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The Radical of an Endo-Restricted Bounded Submodule Related to Prime Submodules

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Abstract

In this paper, we present the concept of the radical of an Endo-Restricted Bounded submodule and establish its characterization, which is regarded as a new notion. In addition, we study the relationship between the radical of submodules and the radical of an Endo-Restricted Bounded submodules and this connection will give us important results in terms of their radical. Furthermore, this article will demonstrate numerous properties and corollaries that elucidate the concept of the Endo-Restricted Bounded submodule's radical. This work includes a new class of T-module as well as a T-submodule called the Endo-Restricted Bounded Module (submodule), written briefly as the Endo-R.B. module (submodule), provided with some examples that illustrate and clarify in a nice way this type of module (submodule). However, our focus will be on the radicals of the endo-R.B. submodule. Prime submodule and scalar module both play a crucial role in many properties that show the relationship between prime submodules and Endo-R.B submodules. Furthermore, some generalizations of prime submodules, such as S-prime submodules, are involved in this research.

Keywords: Endo-R.B Radical submodule, Bounded module, S-Prime submodule, Scalar module.

1. Introduction

The ring in this paper is commutative with identity denoted by T and Ω is a unitary left-T-module. Motivated by the notion of bounded module, where a T-module Ω is called bounded if there exists $x \in \Omega$ such that $ann_T(x) = ann_T(\Omega)$ (1–3). A T-submodule A is called bounded if there exists an element $a \in A$ such that $ann_T(a) = ann_T(A)$ (3). We introduced a new concept of module (submodule) namely an Endo-Restricted Bounded submodule (module) (Endo-R.B submodule (module)) and then we turned to the main purpose, which is the radical of an Endo-R.B submodule that will be defined later in this work with some important properties.

An Endo-R.B submodule is a new type of T-submodule that has not been recognized previously by other authors, where A proper submodule A of a T-module Ω is called Endo-R.B whenever $\varphi(x) \in A$, $\varphi \in End(\Omega)$, $x \in \Omega$ implies that $ann_T(\varphi(x)) = ann_T(A)$. We say that Ω is an Endo-R.B T-module if every proper submodule is an Endo-R.B. Also, prime submodules play a crucial role in order to recall a radical of a submodule and its connection with the radical of an Endo-R.B submodule, where a submodule A of a T-module Ω is said to be prime whenever $rx \in A, x \in \Omega, r \in T$ implies that either $r \in [A :_R \Omega]$ or $x \in A$ (4–7). In addition, we found that if A is an Endo-R.B, then it is not necessary that A be a prime, and the following example shows that:

Let $\Omega = Z_4 \oplus Z_4$ as a $Z-$module and $A = \langle \bar{2} \rangle \oplus \langle \bar{0} \rangle$. Define $\varphi : \Omega \longrightarrow \Omega$ such that $\varphi(\bar{a}, \bar{b}) = (2\bar{a}, \bar{0}), \forall (\bar{a}, \bar{b}) \in \Omega$, then A is an Endo-R.B. However, A is not a prime submodule since if $2(\bar{1}, \bar{2}) \in A$ where $2 \in Z$, $(\bar{1}, \bar{2}) \in \Omega$, but neither $(\bar{1}, \bar{2}) \in A$ nor $2 \in [A :_T \Omega]$ because $2\Omega \nsubseteq A$. Also, we found another example that shows if a submodule A is prime ,then it is not necessary to be an Endo-R.B. Assume that $\Omega = Z_4 \oplus Z_4$ as a Z-module and $A = Z_4 \oplus \langle \bar{0} \rangle$. Define φ as the same way in the previous example.

The radical of a T-submodule denoted by $rad(N)$ and it is the intersection of all prime submodules of Ω that contain N.

In other word, $rad(N) = \{K \mid K \leq \Omega \text{ such that } N \subseteq K\}$(8–11).

In this paper, we prove many properties provided with some crucial conditions, and we will see the relationship between $rad(N)$ and the radical of Endo-R.B submodules.

2. Endo-R.B Submodules and Modules

In this section, we give a brief introduction about Endo-R.B submodules as well as its modules with some examples. We refer to a proper submodule by the symbol $<$ and $End(\Omega)$ is the set of all endomorphisms of a T-module Ω.

2.1.Definition: A proper T-submodule A of Ω is said to be Endo-R.B if there exists an endomorphism of Ω ($\varphi \in End(\Omega)$) and $\varphi(x) \in A$ for some $x \in \Omega$ such that $ann_T(\varphi(x)) = ann_T(A)$.

We need just one an endomorphism φ of Ω with an element x belonging to a T-module Ω defined in some way that satisfies two conditions, $\varphi(x) \in A$ and $ann_T(\varphi(x)) = ann_T(A)$.

