Improved Effective Moment of Inertia Equations for RC Beam Deflection Prediction

Haithm Hasan
Civil Engineering Department, College of Engineering and Architecture, Umm Al-Qura University, Saudi Arabia
Haithm.saad777@gmail.com

Hamdy Elgohary
Structural Engineering Department, Faculty of Engineering, Mansoura University, Egypt
gohary_h@yahoo.com

Rabeea W. Bazuhair
Civil Engineering Department, College of Engineering and Architecture, Umm Al-Qura University, Saudi Arabia
rwbazuhair@uqu.edu.sa (corresponding author)

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ABSTRACT

Accurate deflection calculations are important for ensuring the structural safety and serviceability of Reinforced Concrete (RC) beams. This study uses experimental data from previous research to evaluate the effectiveness of the moment of inertia equations in ACI 318-19 (Bischoff's equation) and SBC 304 (Branson's equation). The results showed that both equations had systematic biases: Bischoff's equation predicted deflections that were 22.8% higher, while Branson's equation predicted deflections that were 14.6% lower, than those measured under service conditions. To address these biases, new equations were developed using nonlinear regression analysis of 78 RC beam specimens from the published literature. The new equations include state-dependent variables related to the concrete strength, reinforcement ratios, and loading state. The proposed state-dependent correction reduced the prediction bias from 16.4% to 4.0% in the service state, from 41.2% to 9.5% in the uncracked state, and from 22.8% to 6.6% in the ultimate state. The statistical validation demonstrated a correlation coefficient greater than 0.95 for all loading states, indicating increased prediction accuracy for structural engineers. The results provide structural design engineers with accurate and realistic deflection prediction capabilities, improving deflection models, which can better support the organizations that adhere to less emphasized design codes. This study addresses a research gap by systematically evaluating the current code-based equations and formulating improved models based on experimental data. These models are then statistically validated. The study follows a structured approach including the background, problem statement, methodology, results, and implications. While the results are promising, the study's scope is limited by the range of the examined beam geometries and loading conditions, suggesting a need for future research to expand the applicability of the models.

Keywords-reinforced concrete; deflection prediction; Bischoff equation; Branson equation; effective moment of inertia

I. INTRODUCTION

Controlling the deflection in the RC structures is one of the most challenging tasks for structural engineers because it affects the serviceability, user comfort, and ongoing structural performance. Deflection is more challenging than the strength-based limit conditions because it depends not only on the elastic modulus, but also on the cracking behavior, tension stiffening, and time-dependent properties of concrete. The accurate deflection estimation is important not only for code acceptance, but also for the structure's performance throughout its intended service life [1, 2]. While present-day codes are evolving toward an objective design approach (there are possibilities for concrete deflection in the next version of ACI 318 and SBC 304), many other design codes use a different approach to account for cracking: the effective moment of inertia (I_e), in conjunction with other codes relating to the transition from the uncracked to fully cracked section behavior. An increasing amount of experimental work shows that significant discrepancies exist between the predicted deflections from the code and actual observations when evaluating the full-load deflections of the RC structures. Much of this discrepancy is due to the intermediate load stages where the cracking behavior starts to control the structure's response [3, 4]. The evolution of the deflection prediction methods has been driven by the need to balance the computational simplicity with accuracy. The first extensively applied method was proposed by Branson, calculating a weighted average of the gross and cracked section properties based on the cracking moment ratio. Implicit in most international codes, this method assumes a cubic relationship between the applied moment and loss in stiffness [5]. Bischoff and Scanlon proposed an alternative equation to address the shortcomings of the Branson approach, such as discontinuity at cracking and poor consideration of tension stiffening. Their method is used in ACI 318-19 and provides a smoother transition between the cracked and uncracked conditions, though it incorporates biases into the deflection prediction [6]. Research has revealed systematic deficiencies in both approaches. Authors in [7] showed that Branson's equation underestimates the deflections of high-strength concrete beams, and authors in [8] revealed similar deficiencies in beams with different reinforcement configurations. Bischoff acknowledged these deficiencies and called for more advanced methods that consider material and geometrical parameters [9]. Several researchers have proposed modifications to existing equations. Authors in [10] showed explicit equations with reinforcement ratio effects, while authors in [11, 12] used them for FRP-strengthened members. However, most of these changes are case-specific and do not offer universal solutions under different loading states and beam configurations. Authors in [13] examined the effect of limited repeated loading on partially prestressed concrete beams, showing that compression-controlled beams exhibited greater resistance to cracking and deformation. This study aims to quantitatively examine the validity of Bischoff's and Branson's equations under different loading conditions using a large experimental database, along with identifying systematic errors and their causes and formulate better equations, incorporating material and geometric parameters. Furthermore, the research aims to confirm the proposed corrections through rigorous statistical testing and provide practical recommendations for contemporary design practices. The study focuses on simply supported RC beams under monotonic loading, with the primary goal of estimating instantaneous deflection without considering time-dependent long-term effects, such as creep and shrinkage, which must be addressed separately in an overall serviceability analysis.

