

An Application of the Balanced Truncation Algorithm Using Linear Mapping to the Robust Control Problem for the Ward-Leonard System in Wind Turbines

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ABSTRACT

The Balanced Truncation Algorithm (BTA) is one of the most used model order reduction algorithms, while multiple new algorithms have been developed based on the former. This paper presents in detail a solution that applies linear mapping to the subdivision process to reduce the order of a stable linear system. To demonstrate the correctness and applicability of the improved algorithm, the latter was applied to reduce the order of a 6th-order controller in a Wind Turbine (WT) system. After evaluating several reduced-order controllers, the 2nd-order controller was identified as the most suitable replacement for the original 6th-order controller. The results show that the system's output response using the 2nd-order controller closely matches that of the original system using the 6th-order controller.

Keywords-balanced truncation algorithm; model order reduction; linear mapping; wind turbine system

I. INTRODUCTION

Numerous order reduction algorithms have been proposed and published in the field of Model Order Reduction (MOR) [1–14], and have been applied to a wide range of problems [15–21]. Among them, the most prominent algorithm groups include the BTA [1, 2] and those based on the Krylov subspace method [3]. For the problem of reducing the order of stable linear systems, most research has focused primarily on BTA-based approaches [1–6]. Authors in [1] first introduced the BTA, which has since served as the foundation for multiple subsequent algorithms [2–9], forming a family of balanced truncation methods. Among these techniques, the Hankel-Norm Approximation [7] and Linear-Quadratic Gaussian (LQG) Balanced Truncation are considered notable [8, 9].

The algorithms within the balanced truncation method group preserve the stability of the original system while achieving low order reduction error, with a well-defined formula available to estimate this error [9]. Authors in [10–12] proposed modified BTA versions to extend its applicability to unstable systems. A common feature among these approaches is the use of coordinate transformation to convert an unstable system into a stable one and vice versa. Similarly aiming to stabilize originally unstable systems, the algorithms proposed in [16, 17] employ linear mapping techniques and have demonstrated favorable order reduction performance.

Building on the findings reported in [13, 14], the current study explores an enhanced BTA version that incorporates linear mapping and applies it to a practical control problem.

The key idea is to use linear mapping to convert the original continuous-time system into a discrete-time form, where the BTA is known to perform efficiently. After reducing the system's order in the discrete domain, an inverse mapping is applied to return to a continuous-time representation. This approach not only helps preserve the Hankel singular values of the original system, but also leads to a reduced-order model with minimal approximation error. To demonstrate its practical value, the proposed method is applied to simplify the controller in a WT system, achieving accurate results with lower computational complexity.

II. BALANCED TRUNCATION ALGORITHM BASED ON LINEAR MAPPING

A. Model Order Reduction Problem

Consider a stable continuous-time linear system with multiple inputs and outputs, described by the following state-space model:

$$\dot{x} = Ax + Bu \quad (1)$$

$$y = Cx + Du \quad (2)$$

where $x \in R^n, u \in R^p, y \in R^q, A \in R^{n \times n}, B \in R^{n \times p}, C \in R^{q \times n}, D \in R^{q \times p}$.

The objective of the MOR problem is to determine a reduced-order state-space model described as:

$$\dot{x}_r = A_r x_r + B_r u \quad (3)$$

$$y_r = C_r x_r + D_r u \quad (4)$$

where $x_r \in R^r, u \in R^p, y_r \in R^q, A_r \in R^{r \times r}, C_r \in R^{q \times r}, D_r \in R^{r \times p}, B_r \in R^{r \times p}, r \leq n$, so that the reduced order models described in (3, 4) can reliably approximate the original system (1, 2), while fulfilling essential criteria, such as minimal approximation error, preservation of key dynamic properties, and computational efficiency.

