Revisiting classical concepts of Linear Elastic Fracture Mechanics - Part III: The stress field in a double-edge notched finite strip by means of the "stress-neutralization" technique

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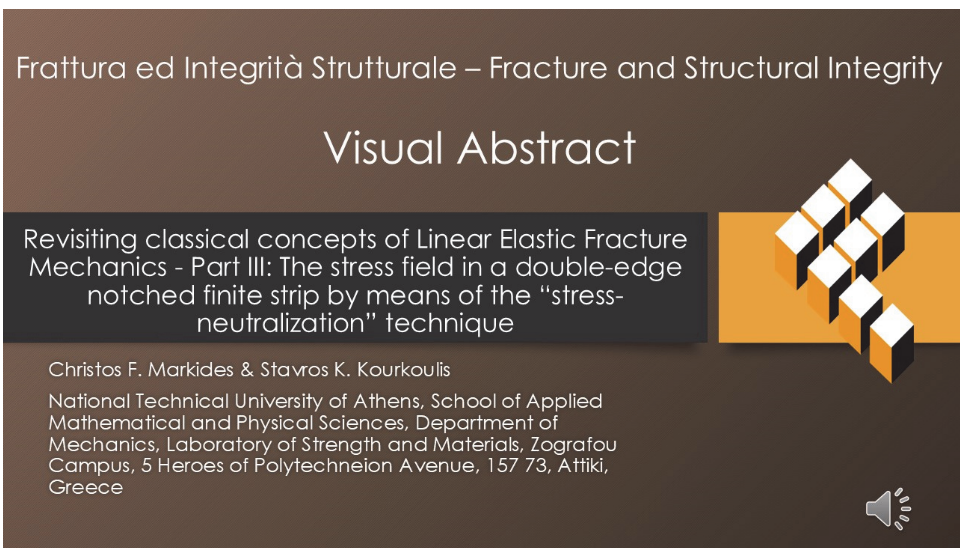

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INTRODUCTION

This is the third part of a short series of papers revisiting classical concepts of Linear Elastic Fracture Mechanics (LEFM) in the frame of alternative analytic approaches, based mainly on the complex potential technique, as it was introduced and formulated by Kolosov [1] and Muskhelishvili [2]. The objective of the series is to provide analytical expressions for the stress field around particular geometric discontinuities in elastic media, under specific loading conditions. In this context, Part-I of the series [3] provided analytical formulae for the Stress Intensity Factors (SIFs) at the tip of a natural (i.e., "mathematical") crack (namely, a crack with zero distance between its lips), in an infinite plate, loaded in its plane, for both opening and closing (contacting each other) crack lips. For the latter case an alternative way was presented for the solution of the familiar problem of 'overlapping' lips [4–6]. In that paper, it was also quantitatively verified that the dimensions of the plate may indeed be comparable to the crack length without significantly affecting the final results (as it was, in fact, tacitly implied by Muskhelishvili [2]), thus relaxing somehow the assumption of the infinite-plate.

Along a similar line of thought, Part-II of the series [7] dealt with the stretching of a strip with finite dimensions, weakened by a single edge notch of parabolic shape, approximating the rounded V-notch configuration. The specific issue concerns the engineering community already from the end of the 19th century, given that the presence of geometrical discontinuities of any kind was found to strongly amplify the local stress field, as it was highlighted by Kirsch [8] and Irwin [9] for circular and elliptical holes, respectively, in infinite domains. Obviously, this amplification of the stress field is crucial for the integrity and safety of any engineering structure, justifying the huge effort devoted since then for the accurate description of the stress field around various types of discontinuities. Initially the efforts were mainly focused on the quantification of the respective Stress Concentration Factors (SCFs), starting with Neuber's pioneering works [10] already from the middle thirties of the 20th century, and, also, the works by Hardrath and Ohman [11] and that by Lin [12] for a finite strip weakened by a pair of symmetrical notches, under the simplifying assumption that the notches were of semicircular shape (although the latter was criticized by Peterson [13] as providing values for the SCF exceeding the ones provided by Neuber).

The attempts for the determination of full-field solutions for the stresses around discontinuities were facilitated after Irwin [14] introduced his seminal solution for the stress field developed in the immediate vicinity of a "mathematical" crack, based on the ideas presented earlier by Westergaard [15]. It sounds perhaps strange, however the solution for the stress field around discontinuities in the form of open 'cracks', i.e., discontinuities in the form of V- or U-shaped notches, was proven much more complicated compared to the respective solution for the stress field around the tips of "mathematical" cracks. It was Williams [16] who presented such a solution for V-shaped notches by means of eigenfunction series expansions. Later on, solutions for the stress field around discontinuities with non-zero tip radius were proposed, also, by Creager and Paris [17] and Glinka and Newport [18], although the latter was found to suffer from insensitivity of its results to the variations of the opening angle of the notch [19].

In spite of the intensive efforts devoted to the determination of full-field solutions for the stresses around V- and U-shaped notches, the issue is by no means closed and it concerns continuously the scientific community due to its paramount practical importance. The relative studies are implemented either numerically or experimentally or adopting hybrid approaches [20, 21]. In this direction it is worth mentioning the milestone contributions by the team of late Professor Lazzarin, who provided analytical solutions for either sharp or rounded V-notches in 1996 [19]. Shortly afterwords they improved their solution for bodies of finite dimensions and then they revised their approach in order to increase its accuracy in case of rounded V-shaped notches with a large opening angle, by "*… adding a component in the polynomial arrangement of potential functions*", adopting the configuration of Fig. 1 [22]. Further on, the team contributed significantly in the development of the "generalized stress intensity factor" for either rounded V- or U-shaped notches, focusing attention on the fracture of notched structural elements, by means of proper either energy- or stress-based fracture criteria [23–25].

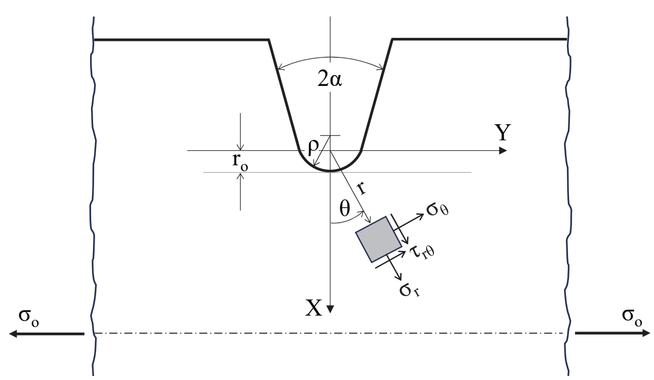

Figure 1: (a) The geometrical configuration of the problem considered by Filippi et al. together with the reference systems adopted for their milestone solution and the definitions for the stress components [22].

In the frame of the above concepts, the approach that was introduced in Part-II of this short series of papers dealt with the single-edge notched finite strip under tension, providing a simple and flexible expression for the respective SCFs, along with the compact expressions for the stress field all over the strip. The results of that study were found in relatively satisfactory agreement to the ones obtained by the well-established approach (analytical and numerical) by Filippi et al. [22], revealing the potentialities of the procedure proposed. The present Part-III of the series completes Part-II by considering instead of a single-edge notched strip, a finite strip weakened by two symmetric, parabolic, stress-free, edge notches, the common axis of which is perpendicular to the direction of the external loading. Though relatively perplex, the solution of this problem might be proven more useful, compared to that for a single-edge notched strip, since it corresponds to a configuration which, due to its symmetry, diminishes bending effects and it is easier to be managed in experimental protocols.

In this direction, using Muskhelishvili's complex potentials technique [2] in combination with a novel procedure for "stress neutralization" of specific areas of the loaded strip, a full-field approximate solution is obtained here in closed form for the problem of a stretched, double-edge notched, linear elastic and isotropic finite strip (Fig. 2a). The notches are of parabolic shape. The radius of curvature at the bases of the parabolas can substitute quite efficiently the radius of the respective rounded V-shaped notches (Fig. 2b), thus approximating according to a very satisfactory manner the configuration of a V-shaped notch with rounded tip. It is mentioned from the very beginning that the role of the exact geometry of the notch (and especially of its tip) is of critical importance for the overall approach to the problem, as it has been highlighted by Lazzarin and Tovo [19]. This aspect of the analysis will be addressed in the "Discussion and Conclusions" section. The basic assumption adopted in the present study is that the ratio of the notch depth over the width of the strip is small, i.e., the notches are assumed to be "shallow". This assumption simplifies significantly the algebraic manipulations.