II. THEORETICAL BASIS FOR EFFECTIVE MOMENT OF INERTIA MODELS IN RC BEAMS

A. Fundamental Concepts

The effective moment of inertia method is a theoretical framework that accounts for the non-linear moment-curvature characteristic of the RC beams under loaded moment. As the loading progresses, the cracking process reduces the flexural stiffness of the section. Therefore, it is necessary to allow for the gross section properties used in the elastic analysis. The transition from uncracked to cracked behavior is a three-stage process: The initial stage is characterized by the uncracked state in which the concrete exhibits complete resistance to all tensile stresses and the entire gross section resists. The second stage is the partially cracked stage during which there is progressive cracking with varying stiffness as the section properties change within limits. The final stage is the cracked stage in which the contribution of concrete to tension is negligible and the reinforcement governs the flexural behavior.

B. Existing Equations

According to Branson's approach, the effective moment of inertia is estimated through:

$$I_e = \left(\frac{M_{cr}}{M_a}\right)^3 I_g + \left[1 - \left(\frac{M_{cr}}{M_a}\right)^3\right] I_{cr} \tag{1}$$

where I_e is the effective moment of inertia, I_g is the gross moment of inertia (uncracked section), I_{cr} is the cracked moment of inertia, M_{cr} is the cracking moment, and M_a is the applied moment. The cubic weighting function is based on the assumption that the stiffness degradation follows a specific mathematical relationship based on the moment ratio. In contrast, Bischoff's formulation addresses the discontinuity issues inherent in Branson's approach through:

$$I_e = \frac{I_{cr}}{1 - \left(\frac{(2/3)M_{cr}}{M_a}\right)^2 \left(1 - \frac{I_{cr}}{I_g}\right)} \tag{2}$$

The second one offers a more gradual transition between the uncracked and cracked states, while more accurately representing the tension stiffening effects through the characteristic 2/3 factor modification that accounts for the concrete's residual tensile capacity after cracking initiation.

III. EXPERIMENTAL DATABASE AND METHODOLOGY

A. Experimental Database

1) Data Sources and Selection Criteria

The experimental database for this study was compiled from two comprehensive research programs: The primary source is [14], where 54 beam specimens were considered and tested under various loading and reinforcement conditions. The present study concentrated on medium-strength concrete beams, with systematic variation of parameters. These parameters included reinforcement ratio, yield strength, span-to-depth ratio, and loading type. The secondary source is from [15], which provided 24 beam specimens that were tested under controlled laboratory conditions. The data extraction process from these sources entailed the digitization of published load-deflection curves, the systematic documentation of the material properties, geometric dimensions, and test conditions. Quality control measures were used to ensure the data consistency, including verification of units, cross-checking of reported values, and validation against expected structural behavior. The final database comprised exclusively of specimens that met this criterion, with a complete documentation of all requisite parameters, as shown in Table I.

2) Material Properties and Test Conditions

The database contains a substantial amount of material properties and geometry parameters, including concrete compressive strengths ranging from 25 MPa to 60 MPa, bottom reinforcement ratios ranging from 0.5% to 3.2%, top reinforcement ratios ranging from 0% to 1.8%, and steel yield strengths ranging from 400 MPa to 500 MPa. All test specimens were subjected to monotonic three-point bending, with a precise measurement of the mid-span deflections at a series of load levels. These levels corresponded to uncracked, service, and maximum conditions. The experimental program employed consistent data acquisition procedures throughout.

