dot{K}_{\Gamma,p(\cdot)}^{\alpha(\cdot),u}(\mathbb{R}^n)} = \sup_{m_0 \in \mathbb{Z}} 2^{-m_0\Gamma} \left(\sum_{k=-\infty}^{m_0} \|2^{k\alpha(\cdot)} g\chi_k\|_{p(\cdot)}^u \right)^{\frac{1}{u}}.$$

For $\Gamma = 0$, variable Herz-Morrey spaces becomes variable Herz spaces.

The next proposition is the generalization of variable exponents Herz spaces in [2].

Proposition 10. *Let α, u, p are as defined in definition 9, then*

$$\|f\|_{M\dot{K}_{\Gamma,p(\cdot)}^{\alpha(\cdot),u}(\mathbb{R}^n)} = \sup_{m_0 \in \mathbb{Z}} 2^{-m_0\Gamma} \left(\sum_{k=-\infty}^{m_0} \|2^{k\alpha(\cdot)} f\chi_k\|_{p(\cdot)}^u \right)^{\frac{1}{u}}$$

$$\approx \max \left\{ \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma} \left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)u} \|f\chi_k\|_{p(\cdot)}^u \right)^{\frac{1}{u}}, \right.$$

$$\sup_{m_0\geq 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma} \left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)u} \|f\chi_k\|_{p(\cdot)}^u \right)^{\frac{1}{u}}$$

$$\left. + \sup_{m_0\geq 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma} \left(\sum_{k=0}^{m_0} 2^{k\alpha_\infty u} \|f\chi_k\|_{p(\cdot)}^u \right)^{\frac{1}{u}} \right\}$$

2. The Atomic Characterization

Let $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of all rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$, and $\mathcal{S}'(\mathbb{R}^n)$ denotes the dual space of $\mathcal{S}(\mathbb{R}^n)$. Let $G_N g$ be the grand maximal function of g defined by

$$G_N g(x) := \sup_{\phi\in\mathcal{A}_N} |\phi_\nabla^*(g)(x)|, \quad x \in \mathbb{R}^n$$

where $\mathcal{A}_N := \left\{ \phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta|\leq N, \forall x\in\mathbb{R}^n} \left| x^\alpha D^\beta \phi(x) \right| \leq 1 \right\}$ and $N > n+1$ and ϕ_∇^* is the nontangential maximal operator defined by

$$\phi_\nabla^*(g)(x) := \sup_{|y-x|<t} |\phi_t * g(y)|,$$

with $\phi_t(x) = t^{-n}\phi\left(\frac{x}{t}\right)$.

Definition 11. Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $0 < q \leq \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 \leq \Gamma < \infty$, and $N > n+1$. The Herz-Morrey-Hardy space with variable exponents $HM\dot{K}_{\Gamma,p(\cdot)}^{\alpha(\cdot),u}(\mathbb{R}^n)$ is defined by

$$HM\dot{K}_{\Gamma,p(\cdot)}^{\alpha(\cdot),u}(\mathbb{R}^n) \quad := \left\{ g \in \mathcal{S}'(\mathbb{R}^n) : \|g\|_{M\dot{K}_{\Gamma,p(\cdot)}^{\alpha(\cdot),u}(\mathbb{R}^n)} := \|G_N g\|_{M\dot{K}_{\Gamma,p(\cdot)}^{\alpha(\cdot),u}(\mathbb{R}^n)} < \infty \right\}.$$

Definition 12. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous both at the origin and infinity, and nonnegative integer $s \geqslant [\alpha_r - n\delta_2]$; here $\alpha_r = \alpha(0)$, if $r < 1$, and $\alpha_r = \alpha_\infty$, if $r \geqslant 1, n\delta_2 \leq \alpha_r < \infty$ and δ_2 as in Lemma 3.

(i) A function a on $\mathbb{R}^n$ is called a central $(\alpha(\cdot), p(\cdot))$ atom, if it satisfies

(1) supp $a \subset B(0,r)$,

(2) $\|a\|_{p(\cdot)} \leq |B(0,r)|^{-\alpha_r/n}$,

(3) $\int_{\mathbb{R}^n} a(x)x^\beta \mathrm{d}x = 0, |\beta| \leq s$.

(ii) A function a on $\mathbb{R}^n$ is called a central $(\alpha(\cdot), p(\cdot))$-atom of restricted type, if it satisfies $2, 3$ and condition given below

(a) supp $\alpha \subset B(0,r)$, $r \geqslant 1$.

Theorem 13. *[28]*
Let $0 < u < \infty, p(\cdot) \in \mathcal{B}(\mathbb{R}^n), 0 \leq \Gamma < \infty$, and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous both at the origin and infinity, $2\lambda \leq \alpha(\cdot), n\delta_2 \leq \alpha(0), \alpha_\infty < \infty$, and δ_2 as in Lemma 3. Then
$f \in HM\dot{K}^{\alpha(\cdot),u}_{\Gamma,p(\cdot)}(\mathbb{R}^n)$ iff $f = \sum_{k=-\infty}^{\infty} \lambda_k a_k$ in the sense of $\mathcal{S}'(\mathbb{R}^n)$, where each a_k is a central $(\alpha(\cdot), p(\cdot))$-atom with support contained in B_k and $\sup\limits_{\vartheta>0} \sup\limits_{m_0\in\mathbb{Z}} 2^{-m_0\Gamma} \sum_{k=-\infty}^{m_0} |\lambda_k|^u < \infty$. Moreover,

$$\|f\|_{HM\dot{K}^{\alpha(\cdot),u}_{\Gamma,p(\cdot)}(\mathbb{R}^n)} \approx \inf\left(\sup_{m_0\in\mathbb{Z}} 2^{-m_0\Gamma}\left(\sum_{k=-\infty}^{m_0} |\lambda_k|^u\right)^{1/u}\right).$$

Theorem 14. *Let $0 \leq \Gamma < \infty$, $0 < u < \infty$, , $q_1(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous both at the origin and infinity. Let α be such that :*

(i) $-\frac{n}{q_1(0)} - v - \frac{n}{s} < \alpha(0) < \frac{n}{q_1'(0)} - v - \frac{n}{s}$

(ii) $-\frac{n}{q_{1\infty}} - v - \frac{n}{s} < \alpha_\infty < \frac{n}{q_{1\infty}'} - v - \frac{n}{s}$.

Then

$$\left\|(|z_1|+1)^{-\lambda(z_1)} [b,\mu_\Phi]^m_\beta f\right\|_{M\dot{K}^{\alpha(\cdot),u}_{\Gamma,q_2(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{HM\dot{K}^{\alpha(\cdot),u}_{\Gamma,q_1(\cdot)}(\mathbb{R}^n)},$$

for $f \in HM\dot{K}^{\alpha(\cdot),u}_{\Gamma,q_1(\cdot)}(\mathbb{R}^n)$.

