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Study on Parameter Estimation of Load-Sharing Parallel Systems under Log-Logistic Component Lifetimes

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Abstract: This article addresses the problem of parameter estimation in load-sharing parallel systems with component lifetimes following log-logistic distribution. By formulating the joint likelihood function for system lifetimes, we obtain robust estimates via maximum likelihood estimation based on a Gauss-Seidel iterative optimization algorithm and then employ bootstrap techniques to construct confidence intervals for the parameters, thereby quantifying the uncertainty of the estimates. Simulation studies demonstrate the high accuracy and stability of the proposed method in handling complex dependency structures, which provides a solid theoretical foundation for the reliability analysis of load-sharing mechanisms in complex engineering systems.

Keywords: Load-sharing Parallel System; Log-Logistic Distribution; Parameter Estimation; Gauss-Seidel Optimization Algorithm; Bootstrap Method.

1. Introduction

Load-sharing mechanisms are crucial in reliability analysis for modern engineering systems. When a component fails, the remaining components dynamically redistribute the load, which significantly affects the system's failure rate. These mechanisms are common in aerospace, power transmission, textile engineering, and materials science. For example, Rosen (1964) noted non-monotonic failure rates in fiber composite parallel systems after a single fiber failure. Carlson and Kardomateas (2005) linked the fiber bundle model in textile engineering and fatigue crack propagation in materials science to load-sharing mechanisms. Amadi (2016) found that transformer bank failures in power systems cause dynamic load redistribution.

Traditional reliability models often assume that component lifetimes have constant or monotonic failure rates. However, these assumptions are inadequate for capturing the complex, non-monotonic failure rate characteristics observed in load-sharing systems. For instance, Lawless (2003) highlighted that the exponential distribution's constant failure rate assumption cannot model the dynamic, non-constant failure rates in load-sharing systems. Khan (2018) highlighted that the Weibull distribution's limitation with its monotonic hazard rate function assumption. Li et al. (2020) mentioned that the Kumaraswamy distribution's restricted parameter ranges limit its flexibility, despite describing non-monotonic failures. Zhong (2024) observed that the Gompertz distribution's high sensitivity to parameters will cause unstable estimation results. Thus, there's an urgent need for new reliability models which are flexible in mathematics and have practical applicability in reliability engineering.

The log-logistic (LL) distribution has attracted attention in reliability analysis for its explicit probability density function and non-monotonic hazard function. Al-Shomrani et al. (2016) employed Markov chain Monte Carlo (MCMC) methods to demonstrate the LL distribution's superiority in survival data analysis, particularly in characterizing a two-stage process of accelerated failure followed by mitigated damage accumulation. Kariuki et al. (2024) introduced the extended log-logistic (ELL) distribution, which overcomes traditional LL distribution shape constraints through a synergistic mechanism of shape and scale parameters, thereby better accommodating complex survival data characteristics. Although the above studies do not directly target load-sharing systems, their theoretical contributions offer valuable insights for modeling such systems with non-monotonic failure rate characteristics.

Maximum likelihood estimation (MLE) is key for analyzing load-sharing systems, but Zhou et al. (2021) found that MLE can produce biases and convergence issues with small sample sizes. To address these limitations, researchers have proposed various strategies. Kong and Ye (2016) improved k-out-of-n system reliability assessment by using interval estimation but assumed known load allocation rules, limiting engineering applicability. Park et al. (2020) proposed an improved EM algorithm framework, decomposing complex likelihood functions to enhance parameter estimation stability. Chang et al. (2019) integrated MCMC methods with Bayesian inference, significantly improving estimation accuracy while considering system uncertainties. However, balancing computational efficiency with estimation accuracy in complex load-sharing systems remains a pressing challenge.

In response to the limitations of existing models in characterizing load-sharing mechanisms, this study introduces the log-logistic distribution with a non-monotonic hazard function to model the failure behavior of components in parallel systems. The Gauss-Seidel iterative optimization method is incorporated, which ensures reliable modeling and robust parameter estimation under complex dependency structures. The effectiveness of the proposed method is demonstrated through simulation experiments.