In this context, the present solution is obtained by means of the solution recently presented in Part-II of this series [7], for the single-edge parabolic notch taking, in addition, advantage of the superposition principle and the "stress -neutralization" concept. The validity of the solution is then proven by the fulfilment of the stress boundary conditions along the sides of the strip and, also, along the periphery of the notches. Moreover, the outcomes of the present solution are considered in juxtaposition to the ones of the solution presented by Filippi et al. [22]. Emphasis is given to the stress field around the base ("crown" or "tip") of the notches and, also, along the axis of symmetry of the strip normal to the loading direction, where the non-zero tensile stress component is given explicitly in terms of a convenient and flexible formula. Taking advantage of this formula, easy-to-use expressions are obtained for the respective SCF, k (for the case of blunt notches), as well as for the respective SIF, K_I (for the case of "mathematical" edge cracks).

THEORETICAL CONSIDERATIONS

The problem

In this study an attempt is made to obtain the stress field in a strip of length 2b and width 2h, stretched by a uniform stress σ_o, and weakened by two symmetric rounded V-shaped notches (Fig. 2a). For mathematical convenience, the rounded V-notches are described as parabolic cavities, for which the curvature at the base of the parabola approximates that of the rounded V-notch (Fig. 2b). The strip lies in the complex z=x+iy plane. The origin of the coordinate system is the focus of the upper parabolic cavity/notch. The two notches have the same length (depth) equal to $d=c+\alpha^2$, where α in the equations of the parabolas (Fig. 2a) dictates the sharpness of the cavities; namely, as α tends to zero the notches tend to the edge "mathematical" cracks. The material of the strip is assumed isotropic and linearly elastic and Muskhelishvili's complex potentials technique [2] is used to solve the problem. As it was mentioned the notches are assumed "shallow" and therefore the stress state at the center of the strip corresponds to that of uniaxial tension, i.e., $\sigma_{xx}=\sigma_o$, $\sigma_{yy}=\sigma_{xy}=0$.

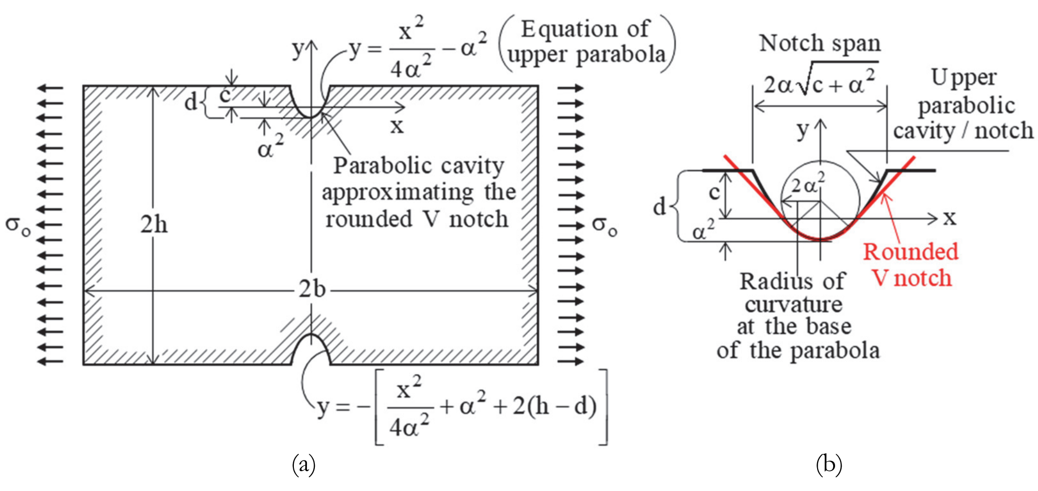

Figure 2: (a) The configuration of the problem; (b) The upper rounded V-shaped notch and the approximating parabolic cavity/notch used in the frame of the present solution.

Outlining the method of solution – The "stress-neutralization" technique

In obtaining the solution of the doubly notched stretched strip, use is made of the solution of the problem of the intact stretched strip which will be denoted as "Problem 1" (Fig. 3a). Namely, at any point of the intact stretched strip, and, therefore, along the points of its two internal parabolic loci shown in Fig.3a, the stress state on normal sections will be that of simple uniaxial tension, namely, $\sigma_{xx}=\sigma_o$, $\sigma_{yy}=\sigma_{xy}=0$. By adding to the points of these two parabolic loci the opposite stress field $\sigma_{xx}=-\sigma_o$, without interfering to the boundary conditions on the outer boundaries of the stretched strip (Fig. 3b), the shaded parabolic regions of the stretched strip (Fig. 3c) become stress free. That is equivalent to stress neutralizing, thus removing these shaded regions from the strip, transforming it into the doubly notched stretched strip in question (Fig. 3d).

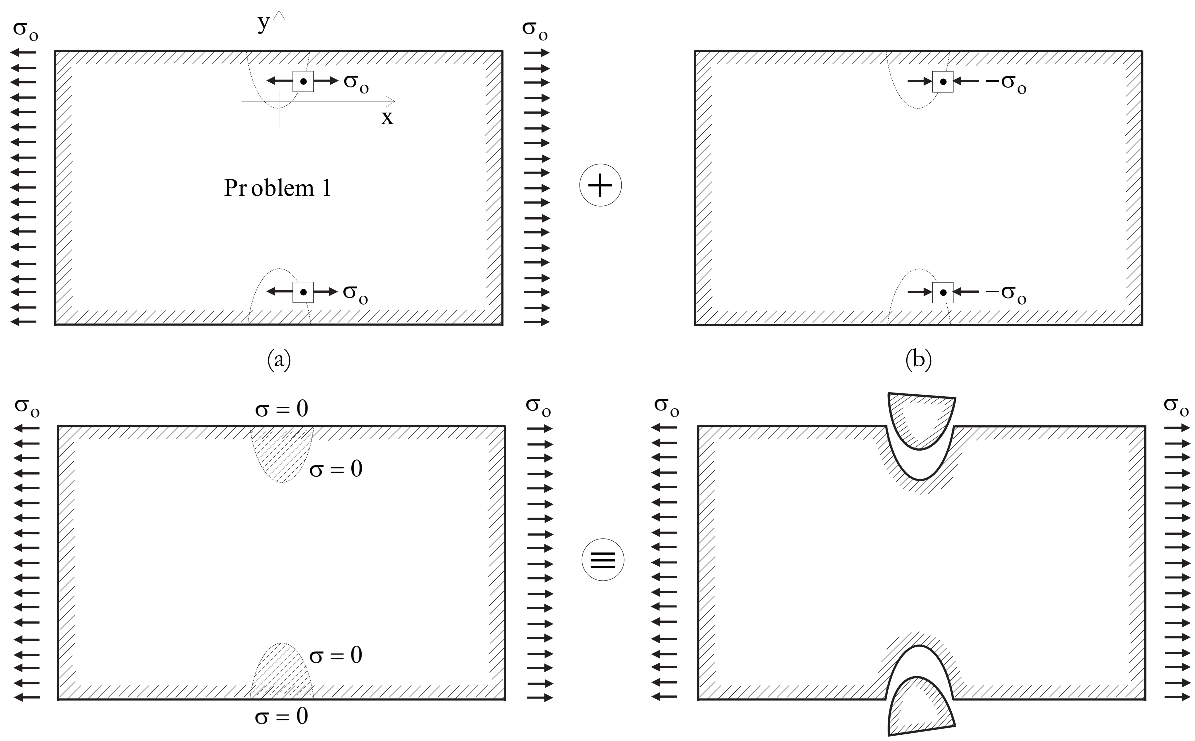

Figure 3: (a) The "Problem 1", i.e., the stretching of the intact strip, and the uniform tensile field σ_o at points of the two internal parabolic loci; (b) Imposing the opposite stress field at the internal parabolic loci; (c) The stretched intact strip with the two stress-free parabolic regions (shown shaded); (d) The double notched stretched strip obtained.

In order to apply the above-described procedure, an auxiliary problem, denoted from here on as "Problem 2" (Fig. 4b), is superposed to "Problem 1" (Fig. 4a). The boundary conditions for "Problem 2" consist of stresses $\sigma_{\eta\eta}$, $\sigma_{\xi\eta}$ along the two parabolic notches corresponding to the $-\sigma_o$ stress field, i.e., the opposite of that occurring in the respective parabolic loci of the intact strip in "Problem 1" (Figs. 4b, 4c). Thus, after superposition of "Problem 2" to "Problem 1", the stretched double notched strip with stress-free notches in question will be obtained (Fig. 4d). The superposition of the notched strip on to the intact one is topologically justified by the fact that the $-\sigma_o$ stress field of "Problem 2" upon superimposed to the intact strip automatically renders it a notched strip as well, since it stress-neutralizes the shadow parabolic regions, discussed previously with regard to Fig. 3. Concerning the sides of the strip, it is seen from the final results that "Problem 2" leaves more or less unaffected the stress state at the strip sides of "Problem 1".