B. Linear Mapping

The state-space model (1, 2) can be transformed into its corresponding transfer function representation:

$$G(s) = C(sI - A)^{-1}B + D \quad (5)$$

The discrete-time transfer function corresponding to the model (1) is given by [16]:

$$G(z) = C_d(zI - A_d)^{-1}B_d + D_d \quad (6)$$

The following transformation from the continuous-time transfer function $G(s)$ to the discrete-time transfer function $G(z)$ is known as a linear mapping. This mapping is defined by:

$$\begin{cases} A_d = (I - A)^{-1}(I + A), B_d = \sqrt{2}(I - A)^{-1}B \\ C_d = \sqrt{2}C(I - A)^{-1}, D_d = D + C(I - A)^{-1}B \end{cases}$$

Similarly, the inverse transformation from the discrete-time transfer function $G(z)$ to the continuous-time transfer function $G(s)$ is referred to as an inverse linear mapping. This transformation is defined by:

$$\begin{cases} A = (I + A_d)^{-1}(A_d - I), B = \sqrt{2}(I + A_d)^{-1}B_d \\ C = \sqrt{2}C_d(I + A_d)^{-1}, D = D_d - C_d(I + A_d)^{-1}B_d \end{cases}$$

Linear mapping makes it easy to convert continuous systems to discrete systems and vice versa. At the same time, in [13], this mapping has been proven to preserve Hankel values, stability, and simplicity.

C. Balanced Truncation Algorithm based on Linear Mapping

Consider the system described by (5).

1) Step 1

Use linear mapping to transform system (5) $G(s)$ to system (6) $G(z)$, as defined by:

$$\begin{cases} A_d = (I - A)^{-1}(I + A) \\ B_d = \sqrt{2}(I - A)^{-1}B \\ C_d = \sqrt{2}C(I - A)^{-1} \\ D_d = D + C(I - A)^{-1}B \end{cases}$$

Following the application of Step 1, the resulting system (6) represents a discrete-time equivalent of the original continuous-time system.

2) Step 2

Solving the following Lyapunov equations will determine the control Gramian matrix P and the observed Gramian matrices Q :

$$Q = \begin{cases} A_d P A_d^T - P + B_d B_d^T = 0 \\ A_d^T Q A_d - Q + C_d^T C_d = 0 \end{cases}$$

3) Step 3

Determine the matrices K_o, K_c by Cholesky decomposition:

$$\begin{cases} P = K_c K_c^T \\ Q = K_o K_o^T \end{cases}$$

4) Step 4

Performing Singular Value Decomposition (SVD) analysis of matrix $K_o^T K_c$ gives the following results:

$$K_o^T K_c = U \Sigma V^T$$

5) Step 5

Determine the state transition matrix using:

$$T = K_c V \Sigma^{-1/2}$$

6) Step 6

Transform the system described by (6) into its balanced (equilibrium) form using the following coordinate transformations:

$$(A_b, B_b, C_b, D_b) = (T^{-1}A_d T, T^{-1}B_d, C_d T, D_d) \quad (7)$$

7) Step 7

The matrices of the balanced equivalent system (7) are partitioned as follows to facilitate the model order reduction process:

$$A_b = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, B_b = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}, C_b = [\hat{C}_1 \quad \hat{C}_2], D_b = [D_d]$$

where $\hat{A}_{11} \in R^{r \times r}, \hat{B}_1 \in R^{r \times p}, \hat{C}_1 \in R^{q \times r}$.

8) Step 8

The reduced-order system is defined as:

$$G_1(z) \sim (\hat{A}_{11}, \hat{B}_1, \hat{C}_1, \hat{D}_1), \hat{D}_1 = D_d$$

9) Step 9

To obtain the continuous-time representation, the inverse linear mapping is applied to the reduced discrete-time system

$G_1(z) \sim (\hat{A}_{11}, \hat{B}_1, \hat{C}_1, \hat{D}_1)$ into the continuous system $G_1(s) \sim (\hat{A}_c, \hat{B}_c, \hat{C}_c, \hat{D}_c)$, specifically:

$$\begin{cases} \hat{A}_c = (I + \hat{A}_{11})^{-1}(\hat{A}_{11} - I), \hat{B}_c = \sqrt{2}(I + \hat{A}_{11})^{-1}\hat{B}_1 \\ \hat{C}_c = \sqrt{2}\hat{C}_1(I + \hat{A}_{11})^{-1}, \hat{D}_c = \hat{D}_1 - \hat{C}_1(I + \hat{A}_{11})^{-1}\hat{B}_1 \end{cases}$$

Following Step 9, the reduced-order system is expressed in the same continuous-time linear form as the original system in equation (5).

