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Pricing Strategy of Convertible Bond with Memory and Jumps

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Abstract: In this paper, a novel pricing formula for convertible bond is developed under Chinese Regulator's framework. This formula generalizes existing results by considering the long-memory property of underling assets and rare market events. An empirical analysis is conducted by the end of this paper.

Keywords: Convertible Bond, Sub-Fractional Brownian Motion, Jump Diffusion, Genetic Algorithm.

1. Introduction

The idea of using mathematical tools to predict the future trend of financial asset is always a hot topic in the field of mathematical finance. The history of such idea can be traced back to the beginning of 20th Century, Bachelier [3] modeled financial asset price dynamics by using Brownian motion. The breakthrough came in 1973, when Black and Scholes [4], Merton [16] developed an explicit expression for European call and put options, which is the so-called BSM model. In the following years, extensive results are developed, see [8,9,11,19] and reference therein. Under Black-Scholes-Merton framework, the financial asset is modelled by a continuous-time Markov process, i.e. The geometric Brownian motion, but such process has limitation in explain the long-memory properties and rare market events.

To overcome such difficulties, In [5], Bojdecki introduced a novel way of describing the asset price, which employed a stochastic differential equation driven by sub-fractional Brownian motion(sfBm). In [18], Tudor found that the sfBm has self-similarity and long- range dependence properties, the sub-fractional Brownian motion degenerated faster than fractional Brownian motion, which shows that the long-memory property of asset pricing. More literature about sub-fractional Brownian motion can be found in [10, 20, 21] and references therein. Moreover, Some rare market events, for example outbreak or war, will have huge impact on the asset price. In terms of mathematics, these sudden changes to the asset prices can be formulated by using a jump diffusion. For instance, Rambeerich and his co-authors [17] developed a numerical approximation of European and American options. In [14] Marina etc, derived the explicit formula for Asian options, where the underlying asset is driven by jump-diffusion. Ji et al. [13] investigated the pricing of barrier option when the underlying asset is driven by a Sub-mixed fractional Brownian motion.

Recently, a class of financial assets, called convertible bond, attracts the eyes of many practitioner in the Mainland, China. The history of convertible bond can be traced back to the middle of 19th century when Jacob Little and Daniel Drew [15] used this instrument to counter volatility risk. There are extensive literature about convertible bonds. In [12], Under BSM framework, Ingersoll developed an explicit formula for pricing the convertible bonds in terms of company's stock prices; Bernnan et al. [6] further extended the previous convertible bond pricing model by introducing the stochastic interest rate, in which the pricing process is driven by a two factor model. More details can be found [1, 2] and reference therein.

The assumptions of BSM framework are restrictive and over-ideal, if all parameters of BSM model rely on the historical data, then there will be a huge gap between theoretical and practical results. To tackle this difficulty, a widely used machine learning algorithm called the genetic algorithm(GA) will be employed in this paper. GA is proved to be efficient in estimating the parameters, and also it provides global optimal estimation of parameters. For instance, Chen et al. [7] employed the GA algorithm to optimize the investment portfolios, while Zelenkov [22] used GA algorithm to reorganize the historical data, which improve the performance in quantitative trading.

To author's best knowledge, there is little literature regrading to the pricing strategy of convertible bond considering both sub-fractional Brownian motion and jump diffusion. The main aim of this paper is to construct the theoretical pricing formula of convertible bond and perform an empirical study by estimating the corresponding parameters via genetic algorithm.

The rest of this paper can be organized as follows, in section 2, some preliminary results will be given. In Section 3, the pricing formula of convertible bonds will be illustrated under classic BSM framework with specified modification. The empirical research involving parameter estimation via GA approach combined with will conduct in the last Section.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}t\}$ $t \geq 0$ satisfying the usual condition (i.e. it is right continuous and $\mathcal{F}0$ contains all P-null sets). Let Bt be a standard m-dimensional Brownian motion. Recall that sfBm in short with Hurst index $H \in (0, 1)$ is a Gaussian process $\xi^H = \{\xi^H(t, \omega)\}$ with a mean zero and covariance

$$E[\xi_t^H \xi_s^H] = t^{2H} + s^{2H} - \frac{1}{2}(|t+s|^{2H} + |t-s|^{2H}), \forall s,t \ge 0$$