In this context, denoting the complex potentials solving "Problem 1" and "Problem 2" by Φ_1, Ψ_1 and Φ_2, Ψ_2, respectively, the solution of the double-notched strip in question (Fig. 4d) will read as:

$$\Phi(z)=\Phi_1(z)+\Phi_2(z), \qquad \Psi(z)=\Psi_1(z)+\Psi_2(z) \tag{1}$$

The solution of the (trivial) "Problem 1" reads as:

$$\Phi_1(z)=\frac{\sigma_o}{4}, \qquad \Psi_1(z)=-\frac{\sigma_o}{2} \tag{2}$$

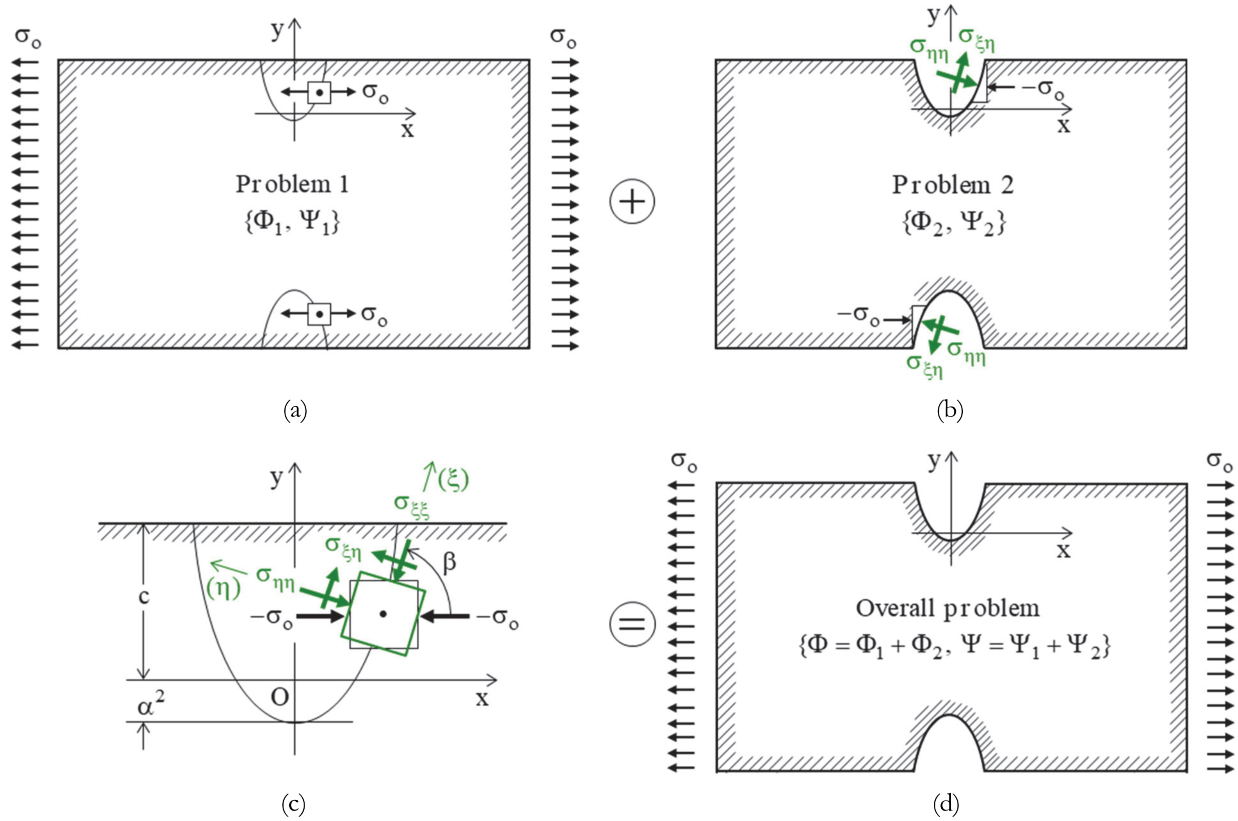

Figure 4: (a) The "Problem 1", i.e., that of the intact stretched strip; (b) The auxiliary "Problem 2"; (c) The $\sigma_{\eta\eta}$, $\sigma_{\xi\eta}$ stress components corresponding to the $-\sigma_o$ stress field (providing the boundary conditions on the notches of "Problem 2"); (d) The overall problem, with its solution obtained by the superposition of the solutions of "Problem 1" and "Problem 2".

The complex potentials Φ_2, Ψ_2, solving the auxiliary "Problem 2" are subjected to determination, according to the method that will be described analytically in next section (*The solution of the auxiliary "Problem 2")*, by means of the superposition of the solutions of two additional auxiliary elementary problems, denoted as "Problem 3" and "Problem 4".

The solution of the auxiliary "Problem 2"

The complex potentials Φ_2, Ψ_2, solving the auxiliary "Problem 2" (Fig. 4b), are here obtained under the assumption of small length $d=c+\alpha^2$ of the notches. In this way Φ_2, Ψ_2 upon superposed to Φ_1, Ψ_1 will provide the solution Φ, Ψ of the overall problem in question, Fig. 4d, leaving, in the limit, the stress field at the center of the strip unaffected, so that the stress state at the center of the strip for the overall problem (Fig. 4d) to be that of simple tension $\sigma_{xx}=\sigma_o$, $\sigma_{yy}=\sigma_{xy}=0$ (holding in the case of the intact strip - "Problem 1"). In this case the two notches will not influence one another, and therefore Φ_2, Ψ_2 can be obtained by simply superposing the solutions of "Problem 3" and "Problem 4" shown in Fig. 5, as:

$$\Phi_2(z)=\Phi_3(z)+\Phi_4(z), \qquad \Psi_2(z)=\Psi_3(z)+\Psi_4(z) \tag{3}$$

The complex potentials Φ_3, Ψ_3 solving "Problem 3", have already been obtained in a recent work of the authors [7], and they can be written as:

$$\Phi_3(z)=\frac{i\sigma_o}{2\pi}\left(\frac{\alpha-\sqrt{iz}}{\sqrt{iz}}\log\frac{\alpha-\sqrt{iz}+i\sqrt{\alpha^2+c}}{\alpha-\sqrt{iz}-i\sqrt{\alpha^2+c}}-\frac{2i\sqrt{\alpha^2+c}}{\sqrt{iz}}\right) \tag{4}$$

$$\Psi_3(z)=\frac{\sigma_o\sqrt{\alpha^2+c}}{2\pi}\left\{\frac{\alpha(3iz-4\alpha^2)}{2z\sqrt{iz}\sqrt{\alpha^2+c}}\log\frac{\alpha-\sqrt{iz}+i\sqrt{\alpha^2+c}}{\alpha-\sqrt{iz}-i\sqrt{\alpha^2+c}}+\frac{4i\alpha^2-z}{z\sqrt{iz}}+\frac{\alpha(4i\alpha^2-5z)-(8i\alpha^2-z)\sqrt{iz}}{z[(\alpha-\sqrt{iz})^2+\alpha^2+c]}\right\} \tag{5}$$

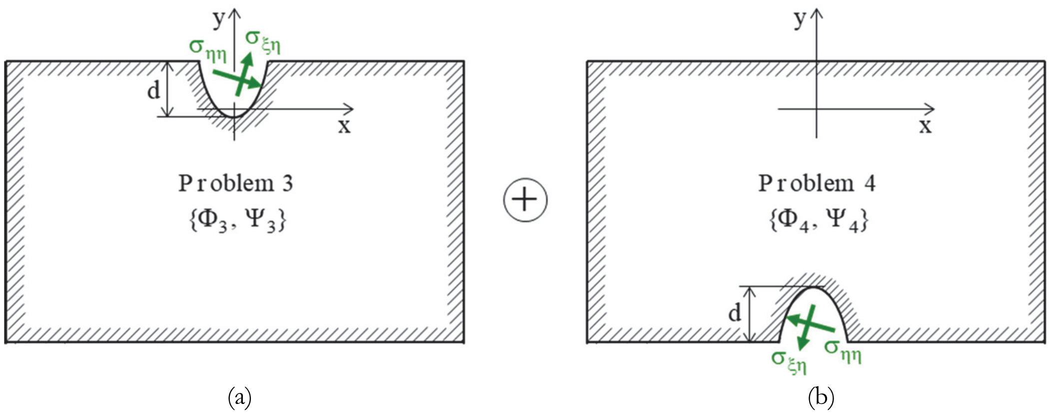

Figure 5: (a) "Problem 3" of the strip with the upper notch; (b) "Problem 4" of the strip with the lower notch.