The final output of the algorithm is the reduced-order continuous-time system, represented in state-space form as $(\hat{A}_c, \hat{B}_c, \hat{C}_c, \hat{D}_c)$.

TABLE I. LOW-ORDER CONTROLLERS OF THE CONTROLLER (9)

Order	Low-order controller $W_r(s)$	Error $\ W_c(s) - W_r(s)\ _{H_\infty}$
5	$\frac{-1.473s^5 + 8.967s^4 + 50.56s^3 + 64.36s^2 + 22.77s + 2.426}{s^5 + 10.09s^4 + 38.36s^3 + 72.13s^2 + 53.16s + 9.26}$	$1.4729 * 10^{-8}$
4	$\frac{-6.851 \cdot 10^{-7}s^4 + 8.967s^3 + 48.35s^2 + 52.38s + 9.831}{s^4 + 9.844s^3 + 35.93s^2 + 63.24s + 37.52}$	$2.7124 * 10^{-6}$
3	$\frac{-0.001103s^3 + 9.008s^2 + 66.75s + 15.71}{s^3 + 11.97s^2 + 40.52s + 59.42}$	0.0025
2	$\frac{0.01309s^2 + 8.706s + 2.089}{s^2 + 4.384s + 7.957}$	0.0131
1	$\frac{-0.001021s + 10.72}{s + 12.23}$	1.1623

III. APPLICATION OF BALANCED TRUNCATION ALGORITHM USING LINEAR MAPPING TO THE WIND TURBINE SYSTEM

A. Control Problem of Wind Turbine System

Wind energy has been increasingly utilized as a renewable power source [15, 21-24]. As a result, WTs have attracted significant attention due to the wind energy advantages: they are clean, sustainable, renewable, relatively easy to deploy, and require minimal land area. WTs play a crucial role in converting the kinetic energy of wind into electrical energy. Various control strategies have been developed to improve WT performance, efficiency, and stability. A robust control structure for the Ward-Leonard system within the WT configuration is presented below, as described in [15, 21] and illustrated in Figure 1.

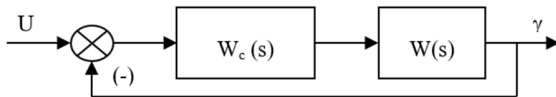


Fig. 1. Robust control structure for the Ward-Leonard system.

The system under control is modeled as:

$$W(s) = \frac{300}{s(s^3 + 184s^2 + 760.5s + 162)} \tag{8}$$

The corresponding robust optimal controller is defined as:

$$W_c(s) = \frac{8.967s^5 + 1663s^4 + 9154s^3 + 1.159 \cdot 10^4 s^2 + 4096s + 436.2}{s^6 + 189.9s^5 + 1852s^4 + 6969s^3 + 1.302 \cdot 10^4 s^2 + 9566s + 1664} \tag{9}$$

High-order controllers often lead to complex control program codes, so the use of a 6th-order controller in a control system may increase the response time of the system, which may cause the control system to fail to meet real-time control requirements. To reduce the control program code complexity and increase the real-time response capability of the control system, the order of the 6th-order controller needs to be reduced.

B. Model Order Reduction of the Controller

By applying the proposed algorithm to reduce the order of the controller defined in (9), the following results were obtained and are displayed in Table I.

According to Table I, the Order Reduction Error (ORE) of the 5th-order and 4th-order controllers is negligible. The ORE of the 3rd-order and 2nd-order controllers remains within an acceptable range, while the 1st-order controller exhibits a significantly large error.

To further evaluate and compare the accuracy of the reduced-order controllers relative to the original controller, both step response and bode response analyses were conducted. The results are depicted in Figure 2.