For H = 1/2, the sfBm degenerate to a standard Brownian motion. In this paper, the underlying asset price will follow the the stochastic differential equation driven by sfBm and a jump process.

$$dS_t = \mu S_t d_t + \sigma S(t) d\xi_t^H + \gamma dN_t, 0 \le t \le T; \quad (2.1)$$

where μ is the average rate of return of the stock price; σ is the volatility of sfBm; dNt is the Poisson compensation proccess; γ is the jump rate of stock price. The initial value is given by S0 = S.

Lemma 2.1 Assume that asset price follows (2.1) with initial value S0 = S, then the it has an explicit solution:

$$S_t = S\exp(\mu t - \sigma^2(1-2^{2H-2})t^{2H} - \frac{1}{2}\gamma^2\lambda t + \sigma\xi_t^H + \gamma N_t). \quad (2.2)$$

Proof: Let $Q(S_t, t)$ be a $C^{2,1}$ function, then apply the gen-

Eralized Itô fomula, we have

$$dQ = (\frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial S}\mu S + \frac{\partial^2 Q}{\partial S^2}\sigma^2 S^2(2-2^{2H-1})Ht^{2H} - \frac{1}{2}\frac{\partial^2 Q}{\partial t^2}\gamma^2 S^2\lambda)dt + \frac{\partial Q}{\partial S}\sigma S d\xi_t^H + \frac{\partial Q}{\partial S}\gamma S dN_t$$

Now, by letting $Q = \ln S$, then

$$\ln S_t = \ln S + (\mu t - \sigma^2(1-2^{2H-2})t^{2H} - \tfrac{1}{2}\gamma^2\lambda t) + \sigma\xi_t^H + \gamma N_t$$

Then, the desired result can be obtained by taking exponential of both sides.

Remark 2.1 According to the Chinese Regulator and Shanghai Stock Exchange(SEC),convertible corporate bonds are a kind of corporate bonds which can be converted into a predetermined amount of the underlying company's equity at certain times. Convertible bonds can be seen as a combination of bonds and stock options. As a result of that, together with the common practise in real financial fields, the price of convertible bond can be divided into four parts:

- The price of bond;
- The price of stock options;
- Put provision;
- Redemption.

Or, equivalently, we have the following equations:

Convertible Bond = Bond + Options + Put provision + Redemption.

3. Asset Pricing Model

In this section, The main result of this paper, the pricing formula of convertible bond will be illustrated.

Theorem 3.1 A company issues a convertible bond at time t = 0, then the convertible bond price V can be formulated as:

$$V = \sum_{t=0}^{n}\frac{I}{(1+r)^t} + \frac{M}{(1+r)^n} + \{S_t[N(d_1) - N(-d_3) - N(d_5)]\} - e^{-r(T-t)}K[N(d_2) - 0.7N(-d_4) - 1.3N(d_6)]\frac{M}{K} \quad (3.1)$$

where

$$d_1 = \frac{\ln(S_t/K) + \sigma^2(1-2^{2H-2})(T^{2H}-t^{2H}) + (\frac{\lambda\gamma^2}{2}+r)(T-t)}{\sqrt{(2-2^{2H-1})(T^{2H}-t^{2H})\sigma^2 + \lambda(T-t)\gamma^2}}$$

$$d_2 = d_1 - \sqrt{(2-2^{2H-1})(T^{2H}-t^{2H})\sigma^2 + \lambda(T-t)\gamma^2}$$

$$d_3 = \frac{\ln(10S_t/7X_{back}) + \sigma^2(1-2^{2H-2})(T^{2H}-t^{2H}) + (\frac{\lambda\gamma^2}{2}+r)(T-t)}{\sqrt{(2-2^{2H-1})(T^{2H}-t^{2H})\sigma^2 + \lambda(T-t)\gamma^2}}$$

$$d_4 = d_3 - \sqrt{(2-2^{2H-1})(T^{2H}-t^{2H})\sigma^2 + \lambda(T-t)\gamma^2}$$

$$d_5 = \frac{\ln(10S_t/13X_{redeem}) + \sigma^2(1-2^{2H-2})(T^{2H}-t^{2H}) + (\frac{\lambda\gamma^2}{2}+r)(T-t)}{\sqrt{(2-2^{2H-1})(T^{2H}-t^{2H})\sigma^2 + \lambda(T-t)\gamma^2}}$$

$$d_6 = d_5 - \sqrt{(2-2^{2H-1})(T^{2H}-t^{2H})\sigma^2 + \lambda(T-t)\gamma^2}$$

