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OPTIMUM CHEBYSHEV LOWPASS FILTER WITH A PAIR OF IMAGINARY AXIS ZEROS

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Abstract. *The paper compares the characteristics of optimum Chebyshev filters with a finite transmission zero pair of arbitrary multiplicity to those of optimum Chebyshev allpole filters. By introducing a transmission zero pair (single or multiple) at a real frequency into the transcendental form of the Chebyshev polynomial, the filter achieves a specified minimum attenuation extreme value in the stopband, thereby improving the cutoff slope. Additionally, the paper presents a new method for deriving a rational polynomial form of the optimum Chebyshev filtering function from its transcendental form, which is essential for determining the poles of the filter's transfer function. The method is straightforward and does not rely on optimization or recursive formulas. The proposed approach is validated and illustrated using an example. Although this approximation is primarily intended for microwave filter applications, it can also be applied to both analog and digital signal processing.*

Key words: *Optimum Chebyshev filters, finite transmission zeros, multiple zeros, half-power bandwidth.*

1. Introduction

In recent publications [1–4], the authors proposed an optimum Chebyshev (C) approximation to design all-pole lowpass filters. Optimum C filters are a variant of C filters that offer an optimum solution for both analog and digital signal processing, where

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the magnitude response of the transfer function is of primary interest. Key features of this approximation include minimum return loss within the half-power bandwidth and maximum out-of-band rejection. In other words, the extremal properties of the Chebyshev approximation are preserved. Since the characteristic function of the lowpass filter[1] represents the ratio of reflected power to transmitted power, the ripple factor minimizes the area below the characteristic function within the half-power passband. The value of the optimum ripple factor, ε_o, is unique for a given filter degree, whether even or odd, and the transfer functions of the optimum C filters can be systematically cataloged [1]. *No all-pole transfer function offers a lower (or equal) return loss with a steeper cutoff slope.*

The small value of ε_o leads to reduced out-of-band attenuation. For example, the ripple factor of the optimum 7th degree C all-pole filter is only $\varepsilon_o = 0.0935$ and decreases as the filter degree increases. Therefore, it is necessary to improve out-of-band attenuation while preserving the passband ripple at ε_o. One or more pairs of poles can be added to its characteristic function to enhance the performance of the stopband. These poles may be distinct or coincident. There are two approaches to incorporating poles into the optimum C characteristic function, depending on whether it is expressed in polynomial or transcendental form.

In the first approach [5–8], a pair of transmission zeros of multiplicity m at the real frequency $\omega = \pm j\omega_0$ is introduced into the characteristic function of the optimum C filter, modifying it as follows: $\Psi_n(\omega) = \varepsilon_o^2 C_n^2(\omega)(\omega_0^2 - 1)^{2m}/(\omega^2 - \omega_0^2)^{2m}$. This modification preserves the filter degree n and the ripple bandwidth $\omega_r = 1$. To determine the pole pair $\pm\omega_0$ of the characteristic function, which also serves as a transmission zero pair, two nonlinear equations must be solved while ensuring the minimum stopband attenuation is satisfied. As a result, the passband ripples are no longer equal but increase toward the passband edge. This approach to improving stopband performance comes at the cost of distorting the extremal properties of the Chebyshev polynomial.

The second approach involves the approximation of generalized C filters [9–13], which exhibit equiripple characteristics in both the passband and stopband. In this case, the filtering function of degree n takes the form of a generalized C filtering function: $C_n(\omega) = \cosh\{\sum_{i=1}^{n} \operatorname{acosh}[(\omega\,\omega_i - 1)/(\omega_i - \omega)]\}$, where ω_i represents the position of the ith transmission zero [14]. Some transmission zeros may coincide at the same location ($\omega_i = \omega_0$), while others may be at infinity. An optimization process is required to determine the number and positions of the transmission zeros. Then, a recursive technique is employed to generate the generalized C transfer and reflection rational polynomials given to the determined driving point impedance of a doubly terminated LC ladder network. Ultimately, this approach results in a lossless coupling matrix resonator network that functions as a microwave bandpass filter [14]. It can be concluded that generalized Chebyshev filters preserve the extremal properties of C polynomials and are primarily designed for microwave filter applications, particularly in coupling matrix resonator networks. Consequently, this approximation is unsuitable for improving the out-of-band attenuation of optimum C filters, as their application is not limited to microwave filters but extends to all types of filters where the magnitude

[1]The characteristic function of the Chebyshev filter is $\Psi_n(\omega^2) = \frac{P_r}{P_L} = \varepsilon^3 C_n^2(\omega)$, where P_r is reflected power and P_L is transmitted power.

response is of primary importance.