TABLE I. SUMMARY OF EXPERIMENTAL DATABASE

Ref.	**Number of samples**	**Load type**	**Section, *b* × *h* (mm)**	***L* (mm)**	***As* (mm^2)**	
[14]	54 samples	Third points (mid span)	100×140, 100 ×200, 100 ×300	2000, 3000	2	Φ10, Φ12, Φ 16
[15]	24 samples	Third points	250 ×300	3500	2	Φ 16

3) *Data Classification*

The experimental results were classified according to three distinct conditions of loading, in relation to the established thresholds of behavior for the RC beams. The uncracked state, therefore, refers to the state of loading where the applied moment is either equal to or less than the cracking moment ($M_a \leq M_{cr}$). This is the elastic condition where the concrete is used to resist the tensile stress and the section is working in accordance with the gross section properties. The service stage is defined as the point at which applied moments occur between 60% and 80% of the yield moment ($0.6\ M_y \leq M_a \leq 0.8\ M_y$). This range is considered typical in working load conditions, where partial cracking may occur while the section maintains sufficient stiffness to meet the serviceability requirements. The state of extreme high loading is analogous to high loading states with applied moment lesser than the ultimate moment capacity, characteristic of near-failure conditions under heavy cracking and severe stiffness degradation. The systematic classification of states is crucial for conducting state-specific analyses and developing equations that address the unique mechanisms governing each loading state.

B. *Methodology*

The comprehensive evaluation process used a systematic comparative analysis approach with deflection calculations based on established equations for each test specimen, statistical analysis based on compared measured values using various performance parameters, bias estimation through systematic error analysis, and parameter correlation activities to ascertain areas of improvement. The statistical performance analysis employed five major measures to generate robust equation assessment: the Mean Absolute Percentage Error (MAPE), a metric used to quantify the accuracy of predictions, the Standard Deviation (SD), a measure of prediction scatter, and the Coefficient of Variation (COV), a scaled variability measure. The Average Absolute Error (AAE) is the mean absolute deviation from the experimental values, and the correlation coefficient (R^2) is a measure of the strength of linear correlation between the predicted and experimental values. The resulting equation modifications were constructed following the application of a disciplined four-step development process. This process included preliminary parameter identification with correlation analysis of the deflection ratios and material/geometric properties, selection of functional forms based on the consideration of the physical behavior as well as statistical goodness of fit, nonlinear regression with XLSTAT software [16] to determine the parameters, and detailed cross-validation with performance checking using derived statistical metrics so that the constructed adjustments are accurate and consistent.

IV. RESULTS AND ANALYSIS

A. *Performance of Existing Equations*

Figure 1 presents a comparison between the predicted and experimental deflection values for both Branson's and Bischoff's equations across all loading states. The scatter plots reveal systematic biases and underscore the necessity for enhanced prediction models.

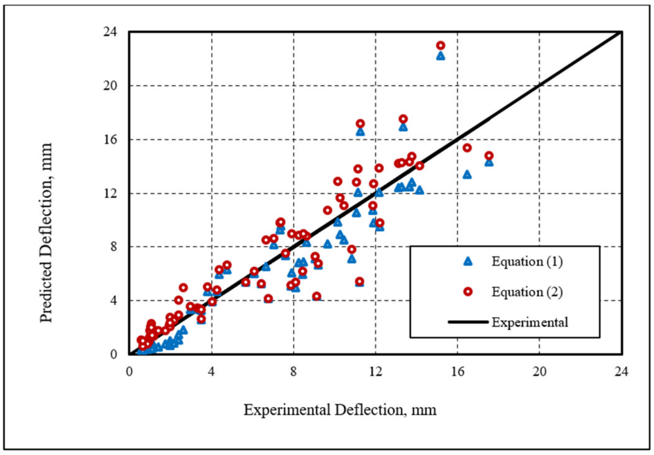

Fig. 1. Comparison of predicted versus experimental deflections.