Remark 3.1 The proof can be divided into four parts, as mentioned in the last section.

To prove this theorem, we will introduce following four lemmas.

First, we consider the bond part of the convertible bond.

Lemma 3.1 Firstly, we consider the bond case. Denote B be the present value of bond, it can be valued in the following form

$$B = \sum_{t=0}^{n}\frac{I_t}{(1+r)^t} + \frac{M}{(1+r)^n} \quad (3.2)$$

where I_t is the cashflow of t-th year while M is the principal of the convertible bond. r be the risk-free rate, while n is the maturity of this bond.

Now, we consider the stock option part of convertible bonds.

Lemma 3.2 Suppose that the underlying asset's price follows (2.2), denote C = C(St, t) be the value of European call option on time t for this asset. Then

$$\frac{\partial C}{\partial t} + rS_t\frac{\partial C}{\partial S} + [Ht^{2H-1}(2-2^{2H-1})\sigma^2 + \frac{\lambda}{2}\gamma^2]S_t^2\frac{\partial^2 C}{\partial S^2} - rC = 0$$

with boundary condition $C(T, S_T) = \max(S_T - K, 0)$.

Given the strike price K, maturity T, the value of European call option can be expressed explicitly,

$$C(T, S_T) = S_t N(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\ln(S_t/K) + \sigma^2(1-2^{2H-2})(T^{2H}-t^{2H}) + (\frac{\lambda\gamma^2}{2}+r)(T-t)}{\sqrt{(2-2^{2H-1})(T^{2H}-t^{2H})\sigma^2 + \lambda(T-t)\gamma^2}}$$

$$d_2 = d_1 - \sqrt{(2-2^{2H-1})(T^{2H}-t^{2H})\sigma^2 + \lambda(T-t)\gamma^2}$$

Proof: Here, we use the idea of Delta hedge to construct a portfolio, which longs one unit of call option and short Δ unit of underlying assets S. At time t, the value of this portfolio is $\Pi_t = C - \Delta S_t$. Within the time interval $(t, t+dt)$, $d\Pi_t = dC - \Delta dS_t$ which implies that

$$d\Pi_t = r\Pi d_t = r(C - \Delta S_t)d_t$$

Recall the fact that sfBm processes the following property

$$(dS_t)^2 = 2\sigma^2 S_t^2 H t^{2H-1}(2-2^{2H-1})dt, \ dt*dB_t^H = 0$$

An application of Itô's lemma yields

$$f(t,S_t) = f(0,0) + \int_0^t\{\frac{\partial f}{\partial x}(x,S_x) + [Hx^{2H-1}(2-2^{2H-1})\sigma^2 + \frac{\lambda}{2}\gamma^2]S^2\frac{\partial^2 f}{\partial S^2}(x,S_x)\}dx + \int_0^t\frac{\partial f}{\partial S}(x,S_x)dS$$

Therefor,

$$dC=\{\frac{\partial C}{\partial t}+[Ht^{2H-1}(2-2^{2H-1})\sigma^2+\frac{\lambda}{2}\gamma^2]S^2\frac{\partial^2 C}{\partial S^2}\}dt+\frac{\partial C}{\partial S}dS$$

(3.4) Combining with the portfolio, we have

$$\begin{aligned} d\Pi &= dC-\Delta dS \\ &=\{\frac{\partial C}{\partial t}+[Hx^{2H-1}(2-2^{2H-1})\sigma^2+\frac{\lambda}{2}\gamma^2]S^2\frac{\partial^2 C}{\partial S^2}\}dt+(\frac{\partial C}{\partial S}-\Delta)dS \end{aligned} \quad (3.5)$$