Examples

1) Suppose that $\Omega = Z_3 \oplus Z, T = Z$ and let $A = \langle \bar{0} \rangle \oplus 3Z$. Then we can find $\varphi: Z_3 \oplus Z \longrightarrow Z_3 \oplus Z$ defined by $\varphi(\bar{a}, b) = (\bar{0}, b)$. It is clear that φ is an endomorphism. Now, let $x = (\bar{1}, 3) \in. \Omega$ Then $\varphi(\bar{1}, 3) = (\bar{0}, 3) \in A$, and hence we see that $ann_T(\bar{0}, 3) = ann_T(A) = \langle \bar{0} \rangle$. We conclude that a submodule A is an Endo-R.B submodule of Ω.

2) Assume $\Omega = Z_4$ as a Z-module and $A = \langle \bar{2} \rangle$. Define $\varphi : Z_4 \longrightarrow Z_4$ as $\varphi(\bar{a}) = \bar{0}, \forall \bar{a} \in Z_4$ and φ is an endomorphism. Then $\varphi(\bar{a}) \in A$, but $ann_Z(A) = ann_Z\langle \bar{2} \rangle = 2Z$ is not equal to $ann_Z(\varphi(\bar{3})) = ann_Z(\bar{0}) = Z$. Therefore, A is not the Endo-R.B submodule.

Now, we are ready to give the definition of the Endo-R.B module with some examples that explain the structure of the definition.

2.2. Definition A T-module Ω is called Endo-R.B module if every proper submodule of Ω is an Endo-R.B submodule.

Examples

(1) Z_P as a Z-module is Endo-R.B where P is a prime number since $(\bar{0})$ is the only proper submodule of Z_P. To show that, take an endomorphism $\varphi \in End(\Omega)$ as $\varphi: Z_3 \longrightarrow Z_3$ such that $\varphi(\bar{a}) = 3\bar{a}, \forall \bar{a} \in Z_3$ then $\varphi(\bar{a}) \in (\bar{0})$ since $3(\bar{0}) = 0, 3(\bar{1}) = 0, 3(\bar{2}) = 0$ and $ann_T(\langle \bar{0} \rangle) = ann_T(\varphi(\bar{a})) = Z$

(2) Let $\Omega = Z_4 \oplus Z_2$ as a Z-module. Define $\varphi: \Omega \longrightarrow \Omega$ as $\varphi(\bar{a}, \bar{b}) = (\bar{a}, 0), \forall (\bar{a}, \bar{b}) \in \Omega$. Then, if we take $A = Z_4 \oplus (\bar{0})$ as a submodule of Ω, we get that $\varphi(\bar{a}, \bar{b}) = (\bar{a}, 0) \in A$. Thus, we conclude that A is not an Endo-R.B Z-submodule because if $x = (\bar{2}, \bar{0})$, then $4Z = ann_Z(A) \neq ann_Z(\bar{2}, \bar{0}) = 2Z$ and, hence, Ω is not an Endo-R.B Z-module.

2.3. Remark

(1) Every Endo-R.B.T-module is bounded, but the converse is not true. To show that, let $\Omega = Z_2 \oplus Z_2$ as a Z-module and let $A = Z_2 \oplus (\bar{0})$ be a submodule of Ω. Define $\psi: \Omega \longrightarrow \Omega$ where $\psi \in End(\Omega)$ by $\psi(\bar{a}, \bar{b}) = (\bar{0}, \bar{0})$. Since $\psi(\bar{a}, \bar{b}) = (\bar{0}, \bar{0}) \in A$, but $2Z = ann_Z(A) \neq ann_Z(\bar{0}, \bar{0}) = Z$. Therefore, Ω is not an Endo-R.B, while Ω is a bounded T-module since there exists an element $x = (\bar{0}, \bar{1}) \in Z_2 \oplus Z_2$ such that $2Z = ann_Z(\Omega) = ann_Z(\bar{0}, \bar{1}) = 2Z$.

(2) Every proper submodule of the Endo-R.B module is also an Endo-R.B.

(3) The intersection of two Endo-R,B submodules of Ω is an Endo-R.B since if $A_1, A_2 \leq \Omega$ and A_1, A_2 are two Endo-R.B submodules. Then, $A_1 \cap A_2 \subseteq A_1$ *and* $A_1 \cap A_2 \subseteq A_2$, which implies that $A_1 \cap A_2$ is an Endo-R.B submodule.

2.4. Proposition Let Ω be a T-module and $N < \Omega$. Then

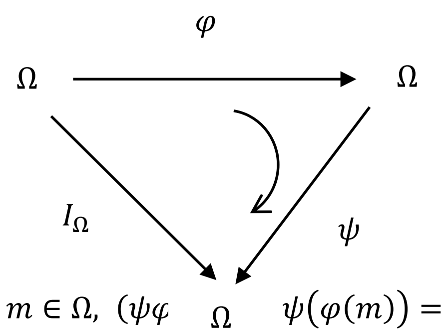
(1) If N is an Endo-R.B submodule of Ω and an epimorphism $\varphi \in End(\Omega)$, then $\varphi(N)$ is an Endo-R.B submodule of Ω.