The complex potentials Φ_4, Ψ_4, which solve "Problem 4", in turn, are easily obtained from Φ_3, Ψ_3 (by means of the non-interacting notches assumption), by modifying Φ_3, Ψ_3, by changing properly the coordinate axes (Fig. 6). Namely, starting from the solution Φ_3, Ψ_3, of "Problem 3" in Fig. 5a (or equivalently in Fig. 6a), then by a 180° clockwise rotation, and an upwards translation of the coordinate axes by an amount 2(h–c), one obtains the solution Φ_4, Ψ_4 of "Problem 4" (Fig. 6c) in terms of Φ_3 and Ψ_3 as:

$$\Phi_4(z) = \Phi_3(-z - i2(h-c)), \qquad \Psi_4(z) = \Psi_3(-z - i2(h-c)) - i2(h-c)\Phi_3'(-z - i2(h-c)) \tag{6}$$

where prime in Φ_3 denotes the first order derivative with respect to z. Clearly, the right-hand sides of Eqns.(6) are directly obtained from Eqns.(4) and (5) substituting in the latter z by –z–i2(h–c).

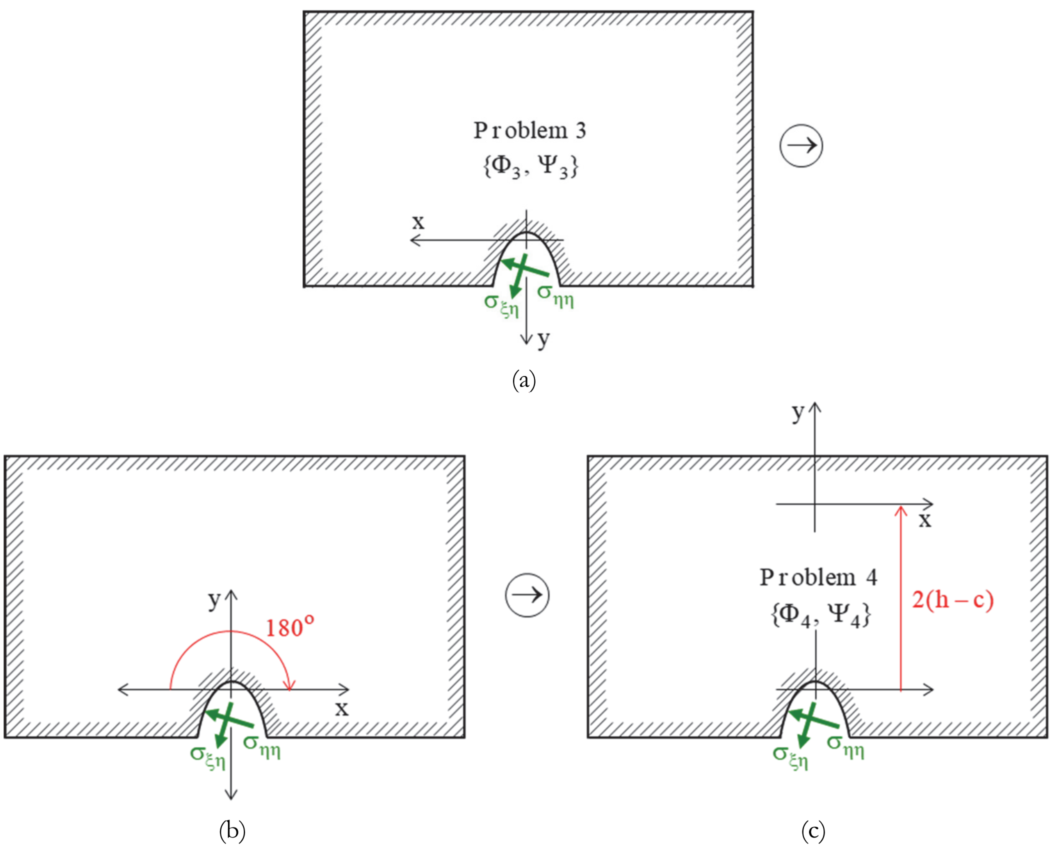

Figure 6: (a) "Problem 3" (with solution Φ_3, Ψ_3); (b) An 180° clockwise rotation of the coordinate axes; (c) A 2(h–c) upwards translation of the coordinate axes, reaching the solution Φ_4, Ψ_4, of "Problem 4".

Thus, taking advantage of Eqns.(3) and (6), the solution of the auxiliary "Problem 2" (Fig. 4b), reads as:

$$\Phi_2(z)=\Phi_3(z)+\Phi_3(-z-i2(h-c)), \qquad \Psi_2(z)=\Psi_3(z)+\Psi_3(-z-i2(h-c))-i2(h-c)\Phi_3'(-z-i2(h-c)) \quad (7)$$

where the general expressions of Φ_3, Ψ_3, are given by Eqns.(4) and (5).

The solution of the overall problem

Substituting from Eqns.(2) and (7) into Eqns.(1), the complex potentials Φ(z) and Ψ(z) solving the overall problem (Fig. 4d) in question, i.e., that of stretching a finite strip weakened by two "shallow" edge notches, are written after some algebra as:

$$\begin{aligned}\Phi(z)=\frac{\sigma_o}{4}+\frac{i\sigma_o}{2\pi}\Bigg[&\frac{\alpha-\sqrt{iz}}{\sqrt{iz}}\log\frac{\alpha-\sqrt{iz}+i\sqrt{\alpha^2+c}}{\alpha-\sqrt{iz}-i\sqrt{\alpha^2+c}}-\frac{2i\sqrt{\alpha^2+c}}{\sqrt{iz}}\\&+\frac{i\alpha-\sqrt{iz-2(h-c)}}{\sqrt{iz-2(h-c)}}\log\frac{i\alpha-\sqrt{iz-2(h-c)}-\sqrt{\alpha^2+c}}{i\alpha-\sqrt{iz-2(h-c)}+\sqrt{\alpha^2+c}}+\frac{2\sqrt{\alpha^2+c}}{\sqrt{iz-2(h-c)}}\Bigg]\end{aligned} \quad (8)$$

$$\begin{aligned}\Psi(z)=-\frac{\sigma_o}{2}+\frac{\sigma_o\sqrt{\alpha^2+c}}{2\pi}\Bigg\{&\frac{\alpha(3iz-4\alpha^2)}{2z\sqrt{iz}\sqrt{\alpha^2+c}}\log\frac{\alpha-\sqrt{iz}+i\sqrt{\alpha^2+c}}{\alpha-\sqrt{iz}-i\sqrt{\alpha^2+c}}+\frac{4i\alpha^2-z}{z\sqrt{iz}}+\frac{\alpha(4i\alpha^2-5z)-(8i\alpha^2-z)\sqrt{iz}}{z[(\alpha-\sqrt{iz})^2+\alpha^2+c]}\\&+\frac{4i\alpha^2+z}{[iz-2(h-c)]^{3/2}}+\frac{\alpha[5iz-8(h-c)-4\alpha^2]-(8i\alpha^2+z)\sqrt{iz-2(h-c)}}{[iz-2(h-c)][(i\alpha-\sqrt{iz-2(h-c)})^2-(\alpha^2+c)]}\\&-\frac{\alpha[3iz-8(h-c)+4\alpha^2]}{2[iz-2(h-c)]^{3/2}\sqrt{\alpha^2+c}}\log\frac{i\alpha-\sqrt{iz-2(h-c)}-\sqrt{\alpha^2+c}}{i\alpha-\sqrt{iz-2(h-c)}+\sqrt{\alpha^2+c}}\Bigg\}\end{aligned} \quad (9)$$

THE STRESS FIELD IN THE DOUBLE-EDGE NOTCHED STRIP UNDER TENSION

Introducing the complex potentials Φ(z) and Ψ(z) from Eqns.(8) and (9) in Muskhelishvili's well-known formulae (see next Eqns.(10), where $\Re$ denotes the real part and over-bar the conjugate complex value), the Cartesian components of the stress field are obtained all over the strip.

$$\sigma_{yy}-i\sigma_{xy}=2\Re\Phi(z)+z\overline{\Phi'(z)}+\overline{\Psi(z)}, \qquad \sigma_{xx}=4\Re\Phi(z)-\sigma_{yy} \quad (10)$$

Along the periphery of the notches use can be made, also, of the curvilinear stress components $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$, $\sigma_{\xi\eta}$ (see Fig. 4c) as they are obtained by employing the familiar transformations:

$$\begin{aligned}\sigma_{\xi\xi}&=\sigma_{xx}\cos^2\beta+\sigma_{yy}\sin^2\beta+2\sigma_{xy}\sin\beta\cos\beta\\ \sigma_{\eta\eta}&=\sigma_{xx}\sin^2\beta+\sigma_{yy}\cos^2\beta-2\sigma_{xy}\sin\beta\cos\beta\\ \sigma_{\xi\eta}&=(\sigma_{yy}-\sigma_{xx})\sin\beta\cos\beta+\sigma_{xy}(\cos^2\beta-\sin^2\beta)\end{aligned} \quad (11)$$

Substituting from Eqns.(8, 9, 10) in Eqns.(11) it can be seen that along the notches only the $\sigma_{\xi\eta}$ component is non-zero, while $\sigma_{\eta\eta}=\sigma_{\xi\eta}=0$, fulfilling the boundary condition for stress free notches.