Then,

$$d\Pi=r(C-\frac{\partial C}{\partial S}S)dt=\{\frac{\partial C}{\partial t}+[Hx^{2H-1}(2-2^{2H-1})\sigma^2+\frac{\lambda}{2}\gamma^2]S^2\frac{\partial^2 C}{\partial S^2}\}dt \quad (3.6)$$

by letting $\Delta=\frac{\partial C}{\partial S}$,

$$\frac{\partial C}{\partial t}+rS_t\frac{\partial C}{\partial S}+[Hx^{2H-1}(2-2^{2H-1})\sigma^2+\frac{\lambda}{2}\gamma^2]S^2\frac{\partial^2 C}{\partial S^2}-rC=0 \quad (3.7)$$

With the Black-Scholes PDE in hand, we can solve this PDE with the boundary condition by using transformation method.

Now let $x=\ln S_t,\tau=T-t,W(\tau,x)=C(t,S_t)$, then we have

$$\frac{\partial C}{\partial t}=\frac{\partial W}{\partial \tau}\bullet\frac{\partial t}{\partial \tau}=-\frac{\partial W}{\partial \tau}$$

$$\frac{\partial C}{\partial S_t}=\frac{\partial W}{\partial x}\bullet\frac{\partial x}{\partial S_t}=\frac{\partial W}{\partial x}\bullet\frac{1}{S_t}$$

$$\frac{\partial^2 C}{\partial S_t^2}=\frac{\partial^2 W}{\partial x^2}\bullet\frac{1}{S_t^2}-\frac{\partial W}{\partial x}\bullet\frac{1}{S_t^2}$$

which implies

$$-\frac{\partial W}{\partial \tau}+(r-Ht^{2H-1}(2-2^{2H-1})\sigma^2-\frac{\lambda}{2}\gamma^2)\frac{\partial W}{\partial x}$$
$$+[Ht^{2H-1}(2-2^{2H-1})\sigma^2+\frac{\lambda}{2}\gamma^2]\frac{\partial^2 W}{\partial x^2}-rW=0.$$

By letting

$$\kappa^2=[Ht^{2H-1}(2-2^{2H-1})\sigma^2+\frac{\lambda}{2}\gamma^2],$$

and $x_1=x+A(\tau),\tau_1=B(\tau),U(\tau_1,x_1)e^{-D(\tau)}=W(\tau,x)$.

we have

$$\frac{\partial W}{\partial x}=e^{-D(\tau)}[A'(\tau)\frac{\partial U}{\partial x_1}+B'(\tau)\frac{\partial U}{\partial \tau_1}-UD'(\tau)],$$

$$\frac{\partial W}{\partial x}=e^{-D(\tau)}\frac{\partial U}{\partial x_1},\frac{\partial^2 W}{\partial x^2}=e^{-D(\tau)}\frac{\partial^2 U}{\partial x_1^2}$$

which implies

$$-A'(\tau)\frac{\partial U}{\partial x_1}-B'(\tau)\frac{\partial U}{\partial \tau_1}+UD'(\tau)+(r-\kappa^2)\frac{\partial U}{\partial x_1}+\kappa^2\frac{\partial^2 U}{\partial x_1^2}-rU=0 \quad (3.8)$$

Assume that

$$A'(\tau)=r-\kappa^2,B'(\tau)=\kappa^2,D'(\tau)=r$$

Then we have

$$\begin{cases} A(\tau)=\int_0^\tau r-[Hs^{2H-1}(2-2^{2H-1})\sigma^2+\frac{\lambda}{2}\gamma^2]ds=(r-\frac{\lambda\gamma^2}{2})\tau-\sigma^2(1-2^{2H-2})\tau^{2H} \\ B(\tau)=\int_0^\tau [Hs^{2H-1}(2-2^{2H-1})\sigma^2+\frac{\lambda}{2}\gamma^2]ds=\frac{\lambda\gamma^2}{2}\tau+\sigma^2(1-2^{2H-2})\tau^{2H} \\ D(\tau)=r\tau \end{cases}$$

which yields

$$\frac{\partial U}{\partial \tau_1}=\frac{\partial^2 U}{\partial x_1^2} \quad (3.9)$$