Proof.
Suppose that $f \in HM\dot{K}^{\alpha(\cdot),u}_{\Gamma,q_1(\cdot)}(\mathbb{R}^n)$. By using Theorem 13, $f = \sum_{i=-\infty}^{\infty} \lambda_i b_i$ converges in $\mathcal{S}'(\mathbb{R}^n)$, where each b_i is a central $(\alpha(\cdot), q_1(\cdot))$-atom with support contained in B_i and

$$\|f\|_{HM\dot{K}^{\alpha(\cdot),u}_{\Gamma,q_1(\cdot)}(\mathbb{R}^n)} \approx \inf\left(\sup_{m_0\in\mathbb{Z}} 2^{-m_0\Gamma}\left(\sum_{i=-\infty}^{m_0} |\lambda_i|^u\right)^{\frac{1}{u}}\right).$$

To keep things simple, we denote $\Lambda = \sup_{m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{i=-\infty}^{m_0} |\lambda_i|^u$. By Proposition 10, we have

$$\left\|(|z_1|+1)^{-\lambda(z_1)} [b,\mu_\Phi]^m_\beta f\right\|^u_{M\dot{K}^{\alpha(\cdot),u}_{\Gamma,q_2(\cdot)}(\mathbb{R}^n)}$$

$$
\begin{aligned}
&\approx \max\left\{ \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u}\left(\sum_{k=-\infty}^{m_0} 2^{k\alpha(0)u}\left\|\left((|z_1|+1)^{-\lambda(z_1)}\left[b,\mu_\Phi\right]_\beta^m f\right)\chi_k\right\|_{q_2(\cdot)}^u\right),\right.\\
&\sup_{m_0\geq 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u}\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)u}\left\|\left((|z_1|+1)^{-\lambda(z_1)}\left[b,\mu_\Phi\right]_\beta^m f\right)\chi_k\right\|_{q_2(\cdot)}^u\right.\\
&\quad\left.\left.+\sum_{k=0}^{m_0} 2^{k\alpha_\infty u}\left\|\left((|z_1|+1)^{-\lambda(z_1)}\left[b,\mu_\Phi\right]_\beta^m f\right)\chi_k\right\|_{q_2(\cdot)}^u\right)\right\}\\
&\lesssim \max\{I, II+III\}.
\end{aligned}
$$

We will find the estimated for I and III and estimate of II can be obtained similarly.

We just need to demonstrate that there is a positive constant C such that $I, II, III \leq C\Lambda$ in order to finish our proof.

Firstly, we will find the estimate of I :

$$
\begin{aligned}
I &= \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u}\left(\sum_{k=-\infty}^{m_0} 2^{k\alpha(0)u}\left\|\left((|z_1|+1)^{-\lambda(z_1)}\left[b,\mu_\Phi\right]_\beta^m f\right)\chi_k\right\|_{q_2(\cdot)}^u\right)\\
&\lesssim \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u}\sum_{k=-\infty}^{m_0} 2^{k\alpha(0)u}\left(\sum_{i=k}^{\infty}|\lambda_i|\left\|\left((|z_1|+1)^{-\lambda(z_1)}\left[b,\mu_\Phi\right]_\beta^m b_i\right)\chi_k\right\|_{q_2(\cdot)}\right)^u\\
&+ \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u}\sum_{k=-\infty}^{m_0} 2^{k\alpha(0)u}\left(\sum_{i=-\infty}^{k-1}|\lambda_i|\left\|\left((|z_1|+1)^{-\lambda(z_1)}\left[b,\mu_\Phi\right]_\beta^m b_i\right)\chi_k\right\|_{q_2(\cdot)}\right)^u\\
&:= I_1 + I_2.
\end{aligned}
$$

Let $k\in\mathbb{Z}$ and $i\leq k$ and a.e. $z_1\in F_k$, $z_2\in F_i$, it is easy to check that $|z_1-z_2|\approx |z_1|\approx 2^k$,

$$
\begin{aligned}
\left|\left(\left[b,\mu_\Phi\right]_\beta^m b_i\right)(z_1)\right| &\leq \left(\int_o^{|z_1|}\left|\int_{|z_1-z_2|\leq t}\frac{\Phi(z_1-z_2)\left[b(z_1)-b(z_2)\right]^m}{|z_1-z_2|^{n-1-\beta(z_1)}}b_i(z_2)dz_2\right|^2\frac{dt}{t^3}\right)^{1/2}\\
&+\left(\int_{|z_1|}^{\infty}\left|\int_{|z_1-z_2|\leq t}\frac{\Phi(z_1-z_2)\left[b(z_1)-b(z_2)\right]^m}{|z_1-z_2|^{n-1-\beta(z_1)}}b_i(z_2)dz_2\right|^2\frac{dt}{t^3}\right)^{1/2}\\
&=: I_{11}+I_{12}.
\end{aligned}
$$

Mean value theorem yields

$$
\left|\frac{1}{|z_1-z_2|^2}-\frac{1}{|z_1|^2}\right| \leq \frac{|z_2|}{|z_1-z_2|^3}. \tag{2.1}
$$

For I_{11}, we get

$$\begin{aligned}
I_{11} \leq & \int\limits_{\mathbb{R}^n} \frac{|\Phi(z_1-z_2)|\,[b(z_1)-b(z_2)]^m}{|z_1-z_2|^{n-1-\beta(z_1)}}\,|b_i(z_2)| \left(\int\limits_{|z_1-z_2|}^{|z_1|} \frac{dt}{t^3} \right)^{1/2} dz_2 \\
\leq & \int\limits_{\mathbb{R}^n} \frac{|\Phi(z_1-z_2)|\,[b(z_1)-b(z_2)]^m}{|z_1-z_2|^{n-1-\beta(z_1)}}\,|b_i(z_2)| \left| \frac{1}{|z_1-z_2|^2} - \frac{1}{|z_1|^2} \right|^{1/2} dz_2 \\
\leq & \int\limits_{\mathbb{R}^n} \frac{|\Phi(z_1-z_2)|\,[b(z_1)-b(z_2)]^m}{|z_1-z_2|^{n-1-\beta(z_1)}}\,|b_i(z_2)| \left| \frac{|z_2}{|z_1-z_2|^3} \right|^{1/2} dz_2 \\
\leq & \frac{2^{l/2}}{|z_1|^{n+\frac{1}{2}} \cdot |z_1|^{-\beta(z_1)}} \int\limits_{F_i} |\Phi(z_1-z_2)|\,[b(z_1)-b(z_2)]^m\,|b_i(z_2)|\,dz_2 \\
\leq & 2^{(i-k)/2} 2^{-kn} |z_1|^{\beta(z_1)} \|b_i\|_{q_1(\cdot)} \|\Phi(z_1-\cdot)\chi_i(\cdot)\|_{q_1'(\cdot)} . \\
\leq & 2^{(i-k)/2} 2^{-kn} |z_1|^{\beta(z_1)} \Bigg\{ |b(z_1)-b_{B_i}|^m \int\limits_{F_i} |\Phi(z_1-z_2)|\,|b_i(z_2)|\,dz_2 \\
& \qquad + \int\limits_{F_i} |b(z_2)-b_{B_i}|^m\,|\Phi(z_1-z_2)|\,|b_i(z_2)|\,dz_2 \Bigg\} \\
\leq & 2^{(i-k)/2} 2^{-kn} |z_1|^{\beta(z_1)} \|b_i(z_2)\|_{q_1(\cdot)} \Big(|b(z_1)-b_{B_i}|^m \|\Phi(z_1-\cdot)\chi_i(\cdot)\|_{q_1'(\cdot)} \\
& \qquad + \|(b(\cdot)-b_{B_i})^m (\Phi(z_1-\cdot)\chi_i(\cdot)\|_{q_1'(\cdot)} \Big) .
\end{aligned}$$