The primary objective of this paper is to enhance the out-of-band rejection of an optimum C all-pole filter by introducing a pair of transmission zeros (single or multiple) in its transcendental form. The attenuation level of the out-of-band lobe is determined by the position of the transmission zero, which is computed using the Newton-Raphson method to solve a system of nonlinear equations. This approach ensures that the extremal properties of the Chebyshev filtering function are preserved. Explicit rational polynomial expressions are derived for the transcendental C function of arbitrary degree, incorporating a pair of transmission zeros with arbitrary multiplicity. The proposed mathematical framework is presented in detail. A multiple transmission zero pair is used because it allows the odd-degree LC ladder network to be both symmetrical and reciprocal.

The validity and efficiency of the proposed method are demonstrated by approximating a seventh-degree rational optimum C filter with a single pair, double pair, and triple pair of real transmission zeros. The properties of these filters are tabulated, and a comparative analysis is provided for all optimum seventh-degree C filters, including the all-pole filter.

2. Rational Optimum C Filters Approximation

The squared magnitude of the transmission coefficient of the optimum C filter, representing the ratio of the transmitted power, P_L, to the power available from the source, P_a, is given by

$$|S_{21}(j\omega)|^2 = \frac{P_L}{P_a} = \frac{1}{1 + \left[\varepsilon_o C_n\left(\frac{\omega}{\omega_r}\right)\right]^2}, \tag{1}$$

where $C_n(\omega/\omega_r) = \cosh[n \operatorname{acosh}(\omega/\omega_r)]$ represents the transcendental form of the C filtering function. When multiplied by ε_o, this function becomes the optimum all-pole C filtering function. Here, ε_o is the ripple factor derived for the optimum C filters [1], while ω_r is the ripple bandwidth, which normalizes the argument of the characteristic function to make it dimensionless.

2.1. Filtering function derivation

The magnitude correction of the transmission coefficient of the optimum C filter to enhance out-of-band rejection begins by introducing a symmetric pole pair of multiplicity m into the transcendental form of the nth degree C filtering function (1), as follows:

$$\begin{aligned} C_{n,m}\left(\frac{\omega}{\omega_r}\right) = \cosh\Big\{ (n-2m)\operatorname{acosh}\left(\frac{\omega}{\omega_r}\right) + m \operatorname{acosh}\left[\alpha\left(\frac{\omega}{\omega_r}\right)\right] \\ + m \operatorname{acosh}\left[\bar{\alpha}\left(\frac{\omega}{\omega_r}\right)\right]\Big\} \end{aligned} \tag{2}$$

where

$$\alpha\left(\frac{\omega}{\omega_r}\right) = \frac{\frac{\omega}{\omega_r}\chi - 1}{\chi - \frac{\omega}{\omega_r}} \quad \text{and} \quad \bar{\alpha}\left(\frac{\omega}{\omega_r}\right) = \frac{\frac{\omega}{\omega_r}\chi + 1}{\chi + \frac{\omega}{\omega_r}}$$

and χ is a dimensionless coefficient. This normalization ensures that the arguments of both hyperbolic functions are also dimensionless. In particular, if χ approaches infinity or if $m = 0$, then (2) reduces to the filtering function C of an all-pole filter. Since ε_o is associated with the optimum C filter, the function $\varepsilon_o C_{n,m}(\omega)$ represents the filtering function of the rational optimum C filter. Without loss of generality, ω_r can be set to one, ensuring that $C_{n,m}(1) = 1$.

The multiple pole pair of the C filtering function (2) is located at an out-of-band frequency, producing a lobe. The frequency response shows ω_m, where the lobe reaches a local minimum attenuation. By setting the first derivative of (2) to zero, the frequency ω_m can be determined in closed form as:

$$\omega_m = \pm\sqrt{\chi^2 + \frac{2m\chi\sqrt{\chi^2-1}}{n-2m}}, \tag{3}$$

where $\omega_r = 1$. The coefficient χ serves as a degree of freedom to adjust the minimum out-of-band insertion loss level (IL_s in dB). Its value is obtained by solving the equation $\varepsilon_o C_{n,m}(\omega_m) = A_{min}$, where $A_{min} = \sqrt{10^{IL_s/10} - 1}$. The process of determining χ such that the filtering function (2) produces A_{min}/ε_o at frequency ω_m is iterative [15]. Given specified values of n and m, the two unknowns, χ and ω_m, may be determined numerically by solving two nonlinear equations. These nonlinear equations can be expressed as:

$$\begin{aligned} f_1(\chi, \omega_m) =& \chi^2 + \frac{2m\chi\sqrt{\chi^2-1}}{n-2m} - \omega_m^2 = 0 \\ f_2(\chi, \omega_m) =& C_{n,m}(\omega_m) + \frac{A_{min}}{\varepsilon_o} = 0 \end{aligned} \tag{4}$$