Then, the payoff of this option at maturity t is given be $U(x_1,0)=\max(e^{x_1}-K,0)$. By applying the heat equation method,

$$\begin{aligned} U(\tau_1,x_1)&=\frac{1}{2\sqrt{\pi\tau_1}}\int_{\ln K}^{+\infty}(e^z-K)e^{-\frac{(z-x_1)^2}{4\tau_1}}dz \\ &=\frac{1}{2\sqrt{\pi\tau_1}}\int_{\ln K}^{+\infty}e^ze^{-\frac{(z-x_1)^2}{4\tau_1}}dz-\frac{K}{2\sqrt{\pi\tau_1}}\int_{\ln K}^{+\infty}e^{-\frac{(z-x_1)^2}{4\tau_1}}dz \\ &=I_1+I_2 \end{aligned} \quad (3.10)$$

where

$$I_1=\frac{1}{2\sqrt{\pi\tau_1}}\int_{\ln K}^{+\infty}e^ze^{-\frac{(z-x_1)^2}{4\tau_1}}dz=\frac{1}{2\sqrt{\pi}}e^{x_1+\tau_1}\int_{\frac{\ln K-x_1-2\tau_1}{\sqrt{2\tau_1}}}^{+\infty}e^{-\frac{t^2}{2}} \quad (3.11)$$
$$=e^{x_1+\tau_1}N(\frac{x_1+2\tau_1-\ln K}{\sqrt{2\tau_1}})$$

by setting $t=\frac{z-x_1-2\tau_1}{\sqrt{2\tau_1}}$ and

$$I_2=\frac{-K}{2\sqrt{\pi\tau_1}}\int_{\ln K}^{+\infty}e^{-\frac{(z-x_1)^2}{4\tau_1}}dz=\frac{-K}{2\sqrt{\pi}}\int_{\frac{\ln K-x_1}{\sqrt{2\tau_1}}}^{+\infty}e^{-\frac{h^2}{2}}dh \quad (3.12)$$
$$=-KN(\frac{x_1-\ln K}{\sqrt{2\tau_1}})$$

by setting $h=\frac{z-x_1}{\sqrt{2\tau_1}}$.Hence,

$$U(\tau_1,x_1)=e^{x_1+\tau_1}N(\frac{x_1+2\tau_1-\ln K}{\sqrt{2\tau_1}})-KN(\frac{x_1-\ln K}{\sqrt{2\tau_1}}) \quad (3.13)$$

and

$$W(\tau,x)=e^{x_1+\tau_1-r\tau}N(\frac{x_1+2\tau_1-\ln K}{\sqrt{2\tau_1}})-Ke^{-r\tau}N(\frac{x_1-\ln K}{\sqrt{2\tau_1}}) \quad (3.14)$$

Substituting x_1 and τ_1 back, which yields

$$W(\tau,x)=e^xN(d_1^*)-Ke^{-r\tau}N(d_2^*) \quad (3.15)$$

where

$$d_1^*=\frac{x-\ln K+\sigma^2(1-2^{2H-2})\tau^{2H}+(\frac{\lambda\gamma^2}{2}+r)\tau}{\sqrt{\sigma^2(2-2^{2H-1})\tau^{2H}+\lambda\gamma^2\tau}}$$

$$d_2^*=\frac{x-\ln K-\sigma^2(1-2^{2H-2})\tau^{2H}+(\frac{\lambda\gamma^2}{2}-r)\tau}{\sqrt{\sigma^2(2-2^{2H-1})\tau^{2H}+\lambda\gamma^2\tau}}$$

By taking $x=\ln S_t,\tau=T-t$ and $W(\tau,x)=C(t,S_t)$.

Remark 3.2 When H=1/2, the explicit formula for European call option with underlying assets driven by standard BM and jump diffusion is recovered. when λ = 0, the formula for European call option with underlying assets driven by sfBm is recovered. When H=1/2, λ=0 the classic BS model has been recovered.