The paper is organized as follows: Section 2 establishes a load-sharing parallel system model based on the LL distribution and derives the joint likelihood function for system lifetimes. Section 3 presents a parameter estimation method combining maximum likelihood estimation with the Gauss-Seidel iterative method and analyzes algorithm convergence. Section 4 evaluates the estimation performance of MLE under different sample sizes through simulation studies and examines its stability by constructing confidence intervals using the bootstrap method. Section 5 presents the conclusion.

2. Model Construction

2.1. Model Description

To develop a load-sharing parallel system model and estimate its parameters, the following assumptions are made.

1. The system comprises n independent and identically distributed (IID) load-sharing parallel subsystems, each consisting of k components. All components collectively bear a constant total load, which is uniformly distributed among the operational components. These n subsystems are constructed under identical experimental conditions and are independent of each other. Within each subsystem, the k components are interconnected through load sharing, and all components follow the same log-logistic distribution.

2. The n independent and identically distributed systems are tested, and their failure times are recorded. Let t_{ij} (for i=1, 2, ..., n; j=1, 2, ..., k) denotes the time interval between the failure of the (j-1)th component and the jth component in the ith system.

3. When a component in the system fails, the load it previously bore is redistributed among the remaining operational components, leading to an increase in the hazard rate of those surviving components.

4. The lifetime of each component initially follows a log-logistic distribution with parameters α_1 and β_1. As components fail successively, the parameters of the lifetime distribution for the remaining components are dynamically updated. For example, after the first component fails, the parameters change to α_2 and β_2; after the second failure, they become α_3 and β_3, and so on, until the last component is characterized by parameters α_k and β_k.

To facilitate the construction of the likelihood function and parameter estimation in subsequent analysis, this paper adopts the survival function of the log-logistic distribution

$$\bar{F}(x) = \frac{1}{1+(\alpha x)^{\beta}}, x>0, \alpha>0, \beta>0, \quad (1)$$

and the probability density function

$$f(x;\alpha,\beta) = \frac{\beta\alpha\left(\alpha x\right)^{\beta-1}}{\left[1+\left(\alpha x\right)^{\beta}\right]^2}, x>0, \alpha>0, \beta>0. \quad (2)$$

Here, the parameters α and β represent the scale and shape parameters, respectively.

2.2. Likelihood Construction

Based on the above assumptions, we construct the likelihood function for parameter estimation. First, consider the case of the first failure time in a single system. Since all components are independent, the earliest failure time among the k components is the minimum of their lifetimes, and its survival function is

$$P(\min\{X_1,...,X_k\}>x) = \prod_{j=1}^{k} P(X_j > x) = [\bar{F}(x;\alpha_1,\beta_1)]^k.$$

Taking the derivative of the above survival function yields the density function of the first failure time

$$f_{\min}(x) = k[\bar{F}(x;\alpha_1,\beta_1)]^{k-1} f(x;\alpha_1,\beta_1).$$

Thus, the likelihood contribution of the first failure time t_{i1} in the i-th system is:

$$L_1(\alpha_1,\beta_1 | t_{i1}) = k[\bar{F}(t_{i1};\alpha_1,\beta_1)]^{k-1} f(t_{i1};\alpha_1,\beta_1).$$

Similarly, for the j-th failure, the number of remaining components is $k-j+1$, and the density function of the failure time can be expressed using parameters (α_j, β_j). Its likelihood contribution is

$$L_j(\alpha_j,\beta_j | t_{ij}) = (k-j+1)[\bar{F}(t_{ij};\alpha_j,\beta_j)]^{k-j} f(t_{ij};\alpha_j,\beta_j).$$

where j=1, 2, …, k. Then, let $\boldsymbol{\alpha}$ = (α_1, α_2, …, α_k), $\boldsymbol{\beta}$ = (β_1, β_2, …, β_k), both belonging to the space $(0,\infty)^k$, $\boldsymbol{t}$ = {t_{ij}, i=1, 2, …, n; j=1, 2, …, k}, and α_j>0, β_j>0. For a single system, the overall likelihood function is the product of the likelihood contributions at each stage, i.e.,