The Newton-Raphson iterative formula can be applied to solve this system of nonlinear equations. In kth iteration, the solution is given by

$$\begin{bmatrix} \chi^{(k+1)} \\ \omega_m^{(k+1)} \end{bmatrix} = \begin{bmatrix} \chi^{(k)} \\ \omega_m^{(k)} \end{bmatrix} - \begin{bmatrix} \dfrac{\partial f_1(\chi^{(k)}, \omega_m^{(k)})}{\partial \chi} & \dfrac{\partial f_1(\chi^{(k)}, \omega_m^{(k)})}{\partial \omega_m} \\ \dfrac{\partial f_2(\chi^{(k)}, \omega_m^{(k)})}{\partial \chi} & \dfrac{\partial f_2(\chi^{(k)}, \omega_m^{(k)})}{\partial \omega_m} \end{bmatrix}^{-1} \times \begin{bmatrix} f_1(\chi^{(k)}, \omega_m^{(k)}) \\ f_2(\chi^{(k)}, \omega_m^{(k)}) \end{bmatrix} \tag{5}$$

The initial value $\chi^{(0)}$ can be any value greater than one, while $\omega_m^{(0)}$ is determined by substituting $\chi^{(0)}$ into $f_1(\chi^{(0)}, \omega_m)$. Note that the Newton-Raphson method is implemented by the Mathematica's built-in function `FindRoot`.

This procedure can be applied to filtering functions with an arbitrary passband ripple. However, if ε_o is used to calculate $A_{\min}$, then $C_{n,m}(\omega)$ is considered the optimum filtering function.

2.2. Rational polynomial derivation

Once χ is determined, the rational polynomial must be derived from the transcendental form of the characteristic function (2) to obtain the transfer function (1). The filtering function can be simplified using the hyperbolic addition formula: $\operatorname{acosh}(\alpha)+\operatorname{acosh}(\bar{\alpha}) = \operatorname{acosh}(\gamma)$, where α and $\bar{\alpha}$ are defined in equation (2), and γ is given by

$$\gamma = \alpha\bar{\alpha} + \sqrt{(\alpha^2-1)(\bar{\alpha}^2-1)}. \tag{6}$$

After simple manipulation, the following expression for $\gamma(\omega)$ can be derived:

$$\gamma(\omega) = \frac{\omega^2\left(2\chi^2-1\right)-\chi^2}{\chi^2-\omega^2}, \tag{7}$$

and (2) can be rewritten in a simplified form as

$$C_{n,m}(\omega) = \cosh\left[(n-2m)\operatorname{acosh}(\omega) + m\operatorname{acosh}(\gamma)\right]. \tag{8}$$

Using the identity $\operatorname{acosh}(x) = \log\left(x+\sqrt{x^2-1}\right)$, the following expression is obtained

$$C_{n,m}(\omega) = \cosh\left[(n-2m)\log\left(\omega+\sqrt{\omega^2-1}\right) + m\log\left(\gamma+\sqrt{\gamma^2-1}\right)\right], \tag{9}$$

or in a more compact form

$$C_{n,m}(\omega) = \cosh\left\{\log\left[\left(\omega+\sqrt{\omega^2-1}\right)^{n-2m} \times \left(\gamma+\sqrt{\gamma^2-1}\right)^{m}\right]\right\}. \tag{10}$$

After `log` and `cosh` operations, we obtain the algebraic form:

$$\begin{aligned} C_{n,m}(\omega) = \frac{1}{2}\Bigg[&\left(\omega+\sqrt{\omega^2-1}\right)^{n-2m}\left(\gamma+\sqrt{\gamma^2-1}\right)^{m} \\ &+\left(\omega-\sqrt{\omega^2-1}\right)^{n-2m}\left(\gamma-\sqrt{\gamma^2-1}\right)^{m}\Bigg]. \end{aligned} \tag{11}$$