To finalize the pricing formula of a convertible bond, we need to consider some critical conditions in the real trading contract. When the stock price is lower then 70% of the strike price K for a period, then the Put provision condition is active, and the holder of convertible bond can sell the bond back to the company who issues such convertible bond. Then the following lemma is introduced:

Lemma 3.3 According to the agreement, the sell back price is X_{back} and option price is R_{back} satisfy the following equation:

$$R_{back}(t,S_t)=-S_tN(-d_3)+X_{back}e^{-(T-t)}N(-d_4) \quad (3.16)$$

where

$$d_3=\frac{\ln(S_t/X_{back})+\sigma^2(1-2^{2H-2})(T^{2H}-t^{2H})+(\frac{\lambda\gamma^2}{2}+r)(T-t)}{\sqrt{(2-2^{2H-1})(T^{2H}-t^{2H})\sigma^2+\lambda(T-t)\gamma^2}}$$

$$d_4 = d_3 - \sqrt{(2-2^{2H-1})(T^{2H}-t^{2H})\sigma^2 + \lambda(T-t)\gamma^2}$$

When the $S_T > 1.3K$, then the redeem condition is active. Then we have the following lemma:

Lemma 3.4 The redeem price and the redeem option price can be calculated as

$$R_{redeem}(t, S_t) = S_t N(d_5) - X_{redeem} e^{-r(T-t)} N(d_6) \quad (3.17)$$

where

$$d_5 = \frac{\ln(S_t / X_{redeem}) + \sigma^2(1-2^{2H-2})(T^{2H}-t^{2H}) + (\frac{\lambda\gamma^2}{2}+r)(T-t)}{\sqrt{(2-2^{2H-1})(T^{2H}-t^{2H})\sigma^2 + \lambda(T-t)\gamma^2}}$$

$$d_6 = d_5 - \sqrt{(2-2^{2H-1})(T^{2H}-t^{2H})\sigma^2 + \lambda(T-t)\gamma^2}$$

Proof of Theorem 3.1: The proof is to formulated above results from all lemmas above, which could be done accordingly. The details can be omit here.

4. Empirical Research

The key problem to solve in this section is to estimate the parameter used in Theorem 3.1. The classic MLE method has its own limitation, hence we adopt the GA algorithm, which is a well-developed algorithm? A typical GA algorithm can be divided into following three parts:

Reproduction It corresponds to natural selection. Random ergodic sampling is used in the search procedure, and the probability of each individual appearing in offspring is calculated according to the fitness value of the individual, and individuals are randomly selected according to this probability to form the offspring population. Real value coding is adopted for each parameter to simplify coding and decoding. For example, each dimension in the four-dimensional vector [A, B, C, D] represents a parameter, and different four-dimensional vectors represent different coding strings, and each coding string is a group of ideal parameters.

Crossover Crossover operation of genetic operation is the core of genetic algorithm. The so-called interchange refers to the two parent individuals to replace part of the structure of the recombination to generate a new individual operation. Through crossover, the search ability of genetic algorithms can be greatly improved. Crossover computing, by randomly swapping genes between two individuals in a population based on crossover rates, can produce new combinations of genes in the hope of putting beneficial genes together. The design of crossover operation mainly consists of three aspects: the determination of crossover location, the way of exchanging a certain part of genes with each other and the determination of crossover probability. Based on the consideration of the length of coding string, the genetic algorithm in this paper adopts the parent-twin method. In this method, after parents are identified, all the genes after that bit are swapped with a random bit, and two offspring are formed after swapping.

Mutation The basic content of mutation operation is to change the gene value on some gene locations of individual strings in the population. For example, a mathematical expression of mutation can be as follows: if there are only two numbers of 0 and 1 in the domain, then mutation changes 0 to 1 and 1 to 0, thus generating a new solution.