$$\begin{aligned} L(\boldsymbol{\alpha},\boldsymbol{\beta}|\boldsymbol{t}) &= \prod_{j=1}^{k}\left\{(k-j+1)[\bar{F}(t_{ij};\alpha_j,\beta_j)]^{k-j} f(t_{ij};\alpha_j,\beta_j)\right\} \\ &= k!\prod_{j=1}^{k}\left[\frac{1}{1+\left(\alpha_j t_{ij}\right)^{\beta_j}}\right]^{k-j} \frac{\beta_j\alpha_j\left(\alpha_j t_{ij}\right)^{\beta_j-1}}{\left[1+\left(\alpha_j t_{ij}\right)^{\beta_j}\right]^2} \\ &= k!\prod_{j=1}^{k}\frac{\beta_j\alpha_j\left(\alpha_j t_{ij}\right)^{\beta_j-1}}{\left[1+\left(\alpha_j t_{ij}\right)^{\beta_j}\right]^{k-j+2}}. \end{aligned}$$

For n independent systems, the joint likelihood function is

$$\begin{aligned} L\left(\boldsymbol{\alpha},\boldsymbol{\beta}|\boldsymbol{t}\right) &= \left(k!\right)^n \prod_{i=1}^{n}\prod_{j=1}^{k}[\bar{F}(t_{ij};\alpha_j,\beta_j)]^{k-j} f(t_{ij};\alpha_j,\beta_j) \\ &= \left(k!\right)^n \prod_{i=1}^{n}\prod_{j=1}^{k}\frac{\beta_j\alpha_j\left(\alpha_j t_{ij}\right)^{\beta_j-1}}{\left[1+\left(\alpha_j t_{ij}\right)^{\beta_j}\right]^{k-j+2}}. \end{aligned}$$

3. Parameter Estimator

Taking the logarithm of the above formula yields the log-likelihood function as follows

$$\begin{aligned} l(\boldsymbol{\alpha},\boldsymbol{\beta}|\boldsymbol{t}) &= n\ln(k!) + \sum_{i=1}^{n}\left[\sum_{j=1}^{k}\left(\ln\left(\alpha_j\right)+\ln\left(\beta_j\right)+\left(\beta_j-1\right)\ln\left(\alpha_j t_{ij}\right)\right)\right] \\ &\quad -\sum_{i=1}^{n}\sum_{j=1}^{k}(k-j+2)\ln\left[1+\left(\alpha_j t_{ij}\right)^{\beta_j}\right]. \end{aligned}$$

To obtain the maximum likelihood estimators of ($\boldsymbol{\alpha}$, $\boldsymbol{\beta}$), we differentiate the log-likelihood function $l(\boldsymbol{\alpha},\boldsymbol{\beta}|\boldsymbol{t})$, resulting in the following likelihood equations with respect to the parameters α_j and β_j

$$\frac{\partial l(\boldsymbol{\alpha},\boldsymbol{\beta}|\boldsymbol{t})}{\partial\alpha_j} = \frac{n\beta_j}{\alpha_j} - \sum_{i=1}^{n}\left(k-j+2\right)\frac{\beta_j}{\alpha_j\left[1+\left(\alpha_j t_{ij}\right)^{-\beta_j}\right]} = 0 \quad (3)$$

$$\frac{\partial l(\boldsymbol{\alpha},\boldsymbol{\beta}|\boldsymbol{t})}{\partial\beta_j} = \frac{n}{\beta_j} + \sum_{i=1}^{n}\ln\left(\alpha_j t_{ij}\right) - \sum_{i=1}^{n}\left(k-j+2\right)\frac{\ln\left(\alpha_j t_{ij}\right)}{1+\left(\alpha_j t_{ij}\right)^{-\beta_j}} = 0 \quad (4)$$

It is evident that an explicit solution for the parameters ($\boldsymbol{\alpha}$, $\boldsymbol{\beta}$) is not attainable. Therefore, the above likelihood equations must be solved numerically by using appropriate iterative techniques for the $2k$ parameters. As described in Ortega & Rheinboldt (2000), the Gauss-Seidel method is particularly suitable for the likelihood equations in this context. The specific steps are as follows