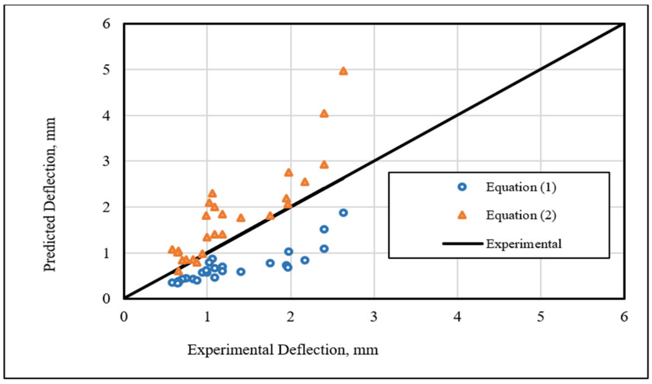

Fig. 2. Deflection comparison in uncracked state.

Figures 2-4 display the pattern of performance, where the optimal performance of Branson's equation is observed in the service state with a MAPE of 14.6%, but with substantial errors when it is in the uncracked state with a MAPE of 45.3%. Conversely, Bischoff's equation exhibits more consistent performance in all the loading states, but with systematic overestimation behaviors, as shown in Table II. It was observed that both equations exhibited increased variability at ultimate loading conditions due to an increase in the nonlinear behavior, which grew in proportion to the structural response, which approached the ultimate capacity.

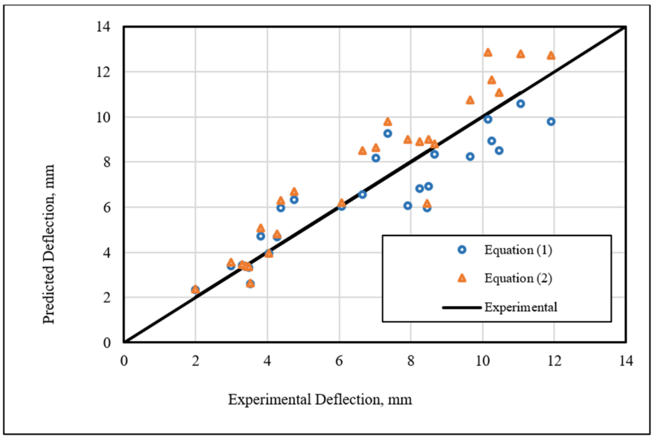

Fig. 3. Deflection comparison in service state.

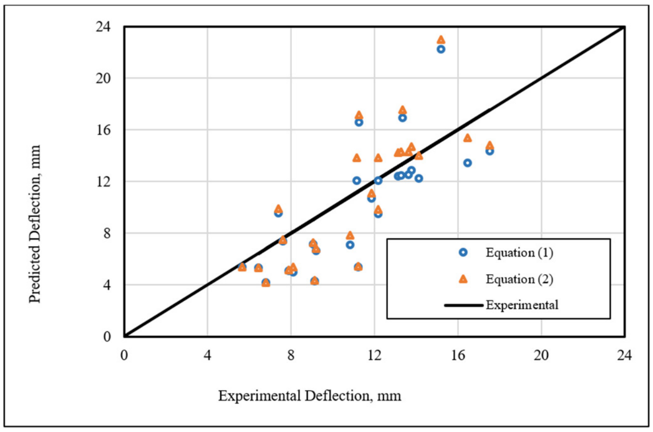

Fig. 4. Deflection comparison in ultimate state.

TABLE II. STATISTICAL PERFORMANCE OF EXISTING EQUATIONS

Loading state	Equation	MAPE (%)	SD (mm)	COV	AAE (mm)	R²
Uncracked	(1)	45.3	0.35	0.51	0.61	0.72
	(2)	41.2	1.82	1.01	0.53	0.68
Service	(1)	14.6	2.41	0.38	0.95	0.89
	(2)	16.4	3.32	0.45	1.04	0.86
Ultimate	(1)	22.9	4.50	0.45	2.45	0.83
	(2)	22.8	4.95	0.46	2.42	0.81

B. Parameter Correlation Analysis

Figures 5-7 present a strong correlation between the accuracy of the deflection prediction and key material and geometric parameters. The analysis of these datasets reveals a strong correlation between the application of these parameters and the modified equations. The regression analysis demonstrated a robust correlation between the accuracy of the deflection prediction and various structural parameters.