4.1. Data

According to the Wind Database, there are more than 300 tradable convertible bond available in the Chinese market. Then according to the characteristics of whether the issuing terms of the convertible bonds are standardized and complete and the issuing quantity, whether the trading of the convertible bonds and the underlying stocks is suspended in the sample interval, whether there is sufficient market price data, financial situation, credit rating, bond maturity and so on. We choose one of the convertible bonds, Hope convertible bond (code 127015), the contents are shown in the table below:

Table 1. Hope convertible bond main contents of relevant clauses

Code	Cred it Rating	Maturi ty Date	Conversi on Price	Redempti on Price
12701 5	AA A	2026.1 .2	19.75	25.68

The compounded return rate of the i_{th} day $r_i = \ln S_i - \ln S_{i-1}$, and the volatility is based on filtered historical data, which covers January 3rd 2020 till January 3rd 2021, in total 236 observations,so the corresponding standard deviation is

$$\sigma_1 = \sqrt{\sum_{i=2}^{T} \frac{(r_i - \mu)^2}{T-1}} = 0.01278,$$

which can be regarded as the daily volatility, with

$$\mu = \sum_{i=2}^{T} \frac{r_i}{T-1} = -0.0008.$$

As a result, the annual volatility $\sigma = \sigma_1 \times \sqrt{236} = 0.196$and the convertible counterpart is 19.63%. Compared to the classic BSM model, the approximated volatility is 45.3. The value of lambda is set to be 22, as the number that the daily volatility is larger than 5%. For the convenience of calculation, the pure bond discount rate in this paper adopts the five-year government's bond benchmark interest rate of the People's Bank of China in 2020, which is r = 5.32%; The risk-free rate is 1.35% for one-year time deposit.

The table below summarizes all the related information:

Table 2. Correlation parameter estimation results

	H	λ	σ	γ	r
B-S Model	0.5	0	0.45	0	1.35%
GA Model	0.77	3	0.22	0.06	1.35%
New Model	0.7	22	0.19	0.05	1.35%

4.2. Simulation Results

In this paper, the value of the Hope convertible bonds is collected on fortnight basis of listing on February 4, 2020, and a total of 12 samples over six months. The estimated parameters of the convertible bond pricing model under the sub-fractional jump diffusion environment were calculated via a Python program, and then the first day convertible bond price was simulated by using the parameters. At the same time, the algorithm has been tested for nearly one year, and the results are good.

The value of the pure bond part of the convertible bond can be calculated from Equation (3.3). Take the Hope convertible bond as an example: the face value is 100 yuan, the duration is 6 years, the coupon interest rate is 0.2% in the first year, and then from 0.4% in the second year, it increases by 0.4% year by year to 2% in the sixth year. The value of the pure bond part of the expected convertible bond can be obtained by discount as follows:

$$B=\sum_{t=0}^{n}\frac{I_t}{(1+r)^t}+\frac{M}{(1+r)^n}=\frac{100\times 0.2\%}{1+5.32\%}+\frac{100\times 0.4\%}{(1+5.32\%)^2}+\frac{100\times 0.8\%}{(1+5.32\%)^3}$$
$$=\frac{100\times 1.2\%}{(1+5.32\%)^4}+\frac{100\times 1.6\%}{(1+5.32\%)^5}+\frac{100\times 2\%}{(1+5.32\%)^6}+\frac{100}{(1+5.32\%)^6}=78.1822$$

for the option part, the number of evolution was set as 1000, the crossover probability was 0.7, the mutation probability was 0.02, and the initial population size was 50 on the genetic algorithm, and the parameters to be estimated were obtained as follows: [H, λ , σ , γ] = [0.77, 3, 0.32, 0.06]. In this case, the value of fitness function, namely the sum of squares of residuals, is 82.56. Figure 1 shows the variation trend of the average adaptive value of the population and the variation trend of the best individual adaptive value of each generation in the evolutionary process.