1. For j=1, 2, …, k, choose appropriate initial values $\alpha_j^{(0)}$, and substitute them into the equation (4) to solve for β_j, denoted as $\beta_j^{(1)}$;
2. Substitute $\beta_j^{(1)}$ into the equation (3) to solve for α_j, denoted as $\alpha_j^{(1)}$;
3. Repeat steps (1) and (2) until α_j and β_j (for j=1, 2, …, k) satisfy the convergence condition:

$$max\left(\left|\alpha_j^{(m+1)}-\alpha_j^{(m)}\right|,\left|\beta_j^{(m+1)}-\beta_j^{(m)}\right|\right)<\epsilon,$$

where ϵ is a pre-set convergence threshold, typically chosen as $\epsilon=10^{-6}$ or smaller.

Next, we further derive the asymptotic distribution of the proposed MLE estimators. For this purpose, we need to compute the second-order partial derivatives of the log-likelihood function with respect to α_j and β_j, denoted respectively as:

$$\frac{\partial^2 l(\boldsymbol{\alpha},\boldsymbol{\beta}|\boldsymbol{t})}{\partial\alpha_j^2},\frac{\partial^2 l(\boldsymbol{\alpha},\boldsymbol{\beta}|\boldsymbol{t})}{\partial\beta_j^2},\frac{\partial^2 l(\boldsymbol{\alpha},\boldsymbol{\beta}|\boldsymbol{t})}{\partial\alpha_j\partial\beta_j},j=1,...,k.$$

However, due to the analytical complexity of these second-order partial derivatives, it is difficult to derive their explicit forms. Therefore, this paper only describes the procedure. According to the asymptotic theory of maximum likelihood estimation, under standard regularity conditions (such as consistency and non-singularity of the information matrix), the MLE of the parameter vector $\Theta=(\alpha_1,\beta_1,\alpha_2,\beta_2,\ldots,\alpha_k,\beta_k)'$ denoted as

$$\hat{\Theta}=\left(\hat{\alpha}_1,\hat{\beta}_1,...,\hat{\alpha}_k,\hat{\beta}_k\right)',$$

follows an asymptotic normal distribution as $n\to\infty$, i.e.,

$$\sqrt{n}\left(\hat{\Theta}-\Theta\right)\xrightarrow{d}N_{2k}\left(0,\mathscr{F}^{-1}(\boldsymbol{\alpha},\boldsymbol{\beta})\right),$$

where $\mathscr{F}(\boldsymbol{\alpha},\boldsymbol{\beta})$ is the Fisher information matrix. This matrix can be expressed in a block form

$$\mathscr{F}(\boldsymbol{\alpha},\boldsymbol{\beta})=\begin{pmatrix}\mathscr{F}^{(1)}(\boldsymbol{\alpha},\boldsymbol{\beta}) & 0 & \cdots & 0\\ 0 & \mathscr{F}^{(2)}(\boldsymbol{\alpha},\boldsymbol{\beta}) & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \mathscr{F}^{(k)}(\boldsymbol{\alpha},\boldsymbol{\beta})\end{pmatrix},$$

where $\mathscr{F}^{(m)}(\boldsymbol{\alpha},\boldsymbol{\beta})=\left(F_{ij}^{(m)}\right)$ is a 2×2 matrix for m=1, 2, …, k, with its elements defined as follows

$$F_{11}^{m}=-E\left[\frac{\partial^2 l(\boldsymbol{\alpha},\boldsymbol{\beta}|\boldsymbol{t})}{\partial\alpha_m^2}\right],$$

$$F_{12}^{m}=F_{21}^{m}=-E\left[\frac{\partial^2 l(\boldsymbol{\alpha},\boldsymbol{\beta}|\boldsymbol{t})}{\partial\alpha_m\partial\beta_m}\right],$$

$$F_{22}^{m}=-E\left[\frac{\partial^2 l(\boldsymbol{\alpha},\boldsymbol{\beta}|\boldsymbol{t})}{\partial\beta_m^2}\right].$$

Since the second-order partial derivatives mentioned above are difficult to obtain explicitly, the observed information matrix is commonly used in practical applications by evaluating the second derivatives of the log-likelihood function at the estimated parameter values $\alpha_m=\hat{\alpha}_m,\beta_m=\hat{\beta}_m$.