Similarly, we can consider I_{12}, we have

$$\begin{aligned}
I_{12} \leq & \int\limits_{\mathbb{R}^n} \frac{|\Phi(z-1-z_2)|}{|z_1-z_2|^{n-1-\beta(z_1)}}\,|b_i(z_2)| \left(\int\limits_{|z_1|}^{\infty} \frac{dt}{t^3} \right)^{1/2} dz_2 \\
\leq & \int\limits_{\mathbb{R}^n} \frac{|\Phi(z_1-z_2)|}{|z_1-z_2|^{n-1-\beta(z_1)}}\,|b_i(z_2)|\,dz_2 \\
\leq & |z_1|^{-n} |z_1|^{\beta(z_1)} \int\limits_{F_i} |\Phi(z_1-z_2)|\,|b_i(z_2)|\,dz_2 \\
\leq & 2^{-kn} |z_1|^{\beta(z_1)} \|b_i(z_2)\|_{q_1(\cdot)} \left\{ |b(z_1)-b_{B_i}|^m \|\Phi(z_1-\cdot)\chi_i(\cdot)\|_{q_1'(\cdot)} + \|(b(\cdot)-b_{B_i})^m (\Phi(z_1-\cdot)\chi_i(\cdot)\|_{q_1'(\cdot)} \right\} .
\end{aligned}$$

We define $q_1(\cdot)$ by the relation $\frac{1}{q_1'(x)} = \frac{1}{q_1(x)} + \frac{1}{s}$. By using Lemma (7) and generalized Hölder's inequality we have

$$\|\Phi(z_1-\cdot)\chi_i(\cdot)\|_{q_1'(\cdot)} \leq \|\Phi(z_1-\cdot)\chi_i(\cdot)\|_{L^s(\mathcal{S}^{n-1})} \|\chi_i(\cdot)\|_{q_1(\cdot)}$$

$$\leq 2^{-iv}\left(\int\limits_{2^{l-1}<|z_2|<2^i}|\Phi(z_1-z_2)|^s|z_2|^{sv}dz_2\right)^{1/s}\|\chi_{B_i}\|_{q_1(\cdot)}$$
$$\leq 2^{-iv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_{B_i}\|_{q_1(\cdot)}.$$

Similarly, by using Lemma (6) we have

$$\begin{aligned}\|(b(\cdot)-b_{B_i})^m(\Phi(z_1-\cdot)\chi_i(\cdot)\|_{q_1'(\cdot)} &\leq \|\Phi(z_1-\cdot)\chi_i(\cdot)\|_s\|(b(\cdot)-b_{B_i})^m\chi_i(\cdot)\|_{q(\cdot)}\\ &\leq C\|f\|_{BMO}^m\|\chi_{B_i}\|_{q(\cdot)}\|\Phi(z_1-\cdot)\chi_i(\cdot)\|_s\\ &\leq C\|f\|_{BMO}^m 2^{-iv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_{B_i}\|_{q_1(\cdot)}.\end{aligned}$$

It is known, see e.g. [36] that

$$\begin{aligned}I^{\beta(\cdot)}\left((b(z_1)-b_{B_i})^m\chi_{B_k}\right)(z_1) &\geq I^{\beta(\cdot)}(\chi_{B_k})(z_1).(\chi_{B_k})(z_1)\\ &=\int_{B_k}\frac{|b(z_1)-b_{B_i}|^m}{|z_1-z_2|^{\beta(z_1)-n}}dz_2.\chi_{B_k}(z_1)\\ &\geq C|b(z_1)-b_{B_i}|^m|z_1|^{\beta(z_1)}.\chi_{B_k}(z_1)\\ &\geq C|b(z_1)-b_{B_i}|^m|z_1|^{\beta(z_1)}.\chi_k(z_1).\end{aligned}$$

Consequently, by using weighted Sobolev estimates [37] we have

$$\begin{aligned}&\left\|(b(z_1)-b_{B_i})^m|z_1|^{\beta(z_1)}\chi_k(z_1)(1+|z_1|)^{-\lambda(z_1)}\right\|_{q_2(\cdot)}\\ &\leq\left\|(1+|z_1|)^{-\lambda(z_1)}(I^{\beta(\cdot)}((b(z_1)-b_{B_i})^m\chi_{B_k})(z_1))\right\|_{q_2(\cdot)}\\ &\leq\|(b(z_1)-b_{B_i})^m\chi_{B_k})(z_1)\|_{q_1(\cdot)}.\end{aligned}$$

Thus we have

$$\begin{aligned}&\left\|\chi_k(1+|z_1|)^{-\lambda(z_1)}[b,\mu_\Phi]^m_\beta b_i\right\|_{q_2(\cdot)}\\ &\leq C2^{-kn}\|b_i\|_{q_1(\cdot)}\Big\{\left\|(b(z_1)-b_{B_i})^m|z_1|^{\beta(z_1)}\chi_k(z_1)(1+|z_1|)^{-\lambda(z_1)}\right\|_{q_2(\cdot)}2^{-iv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_{B_i}\|_{q_1(\cdot)}\\ &+\|f\|^m_{BMO}2^{-iv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_{B_i}\|_{q_1(\cdot)}\left\||z_1|^{\beta(z_1)}\chi_k(z_1)(1+|z_1|)^{-\lambda(z_1)}\right\|_{q_2(\cdot)}\Big\}\\ &\leq C2^{-kn}\|b_i\|_{q_1(\cdot)}\Big\{(k-i)^m\|f\|^m_{BMO}\|\chi_{B_k}\|_{q_1(\cdot)}2^{-iv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_{B_i}\|_{q_1(\cdot)}\\ &+\|f\|^m_{BMO}2^{-iv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_{B_i}\|_{q_1(\cdot)}\|\chi_{B_k}\|_{q_1(\cdot)}\Big\}\\ &\leq C2^{-kn}\|b_i\|_{q_1(\cdot)}(k-i)^m\|f\|^m_{BMO}\|\chi_{B_k}\|_{q_1(\cdot)}2^{-iv}2^{k(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_{B_i}\|_{q_1(\cdot)}\\ &\leq C(k-i)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|f\|^m_{BMO}2^{-kn}2^{-iv}2^{k(v+\frac{n}{s})}\|\chi_{B_k}\|_{q_1(\cdot)}\|\chi_{B_i}\|_{q_1(\cdot)}\|b_i\|_{q_1(\cdot)}.\end{aligned}$$