(2) If N is an Endo-R.B submodule of Ω, then $\varphi^{-1}(N)$ is an Endo-R.B submodule of Ω.

Proof.

(1) Define $\varphi: \Omega \longrightarrow \Omega$ as $\varphi(m) = m, \ \forall m \in \Omega$ and suppose $N < \Omega$, then $\varphi(m) \in N$ because N is an Endo-R.B submodule of Ω.

It is clear that $\varphi(N) < \Omega$; we can define $\psi: \Omega \longrightarrow \Omega$ as $\psi(\varphi(m)) = m, \forall m \in \Omega$. Then, we have the following commutative diagram:



$\psi \circ \varphi = I_\Omega$

Since $\forall m \in \Omega$, $(\psi\varphi$ $\psi(\varphi(m)) = m$, it is obvious that $\psi(\varphi(m)) \in \varphi(N)$. $\psi(\varphi(m)) = \varphi(m) = m \in \Omega = \varphi(N)$ because φ is an epimorphism. Also, it is clear that $ann_T(\varphi(N)) \subseteq ann_T(\psi(\varphi(m))$.

Another containment, let $t \in ann_T(\psi(\varphi((m))$, then $t\psi(\varphi(m)) = 0$ implies that $tI_\Omega(m) = 0$ for all $m \in \Omega$, then $t\Omega = 0$ and since φ is an epimorphism $(\Omega = \varphi(N))$, we obtain $t\varphi(N) = 0$ and hence, $t \in ann_T(\varphi(N))$.

(2) Suppose that $N < \Omega$, then there exists $\varphi: \Omega \longrightarrow \Omega$ defined as:

$\varphi(m) = m, \forall m \in \Omega$ and $\varphi(m) \in N$. Assume $\varphi^{-1}(N) \leq \Omega$ and define:

$$\varphi^{-1}: \Omega \longrightarrow \Omega \text{ as } \varphi^{-1}(\varphi(m)) = m = I_\Omega(m), \forall m \in \Omega.$$

Since I_Ω is an isomorphism, then φ^{-1} is onto and hence, $\varphi^{-1}(N) = \Omega$. From this fact, we get $\varphi^{-1}(\varphi(m)) \in \varphi^{-1}(N)$. It is clear that:

$ann_T(\varphi^{-1}(N)) \subseteq ann_T(\varphi^{-1}(\varphi(m))$

Now, let $t \in ann_T(\varphi^{-1}(\varphi(m)) = ann_T(\varphi(m))$. Thus, $t\varphi^{-1}(m) = 0$ for all $m \in \Omega$, then $t\varphi^{-1}(\Omega) = 0$. Therefore, $t\Omega = 0$ since φ^{-1} is outomor

Then $t \in ann_T(\varphi^{-1}(\varphi(m)) \subseteq ann_T(\varphi^{-1}(N))$.

The next proposition shows that a divisible T-module plays an important role for a cyclic submodule to be an Endo-R.B.

3. The radical of an Endo-R.B T-submodule

In this section, we are ready to focus on the main purpose of this paper and give some definitions and properties that illustrate the notion of the radical of an Endo-R.B submodule.

3.1. Definition

The radical of an Endo-R.B submodule N of a T-module Ω is denoted by Endo-$rad_{\Omega}^{R.B}(N)$ and defined as the intersection of all Endo-R.B submodules of Ω that contains N. If there exists no Endo-R.B submodule of Ω containing N, then we write:

Endo-$rad_{\Omega}^{R.B}(N) = \Omega$.

If Ω =T and N is an ideal of T, then Endo-$rad_{\Omega}^{R.B}(N)$ is the intersection of all Endo-R.B ideals of T containing N.

3.2. Definition

A proper submodule N of an T-module Ω is called an Endo-R.B radical submodule if Endo-$rad_{\Omega}^{R.B}(N) = N$.

3.3. Remark Let N be an Endo-R.B submodule of a T-module Ω, then Endo-$rad_{\Omega}^{R.B}(N)$ is also an Endo-R.B submodule.

Proof.

By the definition of the radical of an Endo-R.B submodule, we have

Endo-$rad_{\Omega}^{R.B}(N)$ =∩ {K| K is an Endo-R.B submodule and $N \subseteq K$}. By using induction and remark (2.5), we conclude that Endo-$rad_{\Omega}^{R.B}(N)$ is an Endo-R.B submodule.

3.4. Proposition Let $f\colon \Omega \longrightarrow \Omega'$ be an epiomorphism and $N < \Omega$ with $\ker f \subseteq N$. Then,

(1) $f\left(\text{Endo} - rad_{\Omega}^{R.B}(N)\right) = \text{Endo} - rad_{\Omega}^{R.B} f(N)$.

(2) $f^{-1}\left(\text{Endo} - rad_{\Omega}^{R.B}(D)\right) = \text{Endo} - rad_{\Omega}^{R.B} f^{-1}(D)$where $D < M'$

Proof.