Next, for simplicity, we study the MLE estimation of the log-logistic distribution when $\beta_1=\beta_2=\ldots=\beta_k=1$. In this case, for n identically and independently distributed load-sharing parallel systems, the log-likelihood function is

$$l(\boldsymbol{\alpha}|\boldsymbol{t})=n\ln(k!)+\sum_{i=1}^{n}\sum_{j=1}^{k}\left[\ln(\alpha_j)-(k-j+2)\ln\left(1+\alpha_j t_{ij}\right)\right].$$

Differentiate the above log-likelihood function with respect to α_j, and set the derivative equal to zero. The resulting equation is

$$\frac{\partial l(\boldsymbol{\alpha}|\boldsymbol{t})}{\partial\alpha_j}=\frac{n}{\alpha_j}-\sum_{i=1}^{n}(k-j+2)\frac{t_{ij}}{1+\left(\alpha_j t_{ij}\right)}=0,j=1,...,k. \quad (5)$$

Obviously, it is not possible to obtain a closed-form solution for the parameter $\boldsymbol{\alpha}$ directly from the likelihood equation mentioned above. Therefore, the Gauss-Seidel method mentioned earlier can be used to obtain its solution, and thus obtain the MLE of $\boldsymbol{\alpha}$.

Based on the result of the MLE, we further construct a joint confidence interval. Under the large sample conditions, according to asymptotic theory, the MLE $\hat{\boldsymbol{\alpha}}$ of the parameter vector $\boldsymbol{\alpha}$ approximately follows a multivariate normal distribution. Its covariance matrix is provided by the inverse of the observed Fisher information matrix evaluated at $\hat{\boldsymbol{\alpha}}$, i.e.,

$$\sqrt{n}\left(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}\right)\xrightarrow{d}N_{2k}\left(0,\sum(\boldsymbol{\alpha})\right),$$

where $\sum(\boldsymbol{\alpha})\approx\mathscr{F}^{-1}(\boldsymbol{\alpha})$.

Therefore, under the large sample assumption, for a given confidence level 1-α (where0<α<1), the joint 100(1-α)% confidence region for the parameter vector $\boldsymbol{\alpha}$ can be approximated as

$$\left(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}\right)'\sum(\boldsymbol{\alpha})^{-1}\left(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}\right)\le\chi_{k,\alpha}^2,$$

where $\chi_{k,\alpha}^2$ denotes the upper α quantile of the chi-squared distribution with k degrees of freedom. This region is a confidence ellipsoid centered at the MLE $\hat{\boldsymbol{\alpha}}$, which visually reflects the joint uncertainty of the estimates α_j.

4. Simulation Study

In this section, we validate the effectiveness of the proposed parameter estimation method through simulation experiments. It is assumed that the lifetimes of the components in the system follow a log-logistic distribution with a single unknown scale parameter. We set α_1=0.1, α_2=0.2 and α_3=0.4, and generate 20 sample observations (n=20) from a system consisting of three components (k=3). The simulated data is shown in Table 1.

To further assess the performance of the method under varying sample sizes, we repeat the experiment for n=20, 50,75, and 100, and compute the parameter estimation bias and mean squared error (MSE) under each condition. The results are presented in Table 2. The experimental results indicate that both the bias and MSE of the parameter estimates decrease with increasing sample size, demonstrating that the proposed method achieves higher estimation accuracy under larger sample sizes.