Therefore, when $0 < u \leq 1$ and $v_1 = n/q_1'(0) - v - \frac{n}{s} - \alpha(0)$, we get

$$
\begin{aligned}
I_1 &= \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=-\infty}^{m_0} 2^{k\alpha(0)u} \left(\sum_{i=k}^{\infty} |\lambda_i| \left\| \left((|z_1|+1)^{-\lambda(z_1)} [b,\mu_\Phi]_\beta^m b_i \right) \chi_k \right\|_{q_2(\cdot)} \right)^u \\
&\lesssim \|f\|_{BMO}^m \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=-\infty}^{m_0} 2^{\alpha(0)ku} \left(\sum_{i=k}^{\infty} |\lambda_i| 2^{-kn} 2^{-iv} 2^{k(v+\frac{n}{s})} \|\chi_{B_k}\|_{q_1(\cdot)} \|\chi_{B_i}\|_{q_1(\cdot)} (k-i)^m 2^{-\alpha_i i} \right)^u \\
&\lesssim \|f\|_{BMO}^m \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=-\infty}^{m_0} 2^{k\alpha(0)u} \\
&\quad \times \left(\sum_{i=k}^{-1} |\lambda_i|^u 2^{-\alpha(0)iu} 2^{u(i-k)(n/q_1'(0)-v-\frac{n}{s})} (k-i)^{mu} \right. \\
&\quad \left. + \sum_{i=0}^{\infty} |\lambda_i|^u 2^{-\alpha_\infty iu} 2^{-ui(n/q_{1\infty}+v+\frac{n}{s})+uk(n/q_1(0)+v+\frac{n}{s})} (k-i)^{mu} \right) \\
&\lesssim \|f\|_{BMO}^m \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=-\infty}^{m_0} \sum_{i=k}^{-1} |\lambda_i|^u 2^{v_1(i-k)u} (k-i)^{mu} \\
&+ \|f\|_{BMO}^m \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=-\infty}^{m_0} 2^{(\alpha(0)+(n/q_1(0)+v+\frac{n}{s}))ku} \sum_{i=0}^{\infty} |\lambda_i|^u 2^{((n/q_{1\infty}+v+\frac{n}{s})-\alpha_\infty)iu} (k-i)^{mu} \\
&\lesssim \|f\|_{BMO}^m \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{i=-\infty}^{-1} |\lambda_i|^u \sum_{k=-\infty}^{i} 2^{v_1(i-k)u} (k-i)^{mu} \\
&+ \|f\|_{BMO}^m \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{i=0}^{\infty} \sum_{j=-\infty}^{i} |\lambda_j|^u 2^{(-ui(n/q_{1\infty}+v+\frac{n}{s}+\alpha_\infty))} (k-i)^{mu} \\
&\lesssim \|f\|_{BMO}^m \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{i=-\infty}^{m_0} |\lambda_i|^u + \Lambda \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{i=0}^{\infty} 2^{(-ui(n/q_{1\infty}+v+\frac{n}{s}+\alpha_\infty))} (k-i)^{mu} \\
&\lesssim \|f\|_{BMO}^m \Lambda.
\end{aligned}
$$

Now we will the estimate for the second case when $1 < u < \infty$. Let $\frac{1}{u} + \frac{1}{u'} = 1$, we obtain

$$
\begin{aligned}
I_1 &= \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=-\infty}^{m_0} 2^{k\alpha(0)u} \left(\sum_{i=k}^{\infty} |\lambda_i| \left\| \left((|z_1|+1)^{-\lambda(z_1)} [b,\mu_\Phi]_\beta^m b_i \right) \chi_k \right\|_{q_2(\cdot)} \right)^u \\
&\lesssim \|f\|_{BMO}^m \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=-\infty}^{m_0} 2^{\alpha(0)ku} \left(\sum_{i=k}^{\infty} |\lambda_i| 2^{-kn} 2^{-iv} 2^{k(v+\frac{n}{s})} \|\chi_{B_k}\|_{q_1(\cdot)} \|\chi_{B_i}\|_{q_1(\cdot)} (k-i)^m 2^{-\alpha_i i} \right)^u
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \|f\|_{BMO}^{m} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u}\\
&\times \sum_{k=-\infty}^{m_0} 2^{k\alpha(0)u} \left(\sum_{i=k}^{-1} |\lambda_i|^u \, 2^{-\alpha(0)iu} 2^{u(i-k)(n/q_1'(0)-v-\frac{n}{s})} (k-i)^{mu} \right.\\
&\left. + \sum_{i=0}^{\infty} |\lambda_i|^u \, 2^{-\alpha_\infty iu} 2^{-ui(n/q_{1\infty}+v+\frac{n}{s})+uk(n/q_1(0)+v+\frac{n}{s})} (k-i)^{mu} \right)\\
&\lesssim \|f\|_{BMO}^{m} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=-\infty}^{m_0} \left(\sum_{i=k}^{-1} |\lambda_i|^u \, 2^{v_1(i-k)u/2} \right) \times \left(\sum_{i=k}^{-1} 2^{\alpha(0)(k-i)u'/2} (k-i)^{mu'/2} \right)^{u/u'}\\
&+ \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=-\infty}^{m_0} 2^{(\alpha(0)n/q_1(0)+v+\frac{n}{s})ku} \left(\sum_{i=0}^{\infty} |\lambda_i|^u \, 2^{-iu(\alpha_\infty+n/q_{1\infty}+v+\frac{n}{s})u/2} \right)\\
&\times \left(\sum_{i=0}^{\infty} 2^{-iu(\alpha_\infty+n/q_{1\infty}+v+\frac{n}{s})u'/2} (k-i)^{mu'/2} \right)^{u/u'}\\
&\lesssim \|f\|_{BMO}^{m} \|f\|_{BMO}^{m} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=-\infty}^{m_0} \left(\sum_{i=k}^{-1} |\lambda_i|^u \, 2^{v_1(i-k)u/2} \right)\\
&+ \|f\|_{BMO}^{m} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=-\infty}^{m_0} 2^{(\alpha(0)n/q_1(0)+v+\frac{n}{s})ku} \left(\sum_{i=0}^{\infty} |\lambda_i|^u \, 2^{-iu(\alpha_\infty+n/q_{1\infty}+v+\frac{n}{s})u/2} \right)\\
&\lesssim \|f\|_{BMO}^{m} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{i=-\infty}^{-1} |\lambda_i|^u \sum_{k=-\infty}^{i} 2^{v_1(i-k)u/2}\\
&+ \|f\|_{BMO}^{m} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \left(\sum_{i=0}^{\infty} |\lambda_i|^u \, 2^{-iu(\alpha_\infty+n/q_{1\infty}+v+\frac{n}{s})u/2} \right)\\
&\lesssim \|f\|_{BMO}^{m} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{i=-\infty}^{-1} |\lambda_i|^u \sum_{k=-\infty}^{i} 2^{v_1(i-k)u/2}\\
&+ \|f\|_{BMO}^{m} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{i=0}^{\infty} \sum_{j=-\infty}^{i} |\lambda_j|^u \, 2^{(-ui(n/q_{1\infty}+v+\frac{n}{s}+\alpha_\infty))/2}\\
&\lesssim \|f\|_{BMO}^{m} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{i=-\infty}^{m_0} |\lambda_i|^u + \|f\|_{BMO}^{m} \Lambda \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{i=0}^{\infty} 2^{(-ui(n/q_{1\infty}+v+\frac{n}{s}+\alpha_\infty))/2}\\
&\lesssim \|f\|_{BMO}^{m} \Lambda.
\end{aligned}
$$