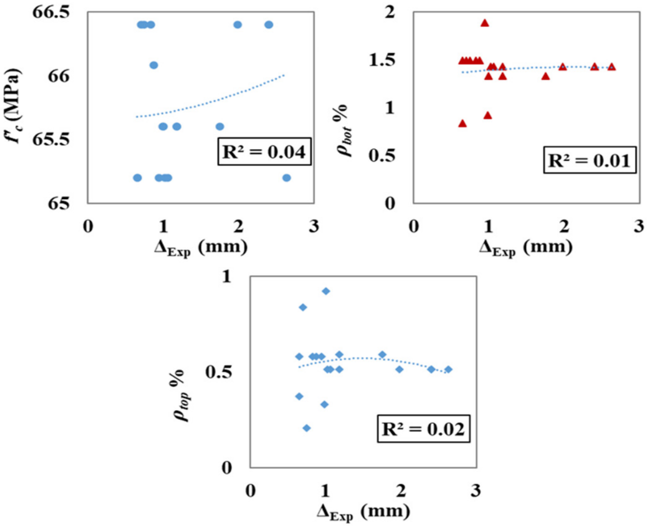

Fig. 5. Parameter correlation analysis for uncracked stage.

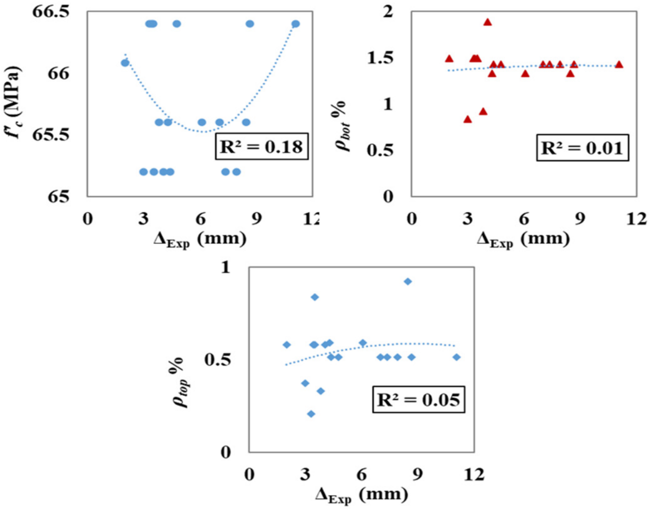

Fig. 6. Parameter correlation analysis for service stage.

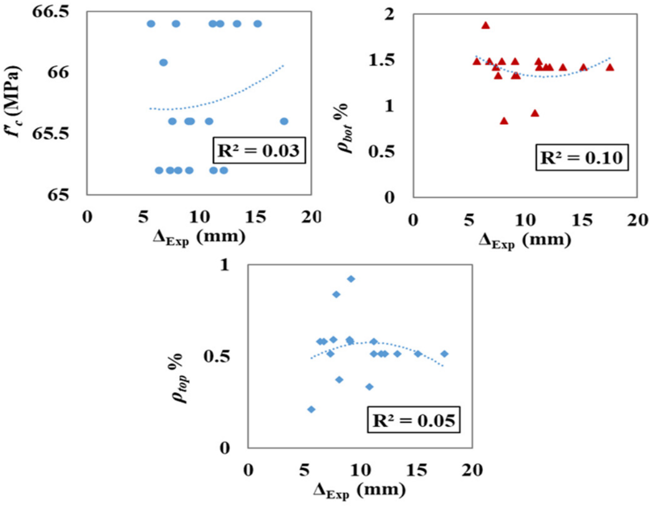

Fig. 7. Parameter correlation analysis for ultimate stage.

The compressive strength of concrete (f_c'), is a significant measure of its resistance to the compressive forces, indicating a high negative correlation with the prediction error. The bottom reinforcement ratio percentage (ρ_{bot}) is a critical metric in this field. A nonlinear relationship has been identified that is responsible for the control of the stiffness transition. The top reinforcement ratio percentage (ρ_{top}) is a critical metric in the field of material science and engineering.