Table 1. Failure time samples

i	t_{i1}	t_{i2}	t_{i3}
1	2.9880591	1.4940295	0.74701476
2	17.0741586	8.5370793	4.26853966
3	4.8628502	2.4314251	1.21571254
4	24.0615608	12.0307804	6.01539020
5	30.3834534	15.1917267	7.59586335
6	0.3782369	0.1891185	0.09455923
7	7.3155493	3.6577746	1.82888732
8	24.9571190	12.4785595	6.23927974
9	7.8938256	3.9469128	1.97345639
10	5.7552712	2.8776356	1.43881780
11	32.6378642	16.3189321	8.15946604
12	5.6903928	2.8451964	1.42259819
13	11.8367554	5.9183777	2.95918884
14	8.4538181	4.2269091	2.11345453
15	0.8972792	0.4486396	0.22431980
16	25.6964345	12.8482172	6.42410862
17	2.4512873	1.2256436	0.61282181
18	0.3481921	0.1740961	0.08704803
19	3.5563181	1.7781590	0.88907951
20	32.3017068	16.1508534	8.07542670

Table 2. Parameter estimation bias and mean squared error (MSE)

		20	50	75	100
α_1	$\widehat{\alpha_1}$	0.108405	0.117066	0.127464	0.106234
	Bias	0.008405	0.017066	0.027464	0.006234
	MSE	0.000071	0.000291	0.000754	0.000039
α_2	$\widehat{\alpha_2}$	0.230243	0.196246	0.203168	0.210520
	Bias	0.030243	-0.003754	0.003168	0.010520
	MSE	0.000915	0.000014	0.000010	0.000111
α_3	$\widehat{\alpha_3}$	0.262784	0.322616	0.365505	0.392723
	Bias	-0.137216	-0.077384	-0.034495	-0.007277
	MSE	0.018828	0.005988	0.001190	0.000053

Furthermore, to analyze the correlation between parameters, we construct confidence ellipses for the parameter pairs (α_1,α_2), (α_1,α_3), and (α_2,α_3) under different confidence levels $1-\alpha$={80%,90%,95%,99%}, as shown in Figures 1–3. The results suggest a positive correlation among α_1, α_2, and α_3.

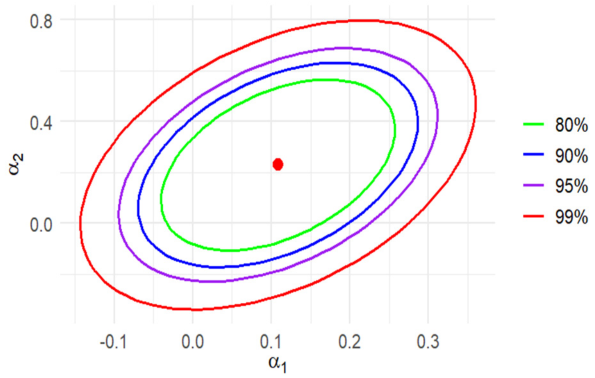

Fig 1. Confidence region for (α1, α2)

After obtaining the parameter point estimates, the Bootstrap method is further employed to construct confidence intervals. By adopting both the boot-p and boot-t methods, the accuracy and uncertainty of the parameter estimates can be evaluated from different perspectives, thereby enhancing the reliability of the results (Efron & Tibshirani, 1986; DiCiccio & Efron, 1996).

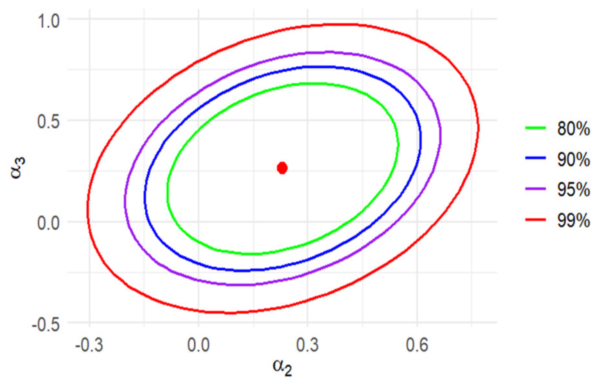

Fig 2. Confidence region for (α2,α3)