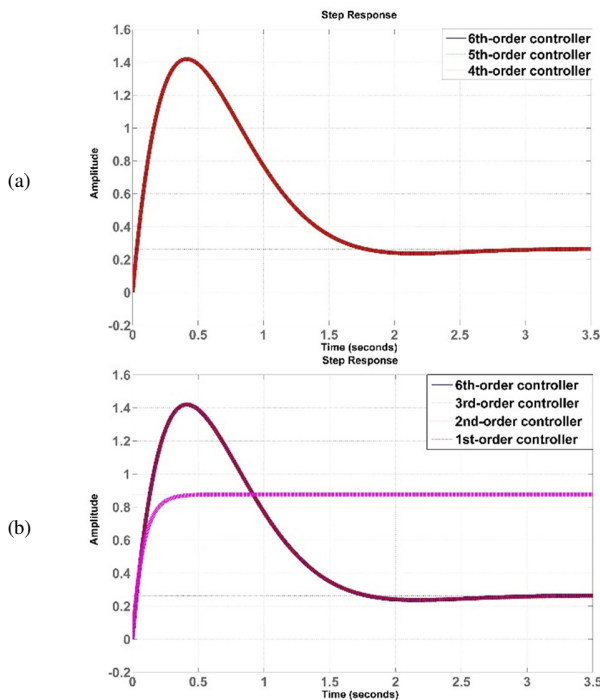


Fig. 2. Step response of the controllers: (a) step response comparison of 6th, 5th, 4th order controllers and (b) 6th, 3rd, 2nd, 1st order controllers.

The results presented in Figure 2 indicate that the step responses of the 5th, 4th, 3rd, and 2nd order controllers are identical to that of the original 6th-order controller. In contrast, the step response of the 1st-order controller exhibits an important deviation from that of the original controller.

The results portrayed in Figure 3 show that the Frequency Amplitude Responses (FARs) of the 5th- and 4th-order controllers are identical to that of the original 6th-order controller.

It is indicated that, in the frequency region $\omega < 1.86 \cdot 10^3$ rad/s, the FAR of the 3rd-order controller is identical to that of the original 6th-order controller. However, for $\omega > 1.86 \cdot 10^3$ rad/s, noticeable differences emerge. Similarly, the 2nd-order controller exhibits identical FAR to the original one for $\omega < 214$ rad/s, while deviations appear when $\omega > 214$ rad/s. In contrast, the FAR of the 1st-order controller is significantly different from that of the original controller across the entire frequency range. Regarding the Frequency Phase Response (FPR), the 5th-, 4th-, 3rd-, and 1st-order controllers display clear deviations compared to the FPR of the original controller. Only the 2nd-order controller preserves phase agreement in the low-frequency region $\omega < 29.6$ rad/s; beyond this point, discrepancies are observed. These findings suggest that the 1st-order controller introduces a substantial reduction error and distortion of the system's dynamic characteristics, rendering it unsuitable for replacing the 6th-order controller.

Based on the above analysis, the choice of reduced-order controller depends on the design requirements. The 5th- and 4th-order controllers are appropriate when very low reduction error is required and when matching the impulse and FARs is critical, even if deviations in phase response are tolerated.

Conversely, the 2nd-order controller is a suitable option when minimizing the controller order is a priority, provided that small reduction errors and phase discrepancies at higher frequencies are acceptable. Therefore, in this study, the 2nd-order controller is selected as the optimal substitute for the original 6th-order controller.