Stresses variation along the strip sides, notch bisector and upper notch

In order to highlight the potentialities of the solution introduced, the above stress formulae will be applied for a specific strip of length 2b=30 cm, width 2h=20 cm, with two short notches of length $d=c+\alpha^2=0.5+0.5^2=0.75$ cm and span 1.73 cm (Figs. 7a, 7b). The notches are free from stresses while the edges of the strip are stretched by a uniform tensile stress σ_{xx} equal to $\sigma_{xx}=\sigma_o=10$ MPa.

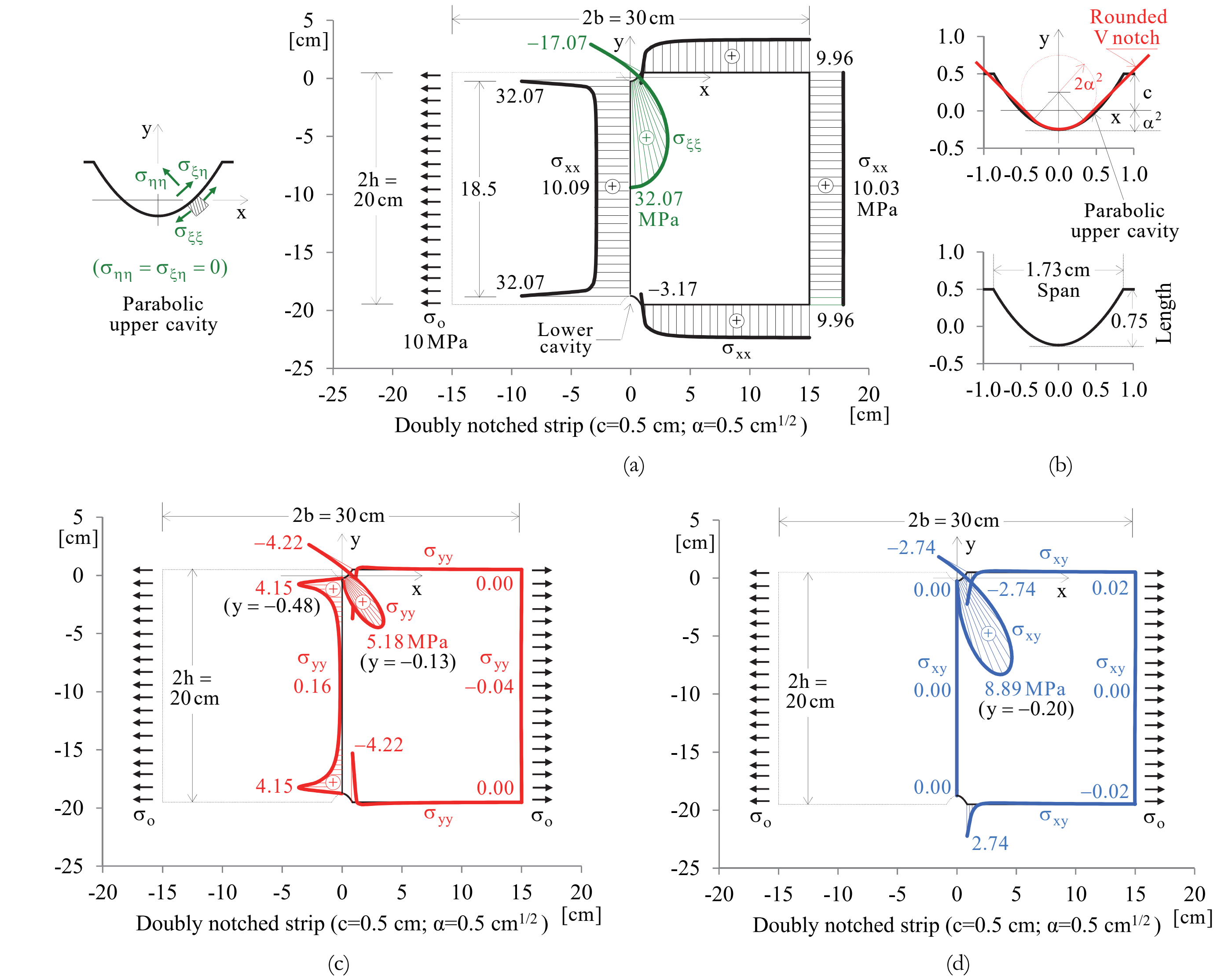

Figure 7: (a) The variation of σ_{xx} along the bisector of the notches and along the sides of the half strip, and the variation of $\sigma_{\xi\xi}$ along half the upper notch; (b) The geometric features of the parabolic cavities (vs the rounded V-notch); (c) The variation of σ_{yy} along the bisector of the notches, along half the strip sides, and along half the upper notch; (d) The variation of σ_{xy} along the bisector of the notches, along half the strip sides, and along half the upper notch.

In Fig. 7a, the variation of the σ_{xx} stress, as obtained from Eqns.(10), is plotted along the bisector of the notches (namely, along y-axis), as well as along the sides of the half (due to the symmetry) upper notch. As it is seen, at the midpoint of the bisector of the notches it holds that σ_{xx}=10.09 MPa, a value almost identical to σ_o holding in the case of the intact strip ("Problem 1"), thus approximating quite satisfactory the demand of non-interacting edge notches, set as a prerequisite while obtaining the present solution. At the bases (tips) of the notches σ_{xx} attains (as it is expected) its maximum value equal to σ_{xx}=32.07 MPa, a reasonable stress amplification (equal to about 3 times σ_o), taking into account the particular geometry of the notch (i.e., that of an almost semi-circular cavity). In addition, σ_{xx} is around 10 MPa all along the sides of the strip. These results show that the short notches assumption suffices both insignificant stress disturbance along the sides of the strip, and, also, effective predictions about the stress concentration at the tips of the notches. The value σ_{xx}=–3.17 MPa at the end points of the notches is to suffice stress equilibrium at the particular points.

The only non-zero curvilinear stress component $\sigma_{\xi\xi}$ is, also, plotted in Fig. 7a, along the upper half notch (green color). Its maximum value $\sigma_{\xi\xi}$=32.07 MPa, attained at the notch base, obviously coincides with the maximum value of σ_{xx} at the same point (where σ_{xx}=$\sigma_{\xi\xi}$). Similarly, the variation of the other two Cartesian stress components σ_{yy} and σ_{xy} are plotted in Figs. 7c and 7d, respectively. The very small value of σ_{yy} (Fig. 7c) at the midpoint of the notches' bisector (0.16 MPa) and the almost zero value attained along the sides of the strip guarantee further the validity of the solution, from the point of view of satisfying the boundary conditions of the problem.

Similar conclusions can be drawn from the variation of the σ_{xy} stress component (Fig. 7d). The non-zero values of σ_{yy} and σ_{xy} along the half upper notch (referring to the interior of the strip) are to suffice stress equilibrium there.

Polar stress variation in the vicinity of the base (tip) of the upper notch

Following previous studies [7], the variation of the polar stress components σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$, along a circular locus centered at the origin of the reference system, and extended from the upper notch boundary to the bisector of the notches (y-axis), is plotted in Fig. 8, for the numerical values of the parameters that were considered in previous paragraph. The radius of the circular locus (colored red in the figure) was considered equal to $r=2.5\alpha^2$. It is easily seen from Fig. 8 that for the specific value of r the locus corresponds to an angle varying in the interval $11.54^o \leq \theta \leq -90^o$. The variations of σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$, closely resemble qualitatively the ones described in ref. [7], highlighting further the potentialities of the present solution.