Substituting (7) into $\sqrt{\gamma^2-1}$ yields the following expression:

$$\sqrt{\gamma^2-1} = \frac{2\chi\omega\sqrt{\chi^2-1}\sqrt{\omega^2-1}}{\chi^2-\omega^2}. \tag{12}$$

Rewriting $\gamma \pm \sqrt{\gamma^2-1}$ in the form $a \pm b\sqrt{\omega^2-1}$ allows (11) to be expressed as the quotient of two algebraic functions:

$$C_{n,m}(\omega) = \frac{1}{2}\,\frac{\begin{array}{c}\left(\omega+\sqrt{\omega^2-1}\right)^{n-2m}\left(a+b\sqrt{\omega^2-1}\right)^{m} \\ +\left(\omega-\sqrt{\omega^2-1}\right)^{n-2m}\left(a-b\sqrt{\omega^2-1}\right)^{m}\end{array}}{(\chi^2-\omega^2)^m} \tag{13}$$

where $a = \omega^2\left(2\chi^2-1\right)-\chi^2$ is the numerator of (7), and $b = 2\chi\omega\sqrt{\chi^2-1}$ is the numerator of (12) excluding the multiplicative factor of $\sqrt{\omega^2-1}$.

Utilization of the binomial expansion formula gives

$$\left(a \pm b\sqrt{\omega^2-1}\right)^m = A_{2m}(\omega) \pm \frac{B_{2m+1}(\omega)}{\sqrt{\omega^2-1}} \tag{14}$$

where

$$A_{2m}(\omega) = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m}{2k} a^{m-2k}\, b^{2k} (\omega^2 - 1)^k$$

and

$$B_{2m+1}(\omega) = \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \binom{m}{2k+1} a^{m-2k-1}\, b^{2k+1} (\omega^2 - 1)^{k+1}$$

are polynomials of degrees $2m$ and $2m+1$, respectively. For $m = 1$ and $m = 2$, these polynomials are

$$\begin{aligned} A_2(\omega) &= \omega^2 \left(2\chi^2 - 1\right) - \chi^2, \\ B_3(\omega) &= 2\chi\omega^3 \sqrt{\chi^2 - 1} - 2\chi\omega\sqrt{\chi^2 - 1}, \end{aligned} \tag{15}$$

and

$$\begin{aligned} A_4(\omega) &= A_2^2(\omega) + 2\chi\omega\sqrt{\chi^2 - 1} B_3(\omega), \\ B_5(\omega) &= 2A_2(\omega) B_3(\omega). \end{aligned} \tag{16}$$

By substituting $A_{2m}(\omega)$ and $B_{2m+1}(\omega)$ into (13), it can be rearranged as follows:

$$C_{n,m}(\omega) = \frac{1}{2} \frac{\begin{array}{c}\left[(\omega + \sqrt{\omega^2 - 1})^{n-2m} + (\omega - \sqrt{\omega^2 - 1})^{n-2m}\right] A_{2m}(\omega) \\ + \left[(\omega + \sqrt{\omega^2 - 1})^{n-2m} - (\omega - \sqrt{\omega^2 - 1})^{n-2m}\right] \frac{B_{2m+1}(\omega)}{\sqrt{\omega^2 - 1}}\end{array}}{(\chi^2 - \omega^2)^m} \tag{17}$$

In the numerator of (17), two hypergeometric functions can be rewritten regarding two Chebyshev polynomials. The first function is $(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n = 2T_n(x)$, where $T_n(x)$ is the Chebyshev polynomial of the first kind, while the second function is $(x + \sqrt{x^2-1})^n - (x - \sqrt{x^2-1})^n = 2U_n(x)\sqrt{x^2-1}$, where $U_n(x)$ is the Chebyshev polynomial of the second kind. Using these relationships, (17) is finally expressed in the form of the rational optimum C filtering function:

$$\begin{aligned} C_{n,m}(\omega) &= \frac{A_{2m}(\omega) T_{n-2m}(\omega) + B_{2m+1}(\omega) U_{n-2m-1}(\omega)}{\left(\chi^2 - \omega^2\right)^m} \\ &= \frac{N_n(\omega)}{D_{2m}(\omega)} \end{aligned} \tag{18}$$

in which the two terms $\sqrt{\omega^2 - 1}$ in the addend cancel each other out. This filtering function is optimum because ε_o is embedded in the calculation of χ. The polynomial $N_n(\omega)$ is a purely odd or purely even function and satisfies the condition given in [14]. It can be noted that $C_{n,m}(1) = 1$, i.e. $\omega_r = 1$.