(1) By the definition of the Endo-R.B radical of submodule, we have

$f\left(\text{Endo} - rad_{\Omega}^{R.B}(N)\right) = f(\cap K)$ where K is an Endo-R.B submodule and $N < K$.

Since $\ker f \subseteq N \subseteq K$, then $f\left(\text{Endo} - rad_{M}^{R.B}(N)\right) = \cap f(K)$, where $f(N) \subseteq f(K)$ and the intersection works over all Endo-R.B submodules $f(K)$ of Ω'.Therefore,

$f\left(\text{Endo} - rad_{\Omega}^{R.B}(N)\right) = \text{Endo} - rad_{\Omega}^{R.B} f(N)$.

(2) Let $D < \Omega'$, then $f\left(\text{Endo} - rad_{\Omega}^{R.B}(D)\right) = \cap K'$, where the intersection is over all Endo-R.B submodules D of Ω' with $D \subseteq K'$. Then

$f^{-1}(\text{Endo} - rad_{\Omega}^{R.B}(D) = f^{-1}(\cap K') = \cap f^{-1}(K')$ where the intersection is over all Endo-R.B submodules $f^{-1}(D)$ of Ω with $f^{-1}(D) \subseteq f^{-1}(K')$.

Hence $f^{-1}\left(\text{Endo} - rad_{\Omega}^{R.B}(D)\right) = \text{Endo} - rad_{\Omega}^{R.B} f^{-1}(D)$

3.5. Proposition Let Ω be an T-module and $N, L < \Omega$. Then, the following statements hold:

(1) $N \subseteq \text{Endo} - rad_{M}^{R.B}(N)$.

(2) If $N \subseteq L$, then Endo-$rad_{\Omega}^{R.B}(N) \subseteq \text{Endo} - rad_{\Omega}^{R.B}(L)$.

(3) Endo-$rad_{\Omega}^{R.B}\left(\text{Endo} - rad_{\Omega}^{R.B}(K)\right) = \text{Endo} - rad_{\Omega}^{R.B}(K)$.

(4) $\text{Endo} - rad_{\Omega}^{R.B}(N \cap L) \subseteq \text{Endo} - rad_{\Omega}^{R.B}(N) \cap \text{Endo} - rad_{\Omega}^{R.B}(L)$.

(5) $\text{Endo} - rad_{\Omega}^{R.B}(N + L) = \text{Endo} - rad_{\Omega}^{R.B}(\text{Endo} - rad_{\Omega}^{R.B}(N) + \text{Endo} - rad_{\Omega}^{R.B}(L))$.

Proof.

(1) By the definition, we have that $\text{Endo}-rad_{\Omega}^{R.B}(N)=\cap K$ where the intersection runs over all Endo-R.B submodules K of Ω with $N\subseteq K$ so that N ⊆ $\text{Endo}-rad_{\Omega}^{R.B}(N)$.

(2) Assume that $N\subseteq L$ and let K be an Endo-R.B submodule of Ω with $L\subseteq K$.

Then $N\subseteq L\subseteq K$ implies that $N\subseteq K$. $\text{Endo}-rad_{\Omega}^{R.B}(N)\subseteq\text{Endo}-rad_{\Omega}^{R.B}(L)$.

(3) Since $\text{Endo}-rad_{\Omega}^{R.B}\left(\text{Endo}-rad_{\Omega}^{R.B}(K)\right)=\cap W$ where the intersection is taken on all over Endo-R.B submodules W of Ω with $\text{Endo}-rad_{\Omega}^{R.B}(K)\subseteq W$ and from no (1), we get that $K\subseteq\text{Endo}-rad_{\Omega}^{R.B}(K)$.

Therefore, $\text{Endo}-rad_{\Omega}^{R.B}\left(\text{Endo}-rad_{\Omega}^{R.B}(K)\right)\subseteq\text{Endo}-rad_{\Omega}^{R.B}(K)$, and by no.(1), we obtain $\text{Endo}-rad_{\Omega}^{R.B}(K)\subseteq\text{Endo}-rad_{\Omega}^{R.B}(\text{Endo}-rad_{\Omega}^{R.B}(K))$.

(4) Let K be an Endo-R.B submodule of Ω containing N and L. Since

$N\cap L\subseteq L\subseteq K$, we have that $\text{Endo}-rad_{\Omega}^{R.B}(N\cap L)\subseteq K$. Thus,

$\text{Endo}-rad_{\Omega}^{R.B}(N\cap L)\subseteq\text{Endo}-rad_{\Omega}^{R.B}(L)$.

Similarly, we have $\text{Endo}-rad_{\Omega}^{R.B}(N\cap L)\subseteq\text{Endo}-rad_{\Omega}^{R.B}(N)$.

Therefore, $\text{Endo}-rad_{\Omega}^{R.B}(N\cap L)\subseteq\text{Endo}-rad_{\Omega}^{R.B}(N)\cap\text{Endo}-rad_{\Omega}^{R.B}(L)$.