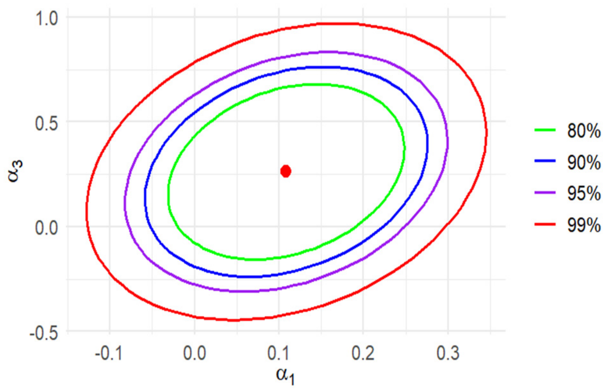

Fig 3. Confidence region for (α1,α3)

4.1. Boot-p Algorithm

1. Based on the known parameter estimates, construct a sample dataset t_{ij} (where i=1, 2, …, n and j=1, 2, …, k) of size n.

2. Randomly draw a bootstrap sample t_{ij}^* of the same size n from the original dataset t_{ij}, and compute the corresponding bootstrap point estimate $\hat{\boldsymbol{\alpha}}^* = (\hat{\alpha}_1^*, \hat{\alpha}_2^*, ..., \hat{\alpha}_k^*)'$.

3. Repeat the sampling and estimation process B times to obtain B sets of bootstrap point estimates $\hat{\boldsymbol{\alpha}}_1^*, \hat{\boldsymbol{\alpha}}_2^*, ..., \hat{\boldsymbol{\alpha}}_B^*$.

4. Sort the B bootstrap point estimates to form an ordered sequence $\hat{\boldsymbol{\alpha}}_{(1)}^* \le \hat{\boldsymbol{\alpha}}_{(2)}^* \le \cdots \le \hat{\boldsymbol{\alpha}}_{(B)}^*$. Based on the given confidence level 1-γ (0<γ<1), the 100(1−γ)% confidence interval for parameter α_j is expressed as $\left[\hat{\boldsymbol{\alpha}}_{(\gamma B/2)}^*, \hat{\boldsymbol{\alpha}}_{((1-\gamma/2)B)}^*\right]$.

4.2. Boot-t Algorithm

1. Based on the known parameter estimates, construct a sample dataset t_{ij} (where i=1, 2, …, n and j=1, 2, …, k) of size n.

2. Randomly draw a bootstrap sample t_{ij}^* of the same size n from the original dataset t_{ij}, and compute the corresponding bootstrap point estimate $\hat{\boldsymbol{\alpha}}^* = (\hat{\alpha}_1^*, \hat{\alpha}_2^*, ..., \hat{\alpha}_k^*)'$.

3. For each bootstrap sample, calculate the pivot quantity $R_j^* = (\hat{\alpha}_j^* - \hat{\alpha}_j) / \hat{\sigma}_j^*$, where $\hat{\sigma}_j^*$ denotes the standard variance of the j-th bootstrap estimate.

4. Repeat steps (2) and (3) a total of B times to obtain B sets of pivot quantities $R_1^*, R_2^*, ..., R_B^*$.

5. Sort the B pivot quantities to form an ordered sequence $R_{(1)}^* \le R_{(2)}^* \le \cdots \le R_{(B)}^*$. Based on the given confidence level 1-γ (0<γ<1), the 100(1−γ) % confidence interval for parameter α_j is expressed as $\left[\hat{\alpha}_j - \hat{\sigma}_j^* R_{((1-\gamma/2)B)}^*, \hat{\alpha}_j - \hat{\sigma}_j^* R_{(\gamma B/2)}^*\right]$.

To validate the effectiveness of the proposed method and demonstrate its performance under different sample sizes, experiments were conducted using the R programming language. The system parameters were set as k=3, α_1=0.1, α_2=0.2 and α_3=0.4, with sample sizes n selected as 20, 50, 75, and 100. By running the Bootstrap procedure 5000 times (B=5000), parameter estimates, standard errors (SE), and the widths of 95% Bootstrap confidence intervals were calculated for each sample size. The results are presented in Table 3.