Polynomial $N_n(\omega)$ can be obtained using only two auxiliary polynomials, derived from the binomial expansion formula (14), along with two known Chebyshev polynomials of the first and second kinds. The degrees of the purely even polynomial $A_{2m}(\omega)$ and the purely odd polynomial $B_{2m+1}(\omega)$ depend solely on the multiplicity of the zero pair of the transmission coefficient and not on the filter degree n. If the number of zero pairs increases, the filter degree n stays the same, eliminating the need for optimization or recursion formulas. As a result, the mathematical framework is significantly simpler than the well-known solution presented in [14] for generalized Chebyshev filters.

To restore the half-power point ω_c as the edge of the passband of the optimum C filter, it is necessary to determine the frequency at which (2) or (18), multiplied by ε_o, first reaches a value of 1. This requires solving the equation[2] $\varepsilon_o C_{n,m}(\omega/\omega_r) = 1$. The solution is given by $\omega_c/\omega_r = \lambda > 1$, where λ is a dimensionless parameter. The new value of the edge of the ripple band, $\omega_r = \omega_c/\lambda$, can be used to renormalize the filtering function (18) to the half-power bandwidth. For convenience and without loss of generality, the half-power point can be set to one ($\omega_c = 1$), resulting in a new ripple band edge of $\omega_r = 1/\lambda < 1$. This value is then used to renormalize the optimum rational C filtering function.

$$\mathcal{C}^{(\varepsilon)}_{n,m}(\omega) = \varepsilon_o \frac{N_n(\omega)}{D_{2m}(\omega)}\bigg|_{\omega\mapsto\omega\lambda} = \frac{\mathcal{N}_n(\omega)}{\mathcal{D}_{2m}(\omega)} \tag{19}$$

ensuring that $\mathcal{C}^{(\varepsilon)}_{n,m}(1) = 1$. The ripple factor ε_o is an embedded parameter in the rational optimum C approximation, arising from the renormalization process related to the half-power bandwidth. This is denoted by the superscript (ε) in (19).

2.3. **Examples and Comparison**

The frequency responses of the filtering functions of the optimum 7th-degree C filter for $m = 1$, 2 and 3, expressed in the rational polynomial form (18) and scaled by ε_o, are shown in Fig. 1. To illustrate that the ripples are equal, the scale within the ripple band $-1 \le \omega \le 1$ is magnified 1000 times. The ripples reach values of $\pm\varepsilon_o$ at the edges of the ripple band $\omega_r = 1$.

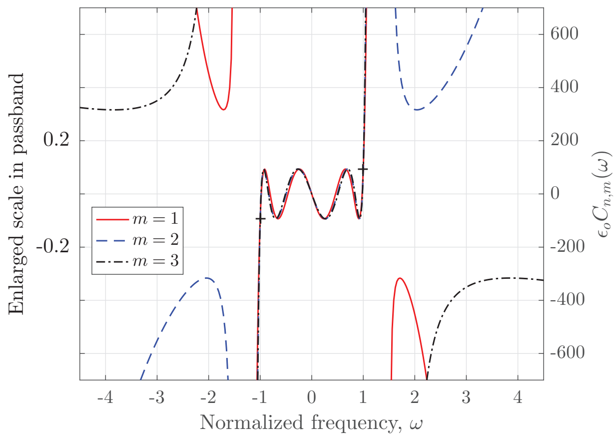

Fig. 1. The 7th-degree optimum C filtering functions, $IL_s = 50$ dB.

The calculation of the rational optimum C filtering function (18), which features a pair of triple poles whose properties are summarized in Table 1, is presented. The

[2]This calculation explicitly states that $\omega_r \neq 1$ can be used for frequency renormalization.

optimum ripple factor is $\varepsilon_o = 0.0935$, while the coefficient $\chi = 1.611042$, is determined to achieve a minimum stopband insertion loss of $IL_s = 50$ dB (or 316.1867 times). The two auxiliary polynomials are given by:

$$A_6(\omega) = 281.86\omega^6 - 473.981\omega^4 + 213.666\omega^2 - 17.484,$$
$$B_7(\omega) = 281.858\omega^7 - 614.885\omega^5 + 415.276\omega^3 - 82.248\omega$$

These polynomials, along with the first-degree Chebyshev polynomial of the first kind and the zero-degree Chebyshev polynomial of the second kind, are used to construct the rational polynomial form of the optimum C filtering function (18), multiplied by ε_o:

$$C_{7,3}(\omega) = 0.0935\frac{-52.712\omega^7 + 101.817\omega^5 - 58.8106\omega^3 + 9.32565\omega}{(\omega^2 - 2.59551)^3} \tag{20}$$

ensuring that $C_{7,3}(1) = 0.00935$.