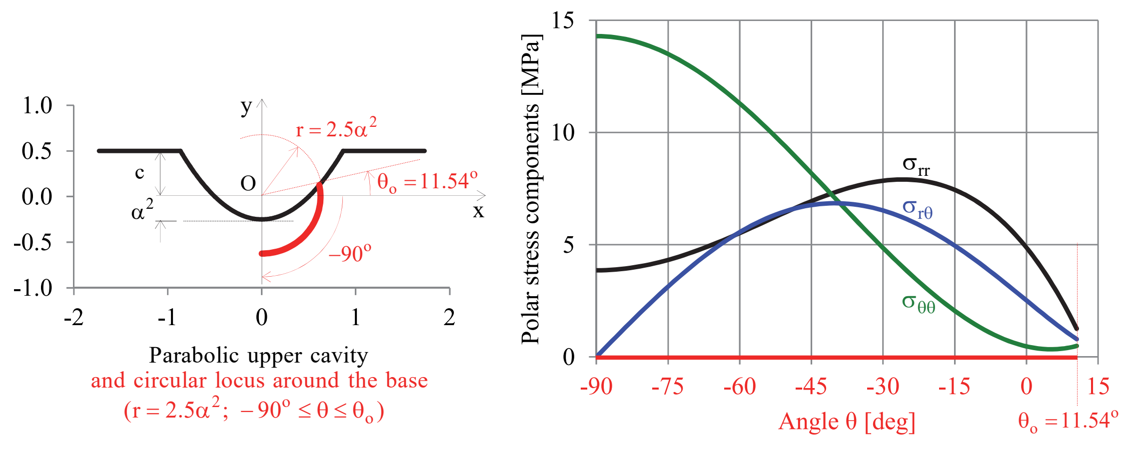

Figure 8: The variation of the σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$, stress components along the red circular locus around the notch base.

ABOUT THE STRESS CONCENTRATION AND THE STRESS INTENSITY AT THE CROWN OF THE NOTCHES

The critical stress at the bases (crowns or tips) of the notches, and the respective Stress Concentration Factor k

Adopting the Cartesian representation of stress, the critical stress component at the base of the upper and the lower notch, i.e., the component quantifying the severity of the stress field, is $\sigma_{xx,cr}=\sigma_{xx}(0, -\alpha^2)=\sigma_{xx}(0, -2(h-c)+\alpha^2)$. Combining Eqns.(8, 9, 10), it is obtained after some algebra that:

$$\frac{\sigma_{xx,cr}}{\sigma_o}=\left\{1+\frac{2}{\pi\sqrt{2(h-c)-\alpha^2}}\left[\left(\sqrt{2(h-c)-\alpha^2}\right)-\alpha\right)\left(2\arctan\frac{\sqrt{2(h-c)-\alpha^2}-\alpha}{\sqrt{c+\alpha^2}}-\pi\right)+\frac{2\sqrt{c+\alpha^2}}{\alpha}\left(\sqrt{2(h-c)-\alpha^2}+\alpha\right)\right]\right\} \tag{12}$$

where the ratio $k=(\sigma_{xx,cr}/\sigma_o)$ represents the Stress Concentration Factor (SCF). Using the simple expression of Eqn.(12), it is relatively easy to explore the variation of the SCF in terms of the parameter α. In this direction, seven different geometries of the two edge parabolic cavities (approximating rounded V-shaped notches) are considered. These geometries are shown schematically in Fig. 9a for the upper notch only, for obvious symmetry reasons. The degree of approximation between the theoretical parabolic cavity and the respective rounded V_j-notch (with green color), j=1, …, 7. is shown in Fig. 9b. It is mentioned that in order to draw Fig. 9b, it was considered that all V_j-notches have the same depth (length), equal to $d=c_j+\alpha_j^2=0.75$ cm (in other words, proper combinations of c_j and α_j were considered so that the depth d of all seven notches to remain constant) for comparison reasons. Indicatively only, Fig. 9c depicts the case of the V_1-edge notches, corresponding to the configuration considered in all previous plots for the stress components.

The variation of the SCF, k, for the above seven geometries, i.e., for the respective values of the parameter a, is plotted in Fig. 9d. It is seen from this figure that the configuration with the V_1-edge notches results in a stress concentration factor

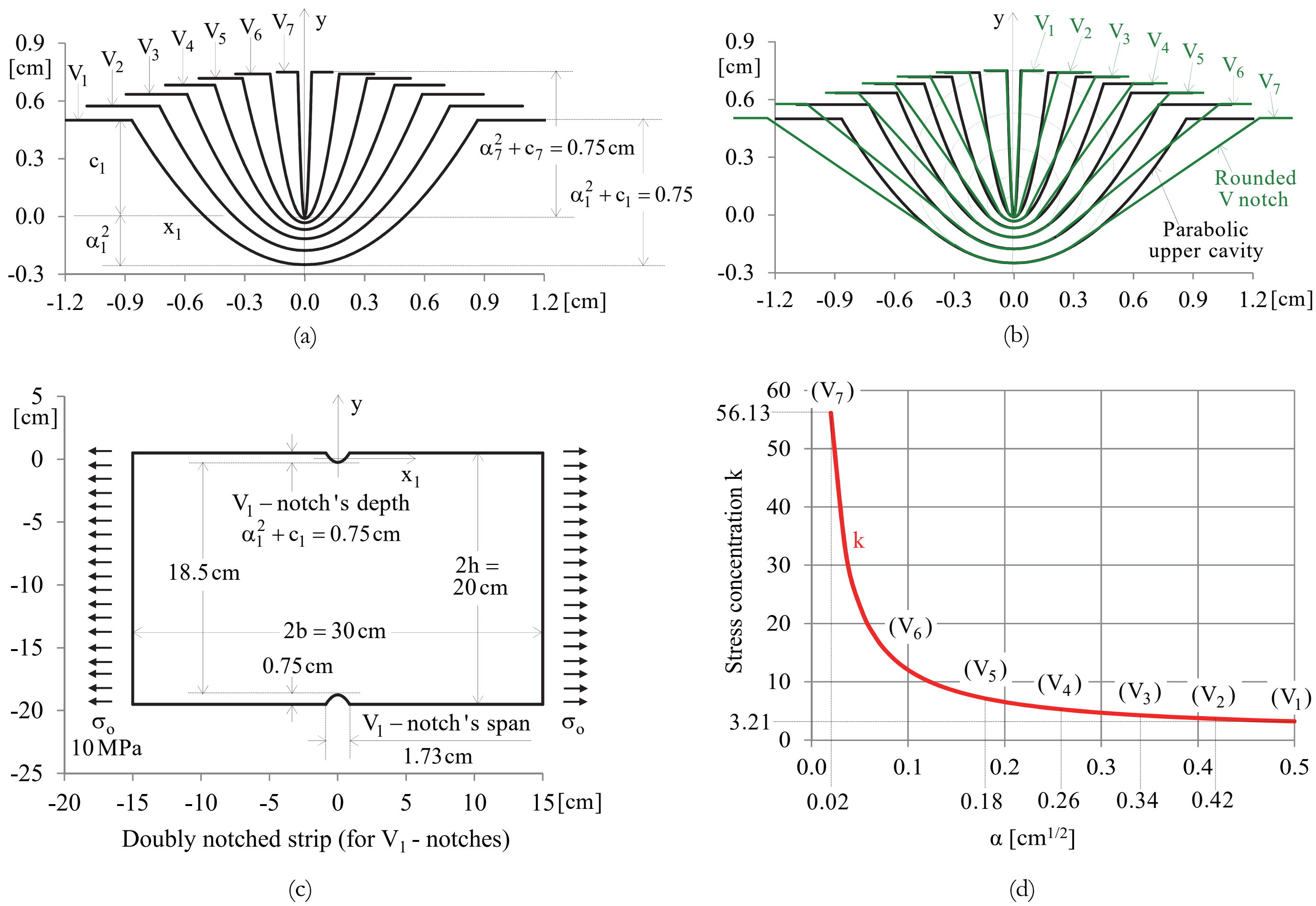

Figure 9: (a) Seven geometries for parabolic notches of the same depth (length) equal to 0.75 cm; (b) The parabolic cavities versus the respective approximated rounded V_j notches; (c) The doubly notched strip in case of the V_1 notches, as a typical example; (d) The variation of the Stress Concentration Factor k versus the parameter α.

equal to k=3.21 (already mentioned previously, with regard to Fig. 7a). Obviously, as α is getting smaller and smaller, k increases monotonically up to the value equal to k=56.13 for the V_7-edge notches, i.e., the case corresponding to a value of α equal to $\alpha=0.02$ cm$^{1/2}$. For $\alpha\to 0$, Eqn.(12) yields $k\to\infty$, which is the case when the edge notches become "mathematical" edge cracks, and the concept of stress concentration k should be replaced by the concept of the stress intensity factor K_I.