(5) Since $N+L\subseteq\text{Endo}-rad_{\Omega}^{R.B}(N)+\text{Endo}-rad_{\Omega}^{R.B}(L)$, then by (2)

$\text{Endo}-rad_{\Omega}^{R.B}(N+L)\subseteq\text{Endo}-rad_{\Omega}^{R.B}(\text{Endo}-rad_{\Omega}^{R.B}(N)+\text{End}-rad_{\Omega}^{R.B}(L))$

Now, let K be an Endo-R.B submodule of Ω containing N+L. Since $N+L\subseteq K$,

then $N\subseteq K$ and $L\subseteq K$. Thus, $\text{Endo}-rad_{\Omega}^{R.B}(N)+\text{Endo}-rad_{\Omega}^{R.B}(L)\subseteq K$. Therefore,

$\text{Endo}-rad_{\Omega}^{R.B}\left(\text{Endo}-rad_{\Omega}^{R.B}(N)+\text{Endo}-rad_{\Omega}^{R.B}(L)\right)\subseteq\text{Endo}-rad_{\Omega}^{R.B}(N+L)$.

Hence, $\text{Endo}-rad_{\Omega}^{R.B}(N+L)=\text{Endo}-rad_{\Omega}^{R.B}(\text{Endo}-rad_{\Omega}^{R.B}(N)+\text{Endo}-rad_{\Omega}^{R.B}(L))$.

Recall a T-module Ω called a multiplication module if for every submodule A of Ω there exists an ideal I of T such that $I\Omega=$ A.(12–14)

Recall a T-module Ω said to be a scalar module if for each $\varphi\in End(\Omega)$ there exists $r\in T$ such that $\varphi(x)=rx,\forall x\in\Omega$ (15–17).

Next, the scalar module plays a crucial role to connect prime and Endo-R.B submodules. We need the next proposition to see this relationship.

3.6. Proposition

Let Ω be a scalar T-module, and A is a prime submodule. Then A is an Endo-R.B.

Proof.

Let $\varphi\in End(\Omega), x\in\Omega$. Since Ω is a scalar module, then for all $\varphi\in End(\Omega)$ there exists $0\neq r\in T$ such that $\varphi(x)=rx,\forall x\in\Omega$. Thus, $\varphi(m)=rm\in A$, $m\in\Omega$. We claim that $ann_T\,\varphi(m)=ann_T(A)$. Let $t\in ann_T\,\varphi(m)$, then

$$t\,\varphi(m)=0$$

$$t(rm)=0$$

$$t\in ann_T(rm)\ \text{for all}\ rm\in A$$

Since A is a prime submodule, then either $m\in A$ or $r\Omega\subseteq A$. We conclude that $t\in ann_T(A)$.

3.7. Proposition

Let Ω be a multiplication finitely generated T-module and $N,L<\Omega$. Then,

$N+L=\Omega$ if and only if $\text{Endo}-rad_{\Omega}^{R.B}(N)+\text{Endo}-rad_{\Omega}^{R.B}(L)=\Omega$.

Proof.

By proposition (3.5), we have that $N\subseteq\text{Endo}-rad_{\Omega}^{R.B}(N)$ and $L\subseteq\text{Endo}-rad_{\Omega}^{R.B}(L)$ so it is obvious that $\text{Endo}-rad_{\Omega}^{R.B}(N)+\text{Endo}-rad_{\Omega}^{R.B}(L)=\Omega$.

Conversely, assume that $\text{Endo}-rad_{\Omega}^{R.B}(N)+\text{Endo}-rad_{\Omega}^{R.B}(L)=\Omega$ and let $N+L\neq\Omega$. Since Ω is finitely generated, then there exists a maximal submodule K of Ω such that $N+L\subseteq K$. K is prime submodule and since Ω is a multiplication T-module, then by (18), Ω is a scalar T-module and hence, by proposition (3.6), we get that K is an Endo-R.B submodule. Therefore,

$\text{Endo}-rad_{M}^{R.B}(N)\subseteq K$ and $\text{Endo}-rad_{\Omega}^{R.B}(L)\subseteq K$.

Thus $\text{Endo}-rad_{\Omega}^{R.B}(N)+\text{Endo}-rad_{\Omega}^{R.B}(L)\subseteq K$ implies that K= Ω which is a contradiction. Hence N+L= Ω.

Recall a submodule N of a –module Ω is said to be completely irreducible if for any two submodules A_1, A_2 of Ω, $A_1\cap A_2\subseteq N$ implies that either $A_1\subseteq N$ or $A_2\subseteq N$ (19,20)

3.8. Proposition

Let Ω be a T-module and $N,L<\Omega$. If every Endo-R.B submodules that contains $N\cap L$ is completely irreducible submodule, then

$\text{Endo}-rad_{\Omega}^{R.B}(N\cap L)=\text{Endo}-rad_{\Omega}^{R.B}(N)\cap\text{Endo}-rad_{\Omega}^{R.B}(L)$.

Proof.