C. Proposed Equation Development

The correlation analysis reveals a particular form for the modified effective moment of inertia, given by:

$$I_e = \frac{I_{cr}}{1-\left(\frac{\frac{2}{3}M_{cr}}{M_A}\right)^{m_{\text{proposed}}}(1-\frac{I_{cr}}{I_g})} \tag{3}$$

with:

$$m_{\text{proposed}} = A + f_c^{\prime B} \times (\rho_{\text{bot}})^C \times \left(1 + (\rho_{\text{top}})^D\right) \tag{4}$$

where $f_c^{\prime}$ is the compressive strength of concrete, ρ_{bot} is the bottom Reinforcement ratio percentage, ρ_{top} is the top reinforcement ratio percentage and A, B, C, and D are the constant values. The determination of the precise values of the constant required the usage of nonlinear regression analysis (XLSTAT 2023.3.0 (1415)) to the considered results, relating the exact values of the deflection to the parameters:

$$m_{\text{Uncracking Stage}} = 1.15 + f_c^{\prime\,-0.35} \times (\rho_{bot})^{3.75} \times \left(1 + (\rho_{top})^{0.75}\right) \tag{5}$$

$$m_{\text{Service Stage}} = 1.7 + f_c^{\prime\,-0.108} \times (\rho_{bot})^{4.75} \times (1 + (\rho_{top})^{3.75}) \tag{6}$$

$$m_{Ultimate\ State} = 1.75 + f_c^{\prime\,0.57} \times (\rho_{bot})^{-5.21} \times (1 + (\rho_{top})^{1.75}) \tag{7}$$

D. Validation Results

The performance improvement achieved by the proposed equations is shown in Figures 8-10, which compare predictions from the original and modified equations against the experimental data for each loading state. The scatter plot presents a comparison of the experimental deflections with predictions from the original Bischoff equation and a proposed modification for the uncracked state. The modified equation demonstrates a substantial enhancement in its alignment with the perfect prediction line (45° diagonal), characterized by a reduction in scatter and an improvement in the correlation coefficient. A comparison of deflection predictions in the service state reveals a substantial improvement with the proposed equation. The reduction in MAPE from 16.4% to 4.0% can be observed through the diminished scatter around the ideal prediction line. The effectiveness of the proposed modifications is proven at high loading levels and the approach is validated across the full loading range.

The proposed modifications exhibit substantial improvements across all loading states, with particularly significant enhancements in the uncracked and service conditions, where an accurate prediction is most critical for design applications, as shown in Table III.

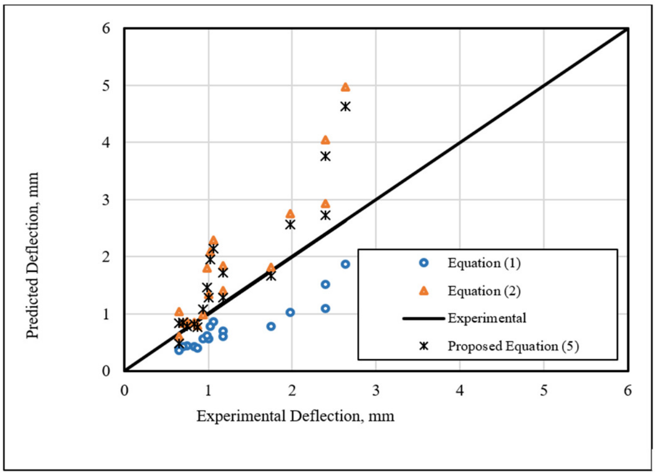

Fig. 8. Improved deflection prediction in uncracked state.