Second, we estimate I_2. Therefore, when $0 < u \leq 1$, we get

$$
I_2 = \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=-\infty}^{m_0} 2^{k\alpha(0)u} \left(\sum_{i=-\infty}^{k-1} |\lambda_i| \left\| \left((|z_1|+1)^{-\lambda(z_1)} [b, \mu_\Phi]_\beta^m \, b_i \right) \chi_k \right\|_{q_2(\cdot)} \right)^u
$$

$$\lesssim \|f\|^m_{BMO} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=-\infty}^{m_0} 2^{\alpha(0)ku} \left(\sum_{i=-\infty}^{k-1} |\lambda_i| \, 2^{-kn} 2^{-iv} 2^{k(v+\frac{n}{s})} \|\chi_{B_k}\|_{q_1(\cdot)} \|\chi_{B_i}\|_{q_1(\cdot)} (k-i)^m 2^{-\alpha_i i} \right)^u$$

$$\lesssim \|f\|^m_{BMO} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=-\infty}^{m_0} 2^{k\alpha(0)u} \left(\sum_{i=-\infty}^{k-1} |\lambda_i|^u \, 2^{-\alpha(0)iu} 2^{u(i-k)(n/q_1'(0)-v-\frac{n}{s})} (k-i)^{mu} \right)$$

$$\lesssim \|f\|^m_{BMO} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=-\infty}^{m_0} \sum_{i=-\infty}^{k-1} |\lambda_i|^u \, 2^{v_1(i-k)u} (k-i)^{mu}$$

$$\lesssim \|f\|^m_{BMO} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{i=-\infty}^{-1} |\lambda_i|^u \sum_{k=-\infty}^{i} 2^{v_1(i-k)u} (k-i)^{mu}$$

$$\lesssim \|f\|^m_{BMO} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{i=-\infty}^{m_0} |\lambda_i|^u$$

$$\lesssim \|f\|^m_{BMO} \, \Lambda.$$

Now we will the estimate for the second case when $1 < u < \infty$. Let $\frac{1}{u} + \frac{1}{u'} = 1$, we obtain

$$I_2 = \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=-\infty}^{m_0} 2^{k\alpha(0)u} \left(\sum_{i=-\infty}^{k-1} |\lambda_i| \left\| \left((|z_1|+1)^{-\lambda(z_1)} [b, \mu_\Phi]^m_\beta \, b_i \right) \chi_k \right\|_{q_2(\cdot)} \right)^u$$

$$\lesssim \|f\|^m_{BMO} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=-\infty}^{m_0} 2^{\alpha(0)ku} \left(\sum_{i=-\infty}^{k-1} |\lambda_i| \, 2^{-kn} 2^{-iv} 2^{k(v+\frac{n}{s})} \|\chi_{B_k}\|_{q_1(\cdot)} \|\chi_{B_i}\|_{q_1(\cdot)} (k-i)^m 2^{-\alpha_i i} \right)^u$$

$$\lesssim \|f\|^m_{BMO} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=-\infty}^{m_0} 2^{k\alpha(0)u} \left(\sum_{i=k}^{-1} |\lambda_i|^u \, 2^{-\alpha(0)iu} 2^{u(i-k)(n/q_1'(0)-v-\frac{n}{s})} (k-i)^{mu} \right)$$

$$\lesssim \|f\|^m_{BMO} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=-\infty}^{m_0} \left(\sum_{i=k}^{-1} |\lambda_i|^u \, 2^{v_1(i-k)u/2} \right) \times \left(\sum_{i=k}^{-1} 2^{\alpha(0)(k-i)u'/2} (k-i)^{mu'/2} \right)^{u/u'}$$

$$\lesssim \|f\|^m_{BMO} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=-\infty}^{m_0} \left(\sum_{i=k}^{-1} |\lambda_i|^u \, 2^{v_1(i-k)u/2} \right)$$

$$\lesssim \|f\|^m_{BMO} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{i=-\infty}^{-1} |\lambda_i|^u \sum_{k=-\infty}^{i} 2^{v_1(i-k)u/2}$$

$$\lesssim \|f\|^m_{BMO} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{i=-\infty}^{-1} |\lambda_i|^u \sum_{k=-\infty}^{i} 2^{v_1(i-k)u/2}$$

$$\lesssim \|f\|^m_{BMO} \sup_{m_0<0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{i=-\infty}^{m_0} |\lambda_i|^u$$

$$\lesssim \|f\|_{BMO}^m \Lambda.$$

Finally, we estimate III :

$$\begin{aligned}
III &= \sup_{m_0\geq 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=0}^{m_0} 2^{k\alpha_\infty u} \left\| \left((|z_1|+1)^{-\lambda(z_1)} [b,\mu_\Phi]_\beta^m f \right) \chi_k \right\|_{q_2(\cdot)}^u \\
&\lesssim \sup_{m_0\geq 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=0}^{m_0} 2^{k\alpha_\infty u} \left(\sum_{i=k}^{\infty} |\lambda_i| \left\| \left((|z_1|+1)^{-\lambda(z_1)} [b,\mu_\Phi]_\beta^m b_i \right) \chi_k \right\|_{q_2(\cdot)} \right)^u \\
&+ \sup_{m_0\geq 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=0}^{m_0} 2^{k\alpha_\infty u} \left(\sum_{i=-\infty}^{k-1} |\lambda_i| \left\| \left((|z_1|+1)^{-\lambda(z_1)} [b,\mu_\Phi]_\beta^m b_i \right) \chi_k \right\|_{q_2(\cdot)} \right)^u \\
&:= III_1 + III_2.
\end{aligned}$$

If $k\in\mathbb{Z}$ and $i\geq k+1$ and a.e. $z_1\in F_k$, $z_2\in F_i$, then $|z_1-z_2|\approx|z_2|\approx 2^i$,

$$\begin{aligned}
|\mu_\Phi(b_i)(z_1)| &\leq \left(\int_o^{|z_2|} \left| \int_{|z_1-z_2|\leq t} \frac{\Phi(z_1-z_2)}{|z_1-z_2|^{n-1-\beta(z_1)}} b_i(z_2) dz_2 \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&+ \left(\int_{|z_2|}^{\infty} \left| \int_{|z_1-z_2|\leq t} \frac{\Phi(z_1-z_2)}{|z_1-z_2|^{n-1-\beta(z_1)}} b_i(z_2) dz_2 \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&=: I_{31} + I_{32}.
\end{aligned}$$

It is easy to find that

$$\begin{aligned}
I_{31} \leq & 2^{(i-k)/2} 2^{-in} |z_1|^{\beta(z_1)} \|b_i(z_2)\|_{q_1(\cdot)} \Big(|b(z_1)-b_{B_i}|^m \|\Phi(z_1-\cdot)\chi_i(\cdot)\|_{q_1'(\cdot)} \\
&+ \|(b(\cdot)-b_{B_i})^m (\Phi(z_1-\cdot)\chi_i(\cdot)\|_{q_1'(\cdot)} \Big).
\end{aligned}$$