It is clear that

$\text{Endo}-rad_{\Omega}^{R.B}(N\cap L)\subseteq\text{Endo}-rad_{\Omega}^{R.B}(N)\cap\text{Endo}-rad_{\Omega}^{R.B}(L)$. Let K be an Endo-R.B submodule such that $N\cap L\subseteq K$. Since K is completely irreducible, then either $N\subseteq K$ or $L\subseteq K$ implies that

$\text{Endo}-rad_{\Omega}^{R.B}(N)\subseteq K$ or $\text{Endo}-rad_{\Omega}^{R.B}(L)\subseteq K$. Hence

$\text{Endo}-rad_{\Omega}^{R.B}(N)\cap\text{Endo}-rad_{\Omega}^{R.B}(L)\subseteq K$ and this holds for any K and the intersection of all K is an $\text{Endo}-rad_{\Omega}^{R.B}(N\cap L)$. Then either $\text{Endo}-rad_{\Omega}^{R.B}(N)\subseteq\text{Endo}-rad_{\Omega}^{R.B}(N\cap L)$ or $\text{Endo}-rad_{\Omega}^{R.B}(L)\subseteq\text{Endo}-rad_{\Omega}^{R.B}(N\cap L)$. Therefore,

$\text{Endo}-rad_{\Omega}^{R.B}(N)\cap\text{Endo}-rad_{\Omega}^{R.B}(L)\subseteq\text{Endo}-rad_{\Omega}^{R.B}(N\cap L)$. Thus

$\text{Endo}-rad_{\Omega}^{R.B}(N\cap L)=\text{Endo}-rad_{\Omega}^{R.B}(N)\cap\text{Endo}-rad_{\Omega}^{R.B}(L)$.

3.9. Proposition

Let Ω be a scalar T-module and $N<M$. Then $\text{Endo}-rad_{\Omega}^{R.B}(L)\subseteq rad(N)$.

Proof.

Let K be a prime submodule that containing N, then by proposition (3.6), K is an Endo-R.B submodule containing N implies that $\text{Endo}-rad_{\Omega}^{R.B}(N)\subseteq K$. Hence,

$\text{Endo}-rad_{\Omega}^{R.B}(N)\subseteq\cap K$ for all Endo-R.B submodules containing N. Therefore,

$\text{Endo}-rad_{\Omega}^{R.B}(N)\subseteq rad(N)$

3.10. Proposition

Let N and L be two submodules of a T-module Ω, then

$N\cap L=\text{Endo}-rad_{\Omega}^{R.B}(N)\cap\text{Endo}-rad_{\Omega}^{R.B}(L)$ if and only if

1. $N\cap L$ is a radical Endo-R.B submodule.
2. $\text{Endo}-rad_{\Omega}^{R.B}(N\cap L)=\text{Endo}-rad_{\Omega}^{R.B}(N)\cap\text{Endo}-rad_{\Omega}^{R.B}(L)$.

Proof.

Suppose that $N\cap L=\text{Endo}-rad_{\Omega}^{R.B}(N)\cap\text{Endo}-rad_{\Omega}^{R.B}(L)$, then

$\text{Endo}-rad_{\Omega}^{R.B}(N)\cap\text{Endo}-rad_{\Omega}^{R.B}(L)\subseteq\text{Endo}-rad_{\Omega}^{R.B}(N\cap L)$ and the inequality

$\text{Endo}-rad_{\Omega}^{R.B}(N\cap L)\subseteq\text{Endo}-rad_{\Omega}^{R.B}(N)\cap\text{Endo}-rad_{\Omega}^{R.B}(L)$ holds by proposition (3.5).

Now, by the assumption and proving part (2), we have $N\cap L$ is a radical Endo-R.B submodule.

Conversely, it is obvious that (1) and (2) prove that:

$N \cap L = \text{Endo} - rad_{\Omega}^{R.B}(N) \cap \text{ Endo} - rad_{\Omega}^{R.B}(L)$.

Recall a submodule A of a T-module Ω called an S-prime if there exists $\varphi \in End(\Omega)$ such that $\varphi(x) \in A$, $x \in \Omega$ implies that either $x \in A$ or $\varphi(\Omega) \subseteq A$ (21–23).

3.11. Lemma

Let Ω be a scalar T-module and A is an Endo-R.B. submodule of Ω. Then A is an S-prime submodule

Proof

Let $\varphi \in End(\Omega)$ and define $\varphi: \Omega \to \Omega$ as $\varphi(x) = rx, \forall x \in \Omega$.

Suppose that $x \notin A$ then, we have to prove that $\varphi(\Omega) \subseteq A$

Since Ω is a scalar and A is an Endo-R.B submodule, then $\varphi(x) \in A$, $\forall x \in \Omega$, which means that

$\varphi(\Omega) \subseteq A$. Therefore, A is an S-prime submodule.

Note that every S-prime submodule is a prime submodule.

3.12. Lemma

Let Ω be a scalar T-module and $N < \Omega$. Then, $rad(N) = \text{Endo} - rad_{M}^{R.B}(N)$.