To achieve a normalized half-power bandwidth of one, the parameter $\lambda = 1.060296$ is calculated for the final renormalization of equation (20) multiplied with the optimum ripple factor. This yields the final rational optimum C filtering function (19):

$$\mathcal{C}_{7,3}^{(\varepsilon)}(\omega) = \frac{\mathcal{N}_n(\omega)}{\mathcal{D}_{2m}(\omega)} = \frac{-79.408\omega^7 + 136.43\omega^5 - 70.098\omega^3 + 9.8872\omega}{(1.1242\omega^2 - 2.59546)^3}$$

ensuring that $\mathcal{C}_{7,3}^{(\varepsilon)}(1) = 1$. For comparison, the properties of other optimum C filtering functions of the 7th-degree are also listed in Table 1. In all approximations, the optimum value of the ripple factor, $\varepsilon_o = 0.0935$, is used to calculate χ, ensuring an insertion loss of $IL_s = 50$ dB at the frequency ω_m.

Table 1. Properties of the 7th-degree optimum C filtering functions

Properties	Characteristic function			
	$m=0$	$m=1$	$m=2$	$m=3$
λ	1.097126	1.077877	1.062136	1.060296
χ	∞	1.500606	1.455552	1.611042
ω_m	none	1.709891	2.042391	3.847784
ω_r	0.911472	0.927748	0.941498	0.943132
Area	0.028670	0.023813	0.019922	0.019531

Table 1 provides a two-part comparison: the first part contrasts the all-pole optimum C characteristic function with its rational counterparts, while the second part compares the rational optimum C functions among themselves. The area below the characteristic function within half power passband[3] (*Area*) is essential for comparison. The first comparison reveals that the all-pole optimum C filtering function has the highest *Area*, i.e. the reflected power, while its ripple band is the smallest. The second comparison shows that as m increases, *Area* decreases, whereas the ripple band expands. Consequently, the area between the ripple band edge frequency ω_r and the half-power frequency $\omega_c = 1$ shrinks.

[3]The area is computed using the integral: $Area = \int_0^1 [\mathcal{C}_{n,m}^{(\varepsilon)}(\omega)]^2 \mathrm{d}\omega$.

3. Transmission Coefficient

The rational squared magnitude function of the transmission coefficient of the optimal C filter is obtained by substituting the all-pole characteristic function with the rational characteristic function—given by equation (19) squared—into equation (1). The transmission coefficient $S_{21}(s)$ can then be determined from (1) using the standard procedure of analytic continuation over the entire s-plane, which corresponds to replacing ω with $-js$ in the given $|S_{21}(j\omega)|^2$. The pole locations of the transmission coefficient can be found by solving for the roots of the polynomial in its denominator[4], given by:

$$\mathcal{D}_{2m}^2(-js) + \mathcal{N}_n^2(-js) = 0. \tag{21}$$

From these roots, the poles $p_i = \sigma_i \pm j\omega_i$, $i = 1,\, 2,\, \ldots, n$, that lie in the left half-plane of s-plane are selected. The pole-zero positions of the 7th-degree transfer functions that correspond to the optimum all-pole Chebyshev and the rational optimum C characteristic functions for $m = 1, 2,$ and 3 are shown in Fig. 2.

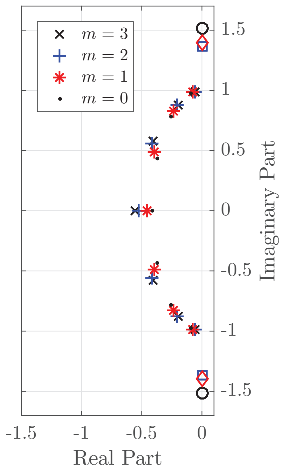

Fig. 2. Pole-zero positions of 7th-degree optimum C filters. The zero locations for $m = 1, 2$ and 3 are represented by $\diamond$, $\square$ and $\circ$, respectively.