Similarly we have

$$I_{32} \leq 2^{-in} |z_1|^{\beta(z_1)} \|b_i(z_2)\|_{q_1(\cdot)} \left\{ |b(z_1)-b_{B_i}|^m \|\Phi(z_1-\cdot)\chi_i(\cdot)\|_{q_1'(\cdot)} + \|(b(\cdot)-b_{B_i})^m (\Phi(z_1-\cdot)\chi_i(\cdot)\|_{q_1'(\cdot)} \right\}.$$

$$\begin{aligned}
&\left\| \chi_k (1+|z_1|)^{-\lambda(z_1)} [b,\mu_\Phi]_\beta^m (g\chi_i) \right\|_{q_2(\cdot)} \\
&\leq C 2^{-in} \|b_i\|_{q_1(\cdot)} \Big\{ \left\| (b(z_1)-b_{B_i})^m |z_1|^{\beta(z_1)} \chi_k(z_1) (1+|z_1|)^{-\lambda(z_1)} \right\|_{q_2(\cdot)} 2^{-iv} 2^{k(v+\frac{n}{s})} \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \|\chi_{B_i}\|_{q_1(\cdot)} \\
&+ \|f\|_{BMO}^m 2^{-iv} 2^{k(v+\frac{n}{s})} \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \|\chi_{B_i}\|_{q_1(\cdot)} \left\| |z_1|^{\beta(z_1)} \chi_k(z_1) (1+|z_1|)^{-\lambda(z_1)} \right\|_{q_2(\cdot)} \Big\}
\end{aligned}$$

$$\le C2^{-in}\left\|b_i\right\|_{q_1(\cdot)}\Big\{(k-i)^m\left\|f\right\|_{BMO}^m\left\|\chi_{B_k}\right\|_{q_1(\cdot)}2^{-iv}2^{k(v+\frac{n}{s})}\left\|\Phi\right\|_{L^s(\mathbb{S}^{n-1})}\left\|\chi_{B_i}\right\|_{q_1(\cdot)}$$
$$+\left\|f\right\|_{BMO}^m 2^{-iv}2^{k(v+\frac{n}{s})}\left\|\Phi\right\|_{L^s(\mathbb{S}^{n-1})}\left\|\,\right\|\chi_{B_i}\left\|_{q_1(\cdot)}\right\|\chi_{B_k}\left\|_{q_1(\cdot)}\right\Big\}$$
$$\le C2^{-in}\left\|b_i\right\|_{q_1(\cdot)}(k-i)^m\left\|f\right\|_{BMO}^m\left\|\chi_{B_k}\right\|_{q_1(\cdot)}2^{-iv}2^{k(v+\frac{n}{s})}\left\|\Phi\right\|_{L^s(\mathbb{S}^{n-1})}\left\|\chi_{B_i}\right\|_{q_1(\cdot)}$$
$$\le C(k-i)^m\left\|\Phi\right\|_{L^s(\mathbb{S}^{n-1})}\left\|f\right\|_{BMO}^m 2^{-in}2^{-iv}2^{k(v+\frac{n}{s})}\left\|\chi_{B_k}\right\|_{q_1(\cdot)}\left\|\chi_{B_i}\right\|_{q_1(\cdot)}\left\|b_i\right\|_{q_1(\cdot)}.$$

Therefore, when $0<u\le 1$ and $v_2=v+\frac{n}{s}+\frac{n}{q_{1\infty}}+\alpha_\infty$, we get

$$\begin{aligned}
III_1 &= \sup_{m_0\ge 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u}\sum_{k=0}^{m_0}2^{k\alpha_\infty u}\left(\sum_{i=k}^{\infty}|\lambda_i|\left\|\left((|z_1|+1)^{-\lambda(z_1)}\,[b,\mu_\Phi]_\beta^m\, b_i\right)\chi_k\right\|_{q_2(\cdot)}\right)^u\\
&\lesssim \left\|f\right\|_{BMO}^m \sup_{m_0\ge 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u}\sum_{k=0}^{m_0}2^{k\alpha_\infty u}\left(\sum_{i=k}^{\infty}|\lambda_i|\,2^{-kn}2^{-iv}2^{k(v+\frac{n}{s})}\left\|\chi_{B_k}\right\|_{q_1(\cdot)}\left\|\chi_{B_i}\right\|_{q_1(\cdot)}(k-i)^m 2^{-\alpha_i i}\right)^u\\
&\lesssim \left\|f\right\|_{BMO}^m \sup_{m_0\ge 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u}\sum_{k=0}^{m_0}\sum_{i=k}^{\infty}|\lambda_i|^u\,2^{v_2(k-i)u}(k-i)^{mu}\\
&\lesssim \left\|f\right\|_{BMO}^m \sup_{m_0\ge 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u}\sum_{i=0}^{\infty}|\lambda_i|^u\sum_{k=0}^{i}2^{v_2(k-i)u}(k-i)^{mu}\\
&\lesssim \left\|f\right\|_{BMO}^m\,\Lambda.
\end{aligned}$$

Now we will the estimate for the second case when $1<u<\infty$. Let $\frac{1}{u}+\frac{1}{u'}=1$, we obtain

$$\begin{aligned}
III_1 &= \sup_{m_0\ge 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u}\sum_{k=0}^{m_0}2^{k\alpha_\infty u}\left(\sum_{i=k}^{\infty}|\lambda_i|\left\|\left((|z_1|+1)^{-\lambda(z_1)}\,[b,\mu_\Phi]_\beta^m\, b_i\right)\chi_k\right\|_{q_2(\cdot)}\right)^u\\
&\lesssim \left\|f\right\|_{BMO}^m \sup_{m_0\ge 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u}\sum_{k=0}^{m_0}2^{k\alpha_\infty u}\left(\sum_{i=k}^{\infty}|\lambda_i|\,2^{-kn}2^{-iv}2^{k(v+\frac{n}{s})}\left\|\chi_{B_k}\right\|_{q_1(\cdot)}\left\|\chi_{B_i}\right\|_{q_1(\cdot)}(k-i)^m 2^{-\alpha_i i}\right)^u\\
&\lesssim \left\|f\right\|_{BMO}^m \sup_{m_0\ge 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u}\sum_{k=0}^{m_0}\sum_{i=k}^{\infty}|\lambda_i|^u\,2^{v_2(k-i)u}(k-i)^{mu}\\
&\lesssim \left\|f\right\|_{BMO}^m \sup_{m_0\ge 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u}\sum_{k=0}^{m_0}\left(\sum_{i=k}^{\infty}|\lambda_i|^u\,2^{v_2(i-k)u/2}\right)\times\left(\sum_{i=k}^{\infty}2^{\alpha(0)(k-i)u'/2}(k-i)^{mu'/2}\right)^{u/u'}\\
&\lesssim \left\|f\right\|_{BMO}^m \sup_{m_0\ge 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u}\sum_{k=0}^{m_0}\left(\sum_{i=k}^{\infty}|\lambda_i|^u\,2^{v_2(i-k)u/2}\right)\\
&\lesssim \left\|f\right\|_{BMO}^m \sup_{m_0\ge 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u}\sum_{i=-\infty}^{-1}|\lambda_i|^u\sum_{k=-\infty}^{i}2^{v_2(i-k)u/2}
\end{aligned}$$

$$\lesssim \|f\|_{BMO}^{m} \sup_{m_0\ge 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{i=0}^{\infty} |\lambda_i|^u \sum_{k=0}^{i} 2^{v_2(k-i)u/2}$$
$$\lesssim \|f\|_{BMO}^{m} \Lambda.$$