Proof.

Since every S-prime submodule is a prime, then the proof is by lemma (3.11) and proposition (3.6).

3.13. Proposition

Let N and L are two submodules of a multiplication finitely generated T-module Ω such that $[N:\Omega] and$ $[L:\Omega]$ are radical ideals, then

$[N \cap L:\Omega] = [\text{Endo} - rad_{\Omega}^{R.B}(N \cap L):\Omega]$.

Proof.

Clearly

$$\begin{aligned} [N \cap L:\Omega] &= [N:\Omega] \cap [N:\Omega] \\ &= \sqrt{[N:\Omega]} \cap \sqrt{[L:\Omega]} \\ &= \sqrt{[N \cap L:\Omega]} \end{aligned}$$

Since Ω is finitely generated, then by theorem 4.4 in (24), we have

$$\sqrt{[N \cap L:\Omega]} = [rad(N \cap L):\Omega].$$

Using lemma (3.12), we conclude that $[N \cap L:\Omega] = [\text{Endo} - rad_{\Omega}^{R.B}(N \cap L):\Omega]$.

3.14. Proposition

Let N and L are two submodules of a multiplication finitely generated T-module Ω. Then

[Endo-$rad_{\Omega}^{R.B}(N \cap L):\Omega$] =[Endo-$rad_{\Omega}^{R.B}(N) \cap$ Endo-$rad_{\Omega}^{R.B}(L):\Omega$]

Proof.

Since Ω is a multiplication finitely generated module, then Ω is a scalar module and hence by lemma (3.12), we have $rad(N) = \text{Endo} - rad_{M}^{R.B}(N)$. Thus,

$$\begin{aligned} [\text{Endo-}rad_{\Omega}^{R.B}(N \cap L):\Omega] &= [rad(N \cap L):\Omega] = \sqrt{[N \cap L:\Omega]} \\ &= \sqrt{[N:\Omega]} \cap \sqrt{[L:\Omega]} = [\text{Endo-}rad_{\Omega}^{R.B}N:\Omega] \cap [\text{Endo-}rad_{\Omega}^{R.B}L:\Omega] \\ &= [\text{Endo} - rad_{\Omega}^{R.B}(N) \cap \text{ Endo} - rad_{\Omega}^{R.B}(L):\Omega]. \end{aligned}$$

3.15. Proposition

Let Ω be a multiplication finitely generated T-module and $N < \Omega$. Then $\sqrt{[N:\Omega]}\Omega = \text{Endo} - rad_{\Omega}^{R.B}(N)$.

Proof.

Let K be an Endo-R.B submodule of Ω containing N. Also, by lemma (3.11), if Ω is a scalar and N is an Endo-R.B, then N is an S-prime and every S-prime is a prime submodule.

Therefore, K is a prime submodule and $[K:\Omega]$ is a prime ideal implies that $[N:\Omega] \subseteq [K:\Omega]$ and $\sqrt{[N:\Omega]} \subseteq [K:\Omega]$. Hence $\sqrt{[N:\Omega]}\Omega \subseteq [K:\Omega]\Omega \subseteq K$ and since K is an arbitrary Endo-R.B submodule containing N so, we have that $\sqrt{[N:\Omega]}\Omega \subseteq \text{Endo} - rad_{\Omega}^{R.B}(N)$.

Since Ω is a multiplication finitely generated T-module then Ω is a scalar module and using lemma (3.12), we have that $rad(N) = \text{Endo} - rad_{\Omega}^{R.B}(N)$. By (24),we get that $rad(N) \subseteq \sqrt{[N:\Omega]}\Omega$.

Therefore, $\sqrt{[N:\Omega]}\Omega = rad(N) = \text{Endo} - rad_{\Omega}^{R.B}(N)$.

Let N be a submodule of a T-module Ω and Q be a multiplicative set of T, then $N(Q) = \{x \in \Omega : \text{rx} \in \text{N for some r} \in \text{T}\}$ is a submodule of Ω contains N (25,26) and the closure of a submodule A is denoted by $Cl(A) = \{ x \in \Omega : [A:x] \leq_e T\}$ (27,28)

3.16. Proposition Let Ω be a T-module and $N < \Omega$. Then

1) $\text{Endo} - rad_{\Omega}^{R.B}(N) \subseteq \text{Endo} - rad_{\Omega}^{R.B}(N(Q))$ where Q is a multiplicative set of T.
2) $\text{Endo} - rad_{\Omega}^{R.B}(N) \subseteq \text{Endo} - rad_{\Omega}^{R.B}(Cl(N))$.
3) $\text{Endo} - rad_{\Omega}^{R.B}(N) \subseteq \text{Endo} - rad_{\Omega}^{R.B}([N:_{\Omega} I])$ for every ideal I of T.

Proof.