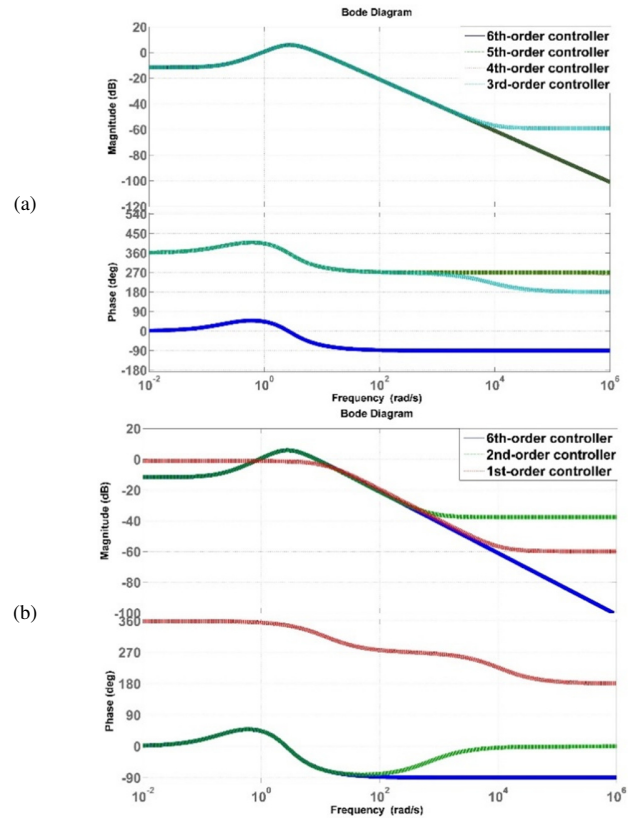


Fig. 3. Bode diagram of (a) 6th and 5th, 4th, 3rd, 2nd order controllers and (b) 6th order controller and 1st order controller.

A comparison between the 2nd-order controller obtained using the standard BTA algorithm, as described in [21], and that obtained using the BTA algorithm based on linear mapping reveals that the latter achieves a lower order reduction error. Specifically, the 2nd-order controller derived via linear mapping yields an error of 0.0131, compared to 0.017 using the conventional BTA approach [21]. This result suggests that converting a continuous-time linear system into its discrete-time counterpart through linear mapping, thereby allowing the BTA algorithm to operate in the discrete domain, can lead to more accurate reduced-order models than applying the algorithm directly in the continuous domain.

C. Control System Using 2nd Order Controller

Based on the control system displayed in Figure 1, the original controller is replaced with a 2nd-order controller and MATLAB/Simulink is utilized to simulate the system. The simulation results of the output response of the control system are depicted in Figure 4.

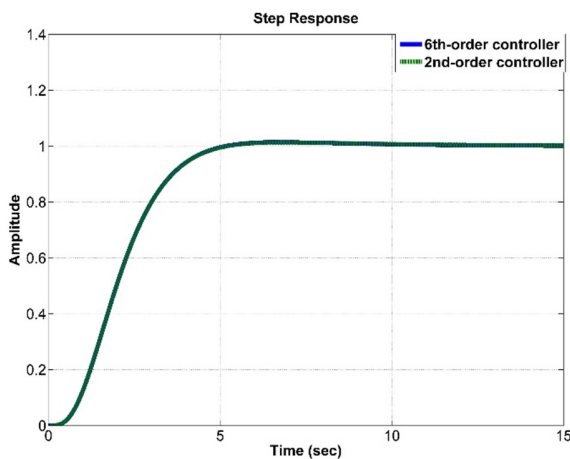


Fig. 4. The output response of the control system.

IV. CONCLUSION

An enhanced version of the Balanced Truncation Algorithm (BTA) incorporating linear mapping was introduced and applied to the control problem of the Ward–Leonard system in Wind Turbine (WT) applications. Linear mapping utilization enables the transformation of continuous-time systems into a discrete-time form and vice versa, allowing BTA to consistently operate in the discrete domain, where it demonstrates higher reduction efficiency. This flexible use of linear mapping constitutes the main innovation of the approach.

The proposed method produced several reduced-order controller candidates, among which the 2nd-order controller was identified as the most appropriate substitute for the original 6th-order controller. Notably, the reduction error achieved using the proposed algorithm (0.0131) was smaller than that obtained via the conventional BTA method described in [21] (0.017). The simulation results confirmed that the 2nd-order controller maintains the same output response as the original, ensuring equivalent dynamic behaviour with significantly lower complexity.

These results demonstrate that BTA combined with linear mapping offers an effective and reliable solution for model order reduction in control system design, particularly when simplicity and real-time performance are critical.

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