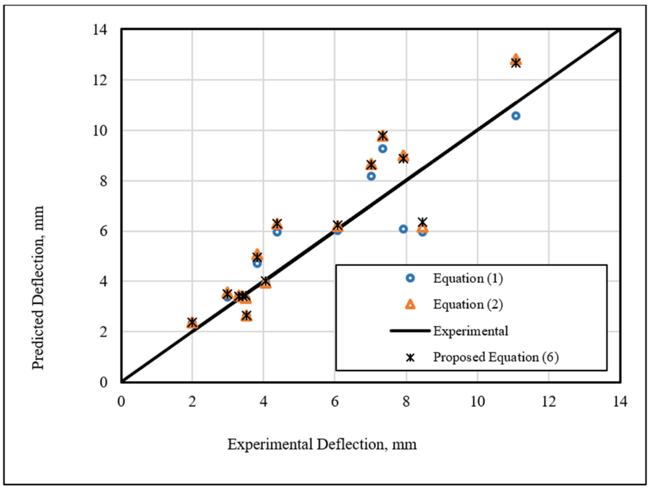

Fig. 9. Enhanced service state deflection prediction.

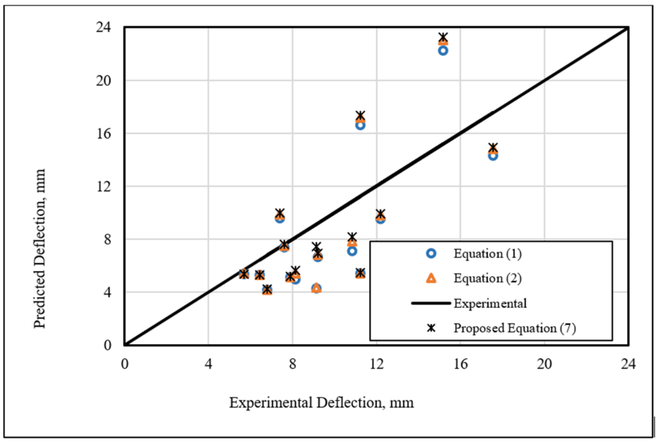

Fig. 10. Ultimate state deflection prediction improvement.

TABLE III. PERFORMANCE OF THE PROPOSED EQUATIONS

State	Metrics	(1)	(2)	Proposed equation
Uncracked	(MAPE)	45.3	41.2	9.5
	(SD)	0.35	1.82	1.08
	(COV)	0.51	1.01	0.63
	(AAE)	0.61	0.53	0.15
	(R^2)	0.72	0.68	0.95
Service	(MAPE)	14.6	16.4	4.0
	(SD)	2.41	3.32	2.79
	(COV)	0.51	1.01	0.63
	(AAE)	0.95	1.04	0.19
	(R^2)	0.89	0.86	0.98
Ultimate	(MAPE)	22.9	22.8	6.6
	(SD)	4.50	4.95	4.88
	(COV)	0.38	0.45	0.48
	(AAE)	2.45	2.42	0.64
	(R^2)	0.83	0.81	0.96

V. CONCLUSIONS

This study validated existing effective moment of inertia equations and developed improved ones for more precise deflection prediction of Reinforced Concrete (RC) beams. The existing equations systematically overpredicted deflections by 16% - 41% according to Bischoff's formula and underpredicted them by 15 % - 45%, depending on the loading conditions, as determined by Branson's formula. The proposed modifications achieved considerable improvements, with a 71% - 77% decrease in the prediction errors across all loading conditions. The new equations used state-dependent descriptions, thereby capturing the fundamental mechanics of cracking and stiffness degradation. The findings of the study indicated that the statistical evidence confirmed the validity of the like adaptations, exhibiting very high correlation coefficients of above 0.95 for both loaded states. Thus, the feasibility of the approach was revealed by using available design parameters that can be readily employed in the current structural design practice. The enhanced equations may inform future discourse on enhancing the serviceability prediction in the design practice. The study's findings contribute to the advancement of the structural design by offering more accurate prediction methods for deflection, enhancing the serviceability and safety of RC structures. The study addresses some of the major limitations of the current design codes while maintaining simplicity in calculation for engineering purposes. Based on the findings of this research it is proposed that practitioners implement these proposed modifications in critical applications where the deflection sensitivity is a primary concern. It is important for code developers to consider including parameter-dependent variables in subsequent code revisions because the demonstrated improvement in accuracy justifies the consideration of more sophisticated approaches. Also, software developers should incorporate these novel equations as supplementary calculation methods within extant design environments. This will provide engineers with enhanced tools without any compatibility loss with the contemporary design practices.

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