1) Since N(Q) is a submodule of Ω contains N, then by proposition (3.5), we have that $\text{Endo} - rad_{\Omega}^{R.B}(N) \subseteq \text{Endo} - rad_{\Omega}^{R.B}\big(N(Q)\big)$.
2) It is clear since $Cl(N)$ is a submodule of Ω contains N.
3) Since for ideal I of T, we have $N \subseteq [N:_{\Omega} I]$. Therefore, the result follows directly from proposition (3.5)

Recall a submodule H of a T-module Ω said to be fully invariant if $\varphi(H) \subseteq H$ for every $\varphi \in End(\Omega)$ (29).

3.17. Proposition

Let N and L be two fully invariant submodules of a –module M and consider $N \oplus L$, then $\text{Endo} - rad_{\Omega}^{R.B}(N) \oplus \text{Endo} - rad_{\Omega}^{R.B}(L) \subseteq \text{Endo} - rad_{\Omega}^{R.B}(N \oplus L)$.

Proof.

Since $N, L \subseteq N \oplus L$, then $\text{Endo} - rad_{\Omega}^{R.B}(N), \text{Endo} - rad_{\Omega}^{R.B}(L) \subseteq \text{Endo} - rad_{\Omega}^{R.B}(N \oplus L)$. Thus,

$\text{Endo} - rad_{\Omega}^{R.B}(N) \oplus \text{Endo} - rad_{\Omega}^{R.B}(L) \subseteq \text{Endo} - rad_{\Omega}^{R.B}(N \oplus L)$.

The set of all Endo-R.B submodules of a T-module Ω is denoted by $Spec^{R.B}(\Omega)$.

Consider the notation:

$V^{R.B}(N) = \{K \mid K \in Spec^{R.B}(\Omega): N \subseteq K\}$ and so $\text{Endo} - rad_{\Omega}^{R.B}(N) = \bigcap_{K \in Spec^{R.B}(\Omega)} K$.

3.18. Proposition

Let Ω be a T-module, then the following holds

(1) $V^{R.B}(0) = Spec^{R.B}(\Omega)$ and $V^{R.B}(\Omega) = \phi$.
(2) $V^{R.B}(k_1) \cap V^{R.B}(K_2) = V^{R.B}(k_1 + K_2)$
(3) $V^{R.B}(N) \cup V^{R.B}(L) \subseteq V^{R.B}(N \cap L)$ for any fully invariant submodule N, L of Ω.

Proof.

(1) and (2) are obvious.

(3) Let N and L are two fully invariant submodules of Ω, then by the definition, we have

$V^{R.B}(N) = \{K \mid K \in Spec^{R.B}(\Omega): N \subseteq K\}$;

$V^{R.B}(L) = \{K \mid K \in Spec^{R.B}(\Omega): L \subseteq K\}$;

$V^{R.B}(N) \cup V^{R.B}(L) = \{K \mid K \in Spec^{R.B}(\Omega): N \subseteq K \text{ or } L \subseteq K\}$;

$V^{R.B}(N \cap L) = \{K \mid K \in Spec^{R.B}(\Omega): N \cap L \subseteq K \}$;

Now, take $h_0 \in V^{R.B}(N) \cup V^{R.B}(L)$, then h_0 is an Endo-R.B submodule such that

$N \subseteq h_0$ or $L \subseteq h_0$. Therefore, $N \cap L \subseteq N \subseteq h_0$ and $N \cap L \subseteq L \subseteq h_0$ and hence $h_0 \in V^{R.B}(N \cap L)$.

Recall the radical of a T-module Ω denoted $rad(\Omega)$ and it is the intersection of all maximal submodules of Ω (30).

3.19. Proposition

Suppose that $rad(\Omega) \leq_e \Omega$ and let $K \subseteq L \subseteq \Omega$ where K is a direct summand of Ω. Then $\text{Endo} - rad_{\Omega}^{R.B}(K) = \text{Endo} - rad_{\Omega}^{R.B}(L)$ if and only if K=L.

Proof.

Since K is a direct summand of Ω, then there exists a submodule K' of Ω such that $\Omega = K \oplus K'$.

Hence, $L = K \oplus (L \cap K')$ so that

$\text{Endo} - rad_{\Omega}^{R.B}(L) = \text{Endo} - rad_{\Omega}^{R.B}(K) \oplus \text{Endo} - rad_{\Omega}^{R.B}(L \cap K')$.

Therefore, $\text{Endo} - rad_{\Omega}^{R.B}(L \cap K') = 0$ and this can be written as $rad(\Omega) \cap (L \cap K') = 0$ implies that $L \cap K' = 0$ since rad(Ω) is an essential submodule of Ω. Thus, K=L.

4. Conclusion

We discussed in this paper the formula of the radical of an Endo-R.B submodule as a new type and proved that it is an Endo-R.B submodule of a T-module Ω. Also, the relationship between prime and Endo-R.B submodules helps us to give many properties. In this paper, we show that the radical of the submodule N and the radical of an Endo-R.B submodule merge under certain conditions that will be useful to other researchers in order to present other results.

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Conflict of Interest

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