Next we have

$$\begin{aligned}
III_2 =& \sup_{m_0\ge 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=0}^{m_0} 2^{k\alpha_\infty u} \left(\sum_{i=-\infty}^{k-1} |\lambda_i| \left\| \left((|z_1|+1)^{-\lambda(z_1)} [b,\mu_\Phi]_\beta^m b_i \right) \chi_k \right\|_{q_2(\cdot)} \right)^u \\
\lesssim& \sup_{m_0\ge 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=0}^{m_0} 2^{k\alpha_\infty u} \left(\sum_{i=-\infty}^{-1} |\lambda_i| \left\| \left((|z_1|+1)^{-\lambda(z_1)} [b,\mu_\Phi]_\beta^m b_i \right) \chi_k \right\|_{q_2(\cdot)} \right)^u \\
&+ \sup_{m_0\ge 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=0}^{m_0} 2^{k\alpha_\infty u} \left(\sum_{i=0}^{k-1} |\lambda_i| \left\| \left((|z_1|+1)^{-\lambda(z_1)} [b,\mu_\Phi]_\beta^m b_i \right) \chi_k \right\|_{q_2(\cdot)} \right)^u \\
\lesssim& III_2^1 + III_2^2.
\end{aligned}$$

Estimate of second term is essentially similar to III_1. For III_2^1, we have

$$\begin{aligned}
&\|(1+|z_1|)^{-\lambda(z_1)} [b,\mu_\Phi]_\beta^m b_i \chi_k\|_{q_2(\cdot)} \\
\le& C(k-i)^m \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \|f\|_{BMO}^m 2^{-kn} 2^{-iv} 2^{k(v+\frac{n}{s})} \|\chi_{B_k}\|_{q_1(\cdot)} \|\chi_{D_l}\|_{q_1(\cdot)} \|b_i\|_{q_1(\cdot)} \\
\le& C(k-i)^m \|\Phi\|_{L^s(\mathbb{S}^{n-1})} \|f\|_{BMO}^m 2^{i(\frac{n}{q_1(0)}-v)} 2^{k(v+\frac{n}{s}-\frac{n}{q'_{1\infty}})} \|b_i\|_{q_1(\cdot)}.
\end{aligned}$$

When $0<u\le 1$, we have

$$\begin{aligned}
III_2^1 =& \sup_{m_0\ge 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=0}^{m_0} 2^{\alpha_\infty ku} \left(\sum_{i=-\infty}^{-1} |\lambda_i| \left\| \left((|z_1|+1)^{-\lambda(z_1)} [b,\mu_\Phi]_\beta^m b_i \right) \chi_k \right\|_{q_2(\cdot)} \|b_i\|_{q_1(\cdot)} \right)^u \\
\lesssim& \|f\|_{BMO}^m \sup_{m_0\ge 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=0}^{m_0} 2^{\alpha_\infty ku} \left(\sum_{i=-\infty}^{-1} |\lambda_i|^u (k-i)^{mu} 2^{lu(\frac{n}{q_1(0)}-v)} 2^{ku(v+\frac{n}{s}-\frac{n}{q'_{1\infty}})} \|b_i\|_{q_1(\cdot)}^u \right) \\
=& \|f\|_{BMO}^m \sup_{m_0\ge 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \\
&\times \sum_{k=0}^{m_0} 2^{\alpha_\infty ku} \left(\sum_{i=-\infty}^{-1} |\lambda_i|^u (k-i)^{mu} 2^{lu(\frac{n}{q_1(0)}-v-\alpha_\infty)} 2^{ku(v+\frac{n}{s}-\frac{n}{q'_{1\infty}})} \right) \\
\lesssim& \|f\|_{BMO}^m \sup_{m_0\ge 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=0}^{m_0} 2^{ku(v+\frac{n}{s}-\frac{n}{q'_{1\infty}})} (k-i)^{mu} \sum_{i=-\infty}^{-1} |\lambda_i|^u 2^{lu(\frac{n}{q_1(0)}-v-\alpha_\infty)} \\
\lesssim& \|f\|_{BMO}^m \sup_{m_0\ge 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{i=-\infty}^{-1} |\lambda_i|^u
\end{aligned}$$

$$\lesssim \|f\|_{BMO}^{m} \Lambda.$$

When $1 < u < \infty$, we have

$$\begin{aligned}
III_2^1 &= \sup_{m_0\geq 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=0}^{m_0} 2^{\alpha_\infty ku} \left(\sum_{i=-\infty}^{-1} |\lambda_i| \left\| \left((|z_1|+1)^{-\lambda(z_1)} [b,\mu_\Phi]_\beta^m b_i \right) \chi_k \right\|_{q_2(\cdot)} \|b_i\|_{q_1(\cdot)} \right)^u \\
&\lesssim \|f\|_{BMO}^{m} \sup_{m_0\geq 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=0}^{m_0} 2^{\alpha_\infty ku} \left(\sum_{i=-\infty}^{-1} |\lambda_i|^u (k-i)^{mu} 2^{lu(\frac{n}{q_1(0)}-v)} 2^{ku(v+\frac{n}{s}-\frac{n}{q'_{1\infty}})} \|b_i\|_{q_1(\cdot)}^u \right) \\
&\lesssim \|f\|_{BMO}^{m} \sup_{m_0\geq 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=0}^{m_0} 2^{\alpha_\infty ku} 2^{ku(v+\frac{n}{s}-\frac{n}{q'_{1\infty}})} \left(\sum_{i=-\infty}^{-1} |\lambda_i|^u (k-i)^{mu} 2^{lu(\frac{n}{q_1(0)}-v-\alpha(0))} \right) \\
&\lesssim \|f\|_{BMO}^{m} \sup_{m_0\geq 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=0}^{m_0} \left(\sum_{i=-\infty}^{-1} |\lambda_i|^u (k-i)^{mu} 2^{lu(\frac{n}{q_1(0)}-v-\alpha(0))} \right) \\
&\lesssim \|f\|_{BMO}^{m} \sup_{m_0\geq 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=0}^{m_0} \left(\sum_{i=0}^{-1} |\lambda_i|^u 2^{i(\frac{n}{q_1(0)}-v-\alpha(0))u/2} \right) \\
&\quad \times \left(\sum_{i=0}^{-1} (k-i)^{mu'/2} 2^{i(\frac{n}{q_1(0)}-v-\alpha(0))u'/2} \right)^{u/(u)'} \\
&\lesssim \|f\|_{BMO}^{m} \sup_{m_0\geq 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{k=0}^{m_0} \left(\sum_{i=0}^{-1} |\lambda_i|^u 2^{i(\frac{n}{q_1(0)}-v-\alpha(0))u/2} \right) \\
&\lesssim \|f\|_{BMO}^{m} \sup_{m_0\geq 0, m_0\in\mathbb{Z}} 2^{-m_0\Gamma u} \sum_{i=-\infty}^{-1} |\lambda_i|^u \\
&\lesssim \|f\|_{BMO}^{m} \Lambda.
\end{aligned}$$

Thus proof of the Theorem is completed.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] M. Izuki. Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization. *Analysis Mathematica*, 36:33–50, 2010.

[2] A. Almeida and D. Drihem. Maximal, potential and singular type operators on Herz spaces with variable exponents. *Journal of Mathematical Analysis and Applications*, 394(2):781–795, 2012.

[3] B. Sultan and M. Sultan. Boundedness of commutators of rough Hardy operators on grand variable Herz spaces. *Forum Mathematicum*, 2023.