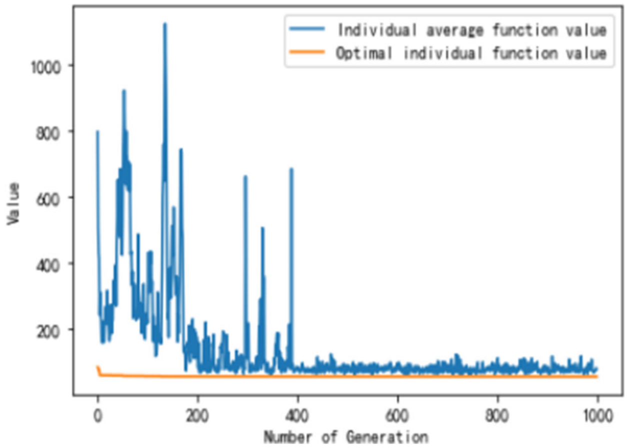

Figure 1: Convergence of individuals and populations

As can be seen from the Figure 1, the average adaptive value of the population fluctuated sharply before the 180 times of genetic algorithm operation, but with the increase of the number of algorithms, the fluctuation of the average adaptive value of the population became smaller and smaller, and showed a decreasing trend. The average adaptive value of the parameters obtained by the genetic algorithm was about 280 at the initial stage, and the average adaptive value stabilized at 80 after 390 generations with the increase of the number of evolutionary times, showing convergence from the population level. The adaptive value of the best individual in the genetic algorithm converges very quickly, indicating that evolution has made the gene significantly improved.

In order to demonstrate the effectiveness of the algorithm, the conclusions obtained should be applied in practice. Firstly, the price march on the first day of listing is compared, and the value calculated by fractional jump-diffusion model is

$$V=C+B=78.1822+8.68*100/19.75=122.11$$

Factors in terms of value of

$$V+B=C+R=78.1822+7.47+5.78(7.48)*100/19.75=113.46$$

According to the BSM model, the value

$$V=C+B=78.1822+6.76*100/19.75=113.46$$

is all around the expected CB listing price of 115.04on the first day of listing.

The theoretical value of each trading day is compared to the actual price:

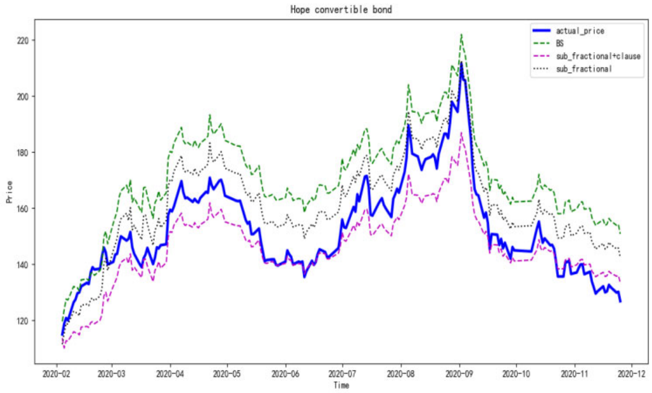

Figure 2: Hope convertible bond pricing results

Through the intuitive analysis of the above Figure 2, it can be seen that the theoretical results of the three models are very close to the actual market price, and the four curves

fit well, that is to say, the theoretical value can well fit the direction of market changes. Through comparison, it is found that the result of the sub-fractional jump-diffusion model is better than that of the B-S model. Due to the existence of the jump, the overall volatility of the underlying asset is shared, so that the pricing result is more stable than that of the B-S model. Then, comparing the sub-fractional jump-diffusion model with clauses, it can be found that the model with clauses added is smoother, which is more consistent with the actual price. Therefore, this paper believes that the sub-fractional jump-diffusion model with clauses added is effective.

Next, this paper introduces the mean square error (MSE), which reflects the degree of deviation between theoretical price and actual price, to compare the advantages and disadvantages of each model quantitatively. The expression of MSE is as follows:

$$MSE=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{C_{model}-C_{actual}}{C_{actual}}\right)^2$$

Table 3. Mean square error of theoretical value of hope convertible bond

	Include Clause		Exclude Clause	
	B-S Model	New Model	B-S Model	New Model
Historical	0.0611	0.0134	0.0874	0.0468
GA	0.0579	0.0056	0.0468	0.0408

The table summarizes the previous results.

As can be seen from the table, the model in this paper is generally superior to the traditional B-S model, the model considering terms factors is superior to the model without considering terms factors, and the parameter effect estimated by genetic algorithm is superior to the parameter estimated by historical data, which further verifies the analysis obtained from Figure 2.

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