The stress intensity factor K_I in the stretched doubly notched strip

Assuming α=0 the two parabolic notches become edge "mathematical" cracks (i.e., edge notches of zero distance between their lips). As it was mentioned before in that case, by means of Eqn.(12), the stress concentration k becomes infinite, and the concept of stress intensity factor K_I is now the adequate quantity to describe the stress state at the tip of the cracks. In this context, a definition is provided next for K_I in the case α=0, as follows:

$$K_I := \lim_{\alpha\to 0}[\sigma_{xx,cr}\cdot\alpha] \tag{13}$$

Substituting from Eqn.(12) in Eqn.(13), yields:

$$K_I := \lim_{\alpha\to 0}[\sigma_{xx,cr}\cdot\alpha] = \frac{4\sigma_o}{\pi}\sqrt{c} = \frac{4}{\pi^{3/2}}\sigma_o\sqrt{\pi c} \tag{14}$$

Noticing that for α=0 the crack length is $d=c+\alpha^2=c$, K_I provided by Eqn.(14) is actually the Stress Intensity Factor at the tip of an internal crack of length 2c under mode I loading in an infinite plate, i.e., $\sigma_o\sqrt{\pi c}$, multiplied by the factor

$4/\pi^{3/2}$=0.72. In other words, between the cases of an infinite plate with a crack 2c, and a strip with two short edge cracks each one of length c, both under the same mode I loading scheme, there is a reduction in K_I , in the latter case by about 28%.

COMPARISON WITH EXISTING SOLUTIONS

A further attempt is here made to evaluate the present solution and check its results with respect to the ones due to the widely acceptable analytical solution by Filippi et al. [22]. It is to be mentioned that both solutions adopt the complex potentials technique, however they differ significantly from each other in the procedure followed, both concerning the type of the curvilinear coordinates used to describe the geometry of the notch, as well as the form of the complex potentials solving the problem. In fact, the present study deals with exact parabolic notches and a relevant net of intersecting orthogonal parabolas providing the curvilinear coordinates at any point of the doubly notched strip, while ref. [22] dealt with curved notches and curvilinear coordinates of general degree. Moreover, ref. [22] considered truncated series forms of φ(z) and ψ(z), obtaining the unknown coefficients by fulfilling the conditions of a stress-free notch. On the other hand, the present study, is based on Muskhelishvili's closed-form general solution for a semi-infinite region bounded by a parabola (which is the reason why the complex potentials due to the present solution are more perplex than those of ref. [22]), and, also, on the superposition principle (in order to stress neutralize-remove two parabolic sectors from the intact stretched strip, thus, transforming it to the doubly notched strip in question). It is worth mentioning, however, that despite the differences between the two solutions, their results, as it will be seen later, are in quite good mutual agreement, further establishing the value of both analytical solutions.
The comparison between the two solutions consists in comparing the stress field along the bisector of the two notches, assuming that they are under mode I loading. For convenience, the analytical formulae by Filippi et al. [22], and the necessary numerical data (presented there in tabulated from) required to plot the stresses, are quoted below, exactly as they are found in ref. [22]. In this context, in the case of mode I loading, the two non-zero stress components along the bisector of the notches ($\theta=0^o$), normalized over the maximum stress σ_{max}, ($\sigma_{max}=\sigma_\theta$ at the base of the rounded V-notch), read as [22]:

$$\frac{\sigma_\theta}{\sigma_{max}}=\left(\frac{r}{r_o}\right)^{\lambda-1}\frac{1+\lambda+\chi_b(1-\lambda)+\dfrac{q}{4(q-1)}\left(\dfrac{r}{r_o}\right)^{\mu-\lambda}\left[\chi_d(1+\mu)+\chi_c\right]}{1+\lambda+\chi_b(1-\lambda)+q\left[(1+\mu)\chi_d+\chi_c\right]/\left[4(q-1)\right]} \tag{15}$$

$$\frac{\sigma_r}{\sigma_{max}}=\left(\frac{r}{r_o}\right)^{\lambda-1}\frac{3-\lambda-\chi_b(1-\lambda)+\dfrac{q}{4(q-1)}\left(\dfrac{r}{r_o}\right)^{\mu-\lambda}\left[\chi_d(3-\mu)-\chi_c\right]}{1+\lambda+\chi_b(1-\lambda)+q\left[(1+\mu)\chi_d+\chi_c\right]/\left[4(q-1)\right]} \tag{16}$$

where

$$q=\frac{2\pi-2\alpha}{\pi},\quad r_o=\frac{q-1}{q}\varrho \tag{17}$$

In the above formulae, r_o is the constant distance of the rounded V-notch "tip" from the origin of the Cartesian reference (corresponding to $-\alpha^2$ in the present notation), and r is the varying distance from the origin, along the common dissector of the two notches (obviously, the interval $0\leq r\leq r_o$ has no physical meaning as it is lying outside the notched strip and is excluded while calculating the stresses, Fig. 1). The notch opening angle 2α was assumed equal to π/4 (see Fig. 10b), corresponding (for mode I loading conditions) to the following set of values of the characteristic parameters of the problem (see the 3rd line of Tab. 1 of ref. [22]):

$$\lambda=0.5050,\ \mu=-0.4319,\ \chi_b=1.1656,\ \chi_c=3.5721,\text{ and }\chi_d=0.0828. \tag{18}$$

In addition, regarding the notched strip geometry (Fig. 10a), the following set of values were considered (see 7th line of Tab. 3 of ref. [22]): Strip width H=6 cm (corresponding to 2h=6 cm in the present notation), notch depth a=1 cm (corresponding to $d=c+\alpha^2=1$ cm in the present notation), $h=H-a=4$ cm (corresponding to $2h-2d=4$ cm in the present notation), and radius of curvature at the rounded V-notch base $\varrho=0.125$ cm, corresponding to $2\alpha^2=0.08$ cm in the present notation (the present $2\alpha^2$ in length dimensions should not be confused with Filippi's et al. notch opening angle 2α in radians). The difference between Filippi's et al. $\varrho=0.125$ cm and the present value $2\alpha^2=0.08$ cm is attributed to the fact that the present study deals with exact parabolic notches the radius of curvature at their bases being $\varrho_p=2\alpha^2$. The particular choice of $\varrho=0.125$ and $2\alpha=\pi/4$ in Filippi's et al. solution, however, yields through Eqns.(17) an $r_o=0.04167$ cm which is very close to $\alpha^2=0.04$ cm provided by the present solution (Fig. 10b). Finally, it should be mentioned that the characteristic dimensions of the parabolic notch (entering in the formulation of the present solution), i.e., $c=0.96$ cm and $\alpha^2=0.04$ cm, were properly chosen in this paragraph in order to be comparable with Filippi's et al. approach for a rounded V-notch. This choice leads to a notch opening span $2\alpha(c+a^2)^{1/2}=0.8$ cm, which is seen to correspond, at least approximately, to the rounded V-notch opening angle of $\pi/4$ selected by the authors for comparing the two solutions.

Substituting accordingly the above values for the problem parameters, in Eqns.(15, 16) of Filippi et al. [22], and Eqns.(8, 9, 10) of the present solution, the variation of the non-zero normal stresses due to both solutions are plotted in juxtaposition to each other in Fig. 10c, along the bisector of the notches (notice that in ref. [22] σ_θ and σ_r correspond to σ_{xx} and σ_{yy} of the present solution, respectively). Regarding the solution by Filippi et al., the variation of the stresses is plotted in an interval $r_o \leq r \leq r=a$, i.e., $0.04167 \leq r \leq 1.0$ cm. On the other hand, regarding the present solution, the corresponding interval is $-\alpha^2 \leq y \leq y=-d$, i.e., $-0.04 \leq r \leq -1.0$ cm. The distinction between the two solutions in Fig. 10 is achieved by using red color for Filippi's et al. solution and black color for the present solution.