The differences in pole positions of all transfer functions are minor. However, the calculation of the critical quality factor, defined as $Q_c = -0.5|p_1|/\sigma_1$, where $p_1 = \sigma_1 \pm j\omega_1$ is the pole pair that is closest to the imaginary axis, is of practical interest. This factor is commonly used to compare the sensitivity of networks and its values are listed in Table 2. The value of Q_c increases with increasing m. However, the difference in Q_c between $m = 2$ and $m = 3$ is minimal, indicating that their passband sensitivities are almost the same.

Fig. 3 compares the steady-state responses, including the insertion loss in dB, given by $IL = 20\log_{10}(|S_{21}(j\omega)|)$, and the group delay responses of the filters, whose

[4] Since $|S_{21}(j\omega)|^2 = \left.\frac{1}{1+\frac{\mathcal{N}_n^2(\omega)}{\mathcal{D}_{2m}^2(\omega)}}\right|_{\omega=-js} = \frac{\mathcal{D}_{2m}^2(-js)}{\mathcal{D}_{2m}^2(-js)+\mathcal{N}_n^2(-js)}$, follows (21).

pole-zero positions are shown in Fig. 2. All filters share the same half-power bandwidth (normalized to 1) and exhibit nearly identical passband characteristics. Table 2 provides the ripple band, $\omega_r = 1/\lambda$, and transmission zero, $\omega_0 = \chi/\lambda$, for each filter. The main difference lies in their stopband behavior.

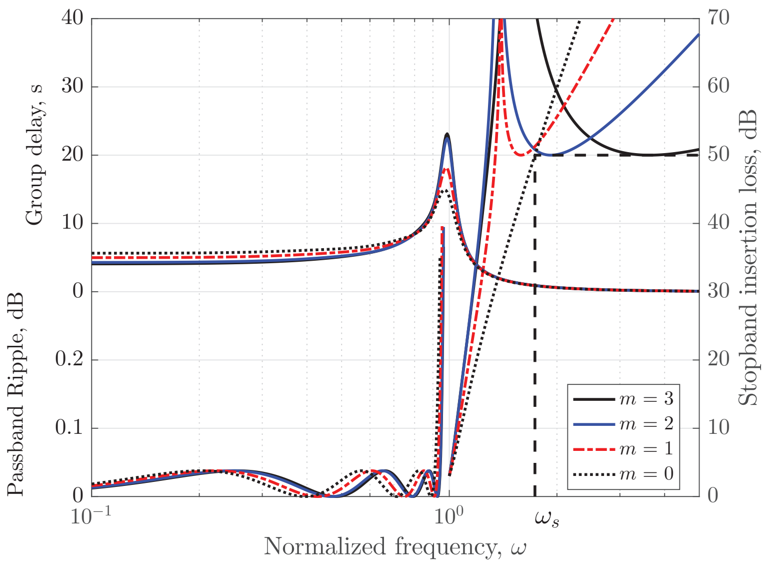

Fig. 3. The steady-state responses of the 7th-degree optimum C lowpass filters for $m = 0$, 1, 2, and 3.

The stopband edge frequencies, ω_s, are defined as the frequencies where the out-of-band attenuation first reaches the specified minimum value of $IL_s = 50$ dB. These values are listed in Table 2. In Fig. 3, the stopband edge frequency is explicitly assigned only for $m = 0$, as it is easily identifiable for the other three filters. It may be seen that all rational optimum C filters have one attenuation lobe, each with a level of 50 dB, at frequencies ω'_m, which are listed in Table 2.

The cutoff slope, commonly defined as the first derivative of the transmission coefficient (1) at the half-power point, of rational optimum C filter can be calculated using the first derivative of its characteristic function as

$$CS = \frac{\mathrm{d}}{\mathrm{d}\omega}|S_{21}(j\omega)|\bigg|_{\omega=1} = -\frac{1}{2\sqrt{2}}\,\frac{\mathrm{d}}{\mathrm{d}\omega}\mathcal{C}^{(\varepsilon)}_{n,m}(\omega)\bigg|_{\omega=1} \tag{22}$$

since $\mathcal{C}^{(\varepsilon)}_{n,m}(1) = 1$. The cutoff slopes of all considered 7th-degree filters are provided in Table 2. Increasing the multiplicity of the transmission zero results in a lower stopband edge frequency and steeper cutoff slope. However, increasing the transmission zero multiplicity beyond two is unnecessary, as the improvement becomes negligible. Notably, the magnitude response of the filter with a triple zero pair exhibits a broad stopband region with extremely high attenuation.

All group delay responses increase monotonically in the passband. The differences become evident only in the peak value near the passband edge, which increases as m

increases. For $m = 2$ and $m = 3$, the group delays are nearly identical. The peak group delay values, τ_m, are calculated and listed in Table 2.