Table 3. Ninety-five percent bootstrap CIs

Estimates	n			
	20	50	75	100
Bootstrap estimates [SE]	$\hat{\alpha}_1 = 0.1008206$	$\hat{\alpha}_1 = 0.1109668$	$\hat{\alpha}_1 = 0.1157372$	$\hat{\alpha}_1 = 0.1200843$
	0.030421	0.026519	0.025917	0.026912
	$\hat{\alpha}_2 = 0.1830094$	$\hat{\alpha}_2 = 0.1929322$	$\hat{\alpha}_2 = 0.1959856$	$\hat{\alpha}_2 = 0.1990104$
	0.036741	0.028989	0.027044	0.026512
	$\hat{\alpha}_3 = 0.2990239$	$\hat{\alpha}_3 = 0.3270942$	$\hat{\alpha}_3 = 0.3436778$	$\hat{\alpha}_3 = 0.3602818$
	0.094801	0.064075	0.054943	0.058538
Bootstrap-p CI {widths}	$\alpha_1 \in [0.0563,0.1765]$	$\alpha_1 \in [0.0682,0.1680]$	$\alpha_1 \in [0.0730,0.1704]$	$\alpha_1 \in [0.0757,0.1745]$
	{0.1202}	0.0998	0.0974	0.0988
	$\alpha_2 \in [0.1114,0.2427]$	$\alpha_2 \in [0.1373,0.2428]$	$\alpha_2 \in [0.1461,0.2485]$	$\alpha_2 \in [0.1504,0.2483]$
	{0.1312}	0.1055	0.1023	0.0979
	$\alpha_3 \in [0.1946,0.5353]$	$\alpha_3 \in [0.2343,0.4694]$	$\alpha_3 \in [0.2618,0.4803]$	$\alpha_3 \in [0.2799,0.5026]$
	0.1353	0.1066	0.0921	0.1284
Bootstrap-t CI {widths}	$\alpha_1 \in [0.0317,0.1670]$	$\alpha_1 \in [0.0627,0.1693]$	$\alpha_1 \in [0.0868,0.1790]$	$\alpha_1 \in [0.0175,0.1459]$
	0.1353	0.1066	0.0921	0.1284
	$\alpha_2 \in [0.2165,0.3616]$	$\alpha_2 \in [0.1561,0.2470]$	$\alpha_2 \in [0.1445,0.2770]$	$\alpha_2 \in [0.1716,0.2725]$
	0.1451	0.0909	0.1325	0.1009
	$\alpha_3 \in [0.0080,0.3305]$	$\alpha_3 \in [0.1361,0.4348]$	$\alpha_3 \in [0.2858,0.4375]$	$\alpha_3 \in [0.2862,0.5021]$
	0.3385	0.2987	0.1517	0.2160

From the results, it can be observed that as the sample size increases, the standard errors of the bootstrap estimates gradually decrease, and the confidence intervals become narrower. This indicates a significant improvement in the precision and stability of the parameter estimates. Notably, the boot-p method demonstrates more consistent and reliable estimation ranges across different parameters, further validating its robustness and applicability in this context. When combined with Table 2, the Gauss-Seidel method demonstrates smaller estimation bias and mean squared error (MSE) with larger sample sizes, indicating its suitability for precise point estimation. In contrast, the Bootstrap method quantifies estimation uncertainty through confidence intervals (CIs). While it exhibits greater fluctuation with smaller samples, its estimates stabilize as the sample size increases. In summary, the Gauss-Seidel method is better suited for high-precision estimation, whereas the Bootstrap method has a comparative advantage in evaluating estimation uncertainty.

5. Summary

This study proposes a novel modeling approach for load-sharing parallel systems based on the log-logistic distribution and combines it with the Gauss-Seidel iterative optimization algorithm to achieve robust parameter estimation in complex dependency structures. By employing log-logistic distribution to characterize the distribution of the component lifetimes, this paper overcomes the limitations of traditional methods in handling non-independent failures and non-monotonic hazard rate characteristics. Simulation results demonstrate that the proposed method significantly improves the accuracy and stability of parameter estimation and explores the impact of different sample sizes on estimation precision. The model not only enhances computational efficiency but also provides effective theoretical support for reliability analysis of load-sharing mechanisms in complex engineering systems.

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