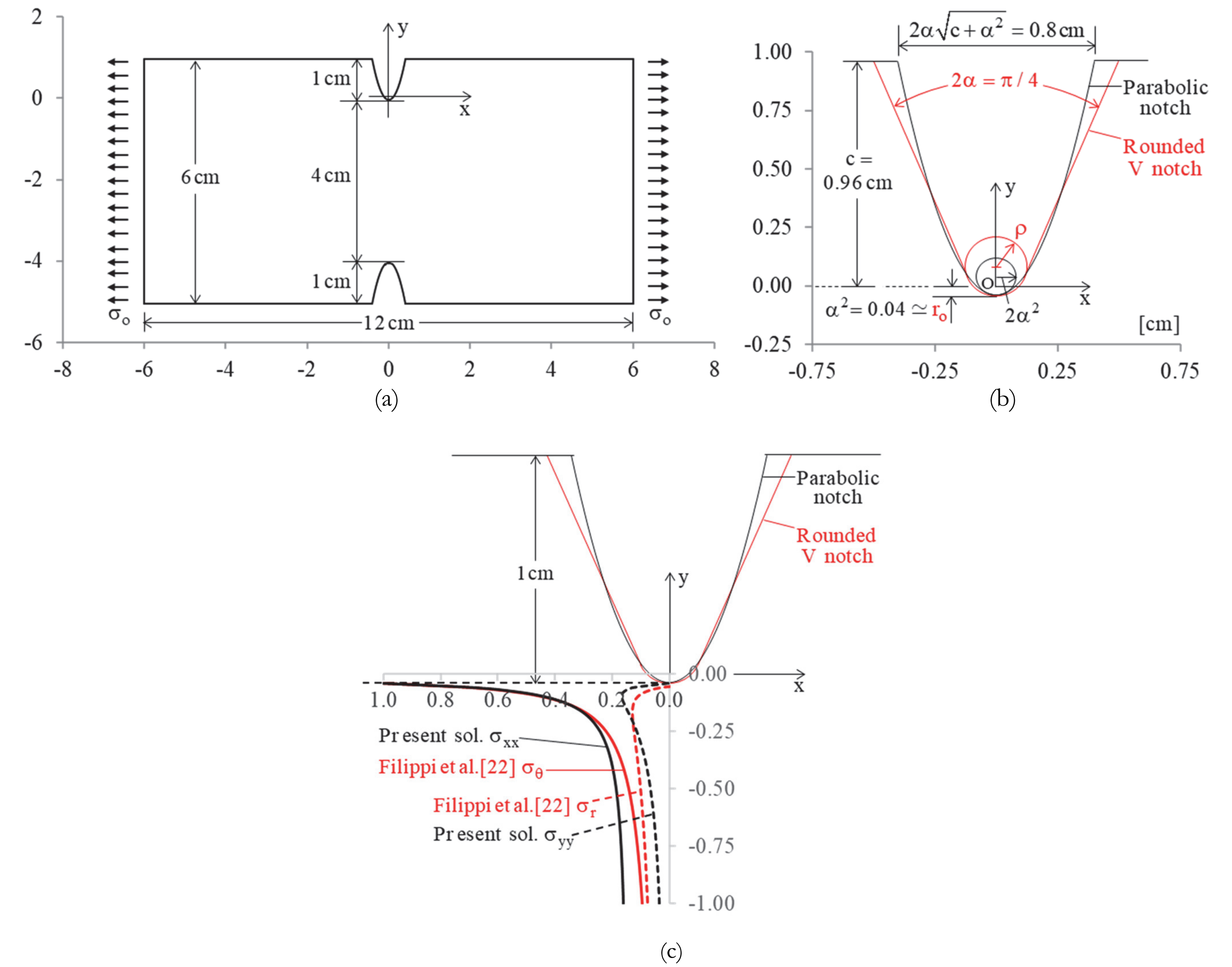

Figure 10: (a) The geometry of the double-notch-strip; (b) Detailed view of the upper notch area: rounded V-notch [22] versus the parabolic notch (present approach); (c) Stresses variation along the notches' dissector in the interval $-0.04 \leq y \leq -1.0$ cm, due to both solutions.

As it is seen from Fig. 10, the two solutions, despite some inevitable differences in the notch geometry, are in quite good qualitative agreement. especially for the respective stress components σ_{xx} and σ_θ in the immediate vicinity of the base of the notch. The more pronounced differences, observed as one is moving away from the base of the notch, may be well attributed to the fact that the present solution is based on the "shallow"-notches assumption, which for the present geometry ceases to be the case, leading to mutual interaction of the two notches, amplifying thus the stress field along the bisector of the notches. In any case, the actual stress state for the configuration of two mutually interacting notches needs a different, much more cumbersome approach, which is beyond the scope of the present study.

DISCUSSION AND CONCLUSIONS

The study presented here is the third part of a short series of papers, dealing with some problems characterized by increased practical engineering interest, which are traditionally confronted in the frame of Linear Elastic Fracture Mechanics. The main feature of these problems is that some critical issues are still open, in spite of the intensive research effort devoted worldwide for their solution. The present Part III, deals with the stretching of a finite strip that is weakened by two symmetric edge notches, and completes Part II, in which the single edge notched strip [7] was discussed. The configuration adopted in this study is more beneficial compared to that confronted in Part II, since it involves symmetric samples, thus, facilitating experimental and numerical protocols dealing with the stress- and displacement-fields around geometrical discontinuities under mode I loading schemes.

In this context, the stress field was obtained here for the stretched double-edge notched finite strip, everywhere on the strip and, also, along the periphery of the notches. Assuming that the material of the strip is isotropic and linearly elastic and that the depth of the notches is relatively small, Muskhelishvili's complex potentials technique [2] was adopted to achieve the solution of the problem in combination with a procedure for "stress neutralization" of specific areas of the loaded strip. The edge parabolic notches of the strip were assumed to approximate rounded V-shaped notches, of closely similar radius of curvature at the base of the parabolas.

At this point it is necessary to discuss some concerns about the analytical solutions for the stress field developed in the immediate vicinity of the base (crown or tip) of a notch. It is generally accepted that the actual geometry of a V-shaped notch is not identical to the theoretical geometry used for analytical schemes employing complex analysis for the solution of the problem. What is, however, of importance is to guarantee that both geometries are characterized by nearly similar radii of curvatures of the tips of the notches. Moreover, for the case of strips of finite dimensions, it is of utmost importance to guarantee that the radius of the tip of the notch is relatively small when compared to the depth (length) of the notch itself. In addition, Filippi et al. [22] emphasized clearly that for the solutions to be sound, the two sides of the notch should be "… *represented by straight lines*". Equally important for obtaining sound analytic solutions is the role of the opening angle, 2α, of the notches. Atzori et al. [26] quantified a zone, in which the stress field in the immediate vicinity of the tip of the notch depends primarily on the radius of the tip of the notch, as being equal to about 0.4 q "*…for plates weakened by semicircular, semi-elliptic and V-shaped notches, with an opening angle ranging from 0 to 135, all subjected to mode I load conditions*" [26], where q is equal to $q=(2\pi-2\alpha)/\pi$ [22, 26] (Eqn. (17)).

In the frame of the above comments, the main assumption adopted in the present study, i.e., that of "shallow" notches sounds definitely reasonable. This assumption simplified significantly the algebraic manipulations providing a quicker and simpler solution, with regard to the case of "deep" mutually interacting notches. The second assumption adopted, i.e., that of considering the notches as "equivalent" parabolas, the radius of curvature of which at their base approximates the radius of the respective rounded V-shaped notches (Fig. 2b), is found to depict in a quite satisfactory manner the configuration of a V-shaped notch with rounded tip, especially for small opening angles, for which the two geometries appear to be almost identical (Fig. 9b).

In this context, taking advantage of the short-notches assumption, the present solution is easily obtained with the aid of the solution presented in Part II [7], and the superposition principle together with the "stress-neutralization" concept. The efficiency of the solution obtained was here assessed by checking the fulfilment of the stress boundary conditions of the problem. As it is seen, even in the case of finite domains, the distribution of the stresses, both along the sides of the strip, and, also, along the periphery of the notches (as they were provided by the present solution) approximate quite satisfactorily the boundary conditions imposed. Moreover, the solution obtained was considered in juxtaposition to the respective one by Filippi et al. [22] and their outcomes were found in very good qualitative and quantitative agreement, in spite of the quite different approaches followed and the different assumptions adopted.

As a next step, and based on the present solution, simple expressions were obtained for both the critical tensile stress at the base (crown or tip) of the notches, and, also, for the respective stress concentration factor k. In addition, when the notches become "mathematical" edge cracks (i.e., discontinuities of zero distance between their lips), a simple formula was, also, provided for the respective stress intensity factor K_I. The formulae for k and K_I can be easily applied by engineers in a wide range of geometries with edge discontinuities varying from almost semi-circular cavities to "mathematical" cracks, provided the length/depth of the notches remains short so as they do not affect each other.

As a concluding remark, it should be mentioned that the procedure of introducing geometric discontinuities by means of "stress neutralization" of the respective regions of the body under study, may provide, also, the general solution for more complicated problems, as it is, for example, the configuration of a finite strip with mutually interacting edge notches, though in that case the solutions presented in Parts II and III need significant modifications.

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