[4] B. Sultan, M. Sultan, and I. Khan. On Sobolev theorem for higher commutators of fractional integrals in grand variable Herz spaces. *Communications in Nonlinear Science and Numerical Simulation*, 126:107489, 2023.

[5] A. Hussain, Naqash Sarfraz, and F. Gurbuz. Sharp weak bounds for p-adic Hardy operators on p-adic linear spaces. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 71(4):919–929, 2022.

[6] A. Hussain, N. Sarfraz, et al. The boundedness of commutators of rough p-adic fractional Hardy type operators on Herz-type spaces. *J. Inequal. Appl.*, 2021:123, 2021. Article ID 123.

[7] A. Hussain, N. Sarfraz, Ilyas Khan, and A. M. Alqahtani. Estimates for Commutators of Bilinear Fractional p-Adic Hardy Operator on Herz-Type Spaces. *J. Funct. Spaces*, 2021:6615604, 2021. Article ID 6615604.

[8] A. Ajaib and A. Hussain. Weighted CBMO estimates for commutators of matrix Hausdorff operator on the Heisenberg group. *Open Mathematics*, 18:496–511, 2020.

[9] M. Sultan, B. Sultan, and A. Hussain. Grand Herz–Morrey spaces with variable exponent. *Mathematical Notes*, 114(5):957–977, 2023.

[10] A. Hussain and G. Gao. Multilinear singular integrals and commutators on Herz space with variable exponent. *ISRN Mathematical Analysis*, 2014:1–10, 2014.

[11] A. Hussain, I. Khan, and A. Mohamed. Variable Herz–Morrey estimates for rough fractional Hausdorff operator. *Journal of Inequalities and Applications*, 2024:33, 2024.

[12] J. Younas, A. Hussain, H. Alhazmi, A. F. Aljohani, and I. Khan. BMO estimates for commutators of the rough fractional Hausdorff operator on grand-variable-Herz–Morrey spaces. *AIMS Mathematics*, 9(9):23434–23448, 2024.

[13] M. Sultan and B. Sultan. A note on the boundedness of Marcinkiewicz integral operator on continual Herz–Morrey spaces. *Filomat*, 39(6):2017–2027, 2025.

[14] B. Sultan, F. Azmi, M. Sultan, M. Mehmood, and N. Mlaiki. Boundedness of Riesz potential operator on grand Herz–Morrey spaces. *Axioms*, 11(11):583, 2022.

[15] B. Sultan, M. Sultan, M. Mehmood, F. Azmi, M. A. Alghafli, and N. Mlaiki. Boundedness of fractional integrals on grand weighted Herz spaces with variable exponent. *AIMS Mathematics*, 8(1):752–764, 2023.

[16] B. Sultan, F. Azmi, M. Sultan, T. Mahmood, N. Mlaiki, and N. Souayah. Boundedness of fractional integrals on grand weighted Herz–Morrey spaces with variable exponent. *Fractal and Fractional*, 6(11):660–670, 2022.

[17] B. Sultan, M. Sultan, Q. Q. Zhang, and N. Mlaiki. Boundedness of Hardy operators on grand variable weighted Herz spaces. *AIMS Mathematics*, 8(10):24515–24527, 2023.

[18] M. Sultan, B. Sultan, A. Khan, and T. Abdeljawad. Boundedness of Marcinkiewicz integral operator of variable order in grand Herz–Morrey spaces. *AIMS Mathematics*, 8(9):22338–22353, 2023.

[19] B. Sultan and M. Sultan. Boundedness of higher order commutators of Hardy operators on grand Herz–Morrey spaces. *Bulletin des Sciences Mathématiques*, 190:103390,

2024.

[20] M. Sultan and B. Sultan. Boundedness of sublinear operators on grand central Orlicz–Morrey spaces. *Bulletin des Sciences Mathématiques*, 205:103704, 2025.

[21] M. Sultan and B. Sultan. Λ-Central Musielak–Orlicz–Morrey spaces. *Arabian Journal of Mathematics*, 14:357–363, 2025.

[22] B. Sultan, M. Sultan, and A. Hussain. Boundedness of the Bochner–Riesz operators on the weighted Herz–Morrey type Hardy spaces. *Complex Analysis and Operator Theory*, 19:49, 2025.

[23] B. Sultan, A. Hussain, and M. Sultan. Characterization of generalized Campanato spaces with variable exponents via fractional integrals. *Journal of Pseudo-Differential Operators and Applications*, 16:22, 2025.

[24] B. Sultan, M. Sultan, A. Khan, and T. Abdeljawad. Boundedness of commutators of variable Marcinkiewicz fractional integral operator in grand variable Herz spaces. *Journal of Inequalities and Applications*, 2024:93, 2024.

[25] B. Sultan and M. Sultan. Sobolev-type theorem for commutators of Hardy operators in grand Herz spaces. *Ukrainian Mathematical Journal*, 76:1196–1213, 2024.

[26] M. Sultan, B. Sultan, and R. E. Castillo. Weighted composition operator on Gamma spaces with variable exponent. *Journal of Pseudo-Differential Operators and Applications*, 15:46, 2024.

[27] M. Sultan and B. Sultan. A note on the boundedness of higher order commutators on fractional integrals in grand variable Herz–Morrey spaces. *Kragujevac Journal of Mathematics*, 50(7):1063–1080, 2026.

[28] J. Xu and X. Yang. Herz–Morrey–Hardy spaces with variable exponents and their applications. *Journal of Function Spaces*, 2015:19, 2015.

[29] T. Anh, J. Cao, L. D. Ky, D. Yang, and S. Yang. Weighted Hardy spaces associated with operators satisfying reinforced off-diagonal estimates. *Taiwanese Journal of Mathematics*, 17(4):1127–1166, 2013.

[30] X. Fu, H. Lin, D. Yang, and D. Yang. Hardy spaces H^p over non-homogeneous metric measure spaces and their applications. *Science China Mathematics*, 58(2):309–388, 2015.

[31] R. Gong, J. Li, and L. Yan. A local version of Hardy spaces associated with operators on metric spaces. *Science China Mathematics*, 56(2):315–330, 2013.

[32] S. Samko. Variable exponent Herz spaces. *Mediterranean Journal of Mathematics*, 10(4):2007–2025, 2013.

[33] M. Izuki. Boundedness of commutators on Herz spaces with variable exponent. *Rendiconti del Circolo Matematico di Palermo*, 59:199–213, 2010.

[34] D. Cruz-Uribe and A. Fiorenza. *Variable Lebesgue Spaces: Foundations and Harmonic Analysis*. Applied and Numerical Harmonic Analysis. Birkhäuser, Heidelberg, 2013.

[35] B. Muckenhoupt and R. L. Wheeden. Weighted norm inequalities for singular and fractional integrals. *Transactions of the American Mathematical Society*, 161:249–258, 1971.

[36] J. L. Wu and W. J. Zhao. Boundedness for fractional Hardy-type operator on variable-

exponent Herz–Morrey spaces. *Kyoto Journal of Mathematics*, 56(4):831–845, 2016.

[37] V. Kokilashvili and S. Samko. On Sobolev theorem for Riesz-type potentials in the Lebesgue spaces with variable exponent. *Zeitschrift für Analysis und ihre Anwendungen*, 22:899–910, 2003.