Another important parameter to consider is the reflection coefficient

$$|S_{11}(j\omega)|^2 = \frac{\left[\mathcal{C}_{n,m}^{(\varepsilon)}(\omega)\right]^2}{1 + \left[\mathcal{C}_{n,m}^{(\varepsilon)}(\omega)\right]^2} \tag{23}$$

The return loss frequency responses in decibels, $RL = 20\log_{10}(|S_{11}(j\omega)|)$, of the 7th-degree optimum C filters with for $m = 0$, 1, 2 and 3 are shown in Fig. 4. The return loss levels of all filters are determined with ε_o and exhibit the equiripple behavior. The differences in the spacing of reflection zeros among 7th-degree optimum C filters are sufficient for functional tuning. However, the optimum all-pole C filter has the widest separation, while for the rational optimum C filters, this separation slightly decreases as m increases. Additionally, as the filter degree increases, the zero separation decreases.

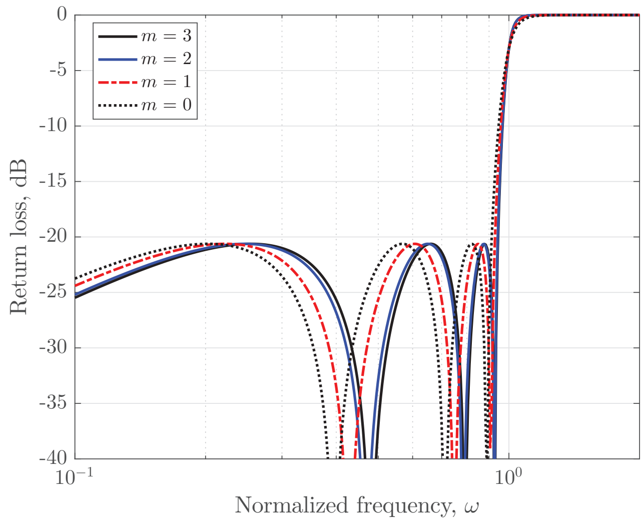

Fig. 4. The return loss responses of the 7th-degree optimum C lowpass filters for $m = 0$, 1, 2, and 3.

Table 2. Properties of 7th-degree optimum C filters

Properties	Filters			
	$m = 0$	$m = 1$	$m = 2$	$m = 3$
$\omega_0 = \chi/\lambda$	∞	1.392187	1.370401	1.519427
$\omega'_m = \omega_m/\lambda$	none	1.586262	1.922885	3.628922
$\omega_r = 1/\lambda$	0.911472	0.927748	0.941498	0.943132
Q_c	5.3421	6.6824	8.3906	8.6179
ω_s	1.7358	1.3457	1.2674	1.2892
CS	16.94209	21.55732	27.22896	27.85117
τ_m	14.9337	18.1713	22.4434	23.1394

The magnitude correction of the transmission coefficient can be applied to both odd- and even-degree filters. Introducing a single transmission zero significantly enhances the out-of-band rejection of the all-pole filter by steepening the cutoff slope and lowering the stopband edge frequency. While increasing the multiplicity of the transmission zero provides a slight improvement in the cutoff slope, extending it beyond two is unnecessary, as it leads to higher sensitivity and increased network complexity. However, a triple-zero pair results in a magnitude response with a wide stopband region and extremely high attenuation in the frequency band around the transmission zero.

4. Conclusion

This paper presents an efficient and straightforward procedure for correcting the magnitude function in optimal C filters, specifically designed for low-pass filters that prioritize magnitude specifications. The Chebyshev filtering function is modified by introducing a symmetric pair of poles of arbitrary multiplicity into its transcendental form, preserving the extremal properties of the passband's equal-ripple behavior.

A new and efficient method is proposed for deriving a rational polynomial from the transcendental form of the optimum C filtering function that exhibits a transmission zero pair, single or multiple, at a finite frequency. As an example, the derivation of a 7th-degree rational polynomial from the transcendental optimum Chebyshev filtering function is demonstrated for cases of single, double, and triple pole pairs.

The proposed approach enhances the cutoff slope and slightly reduces reflection power at the filter's input terminals compared to the optimum C all-pole filter. However, a single or double transmission pole pair is generally preferred for filter design, as increasing the multiplicity provides minimal improvements in the cutoff slope and return loss while increasing hardware complexity, sensitivity, and tuning time.

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