

# THEORY OF FIFTH-ORDER STOKES WAVES IN A LINEAR SHEAR CURRENT

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## INTRODUCTION

In coastal and deep-water areas, shear currents are often generated by wind action. This leads to a strong vertical shearing action within the upper layer of the water body. Therefore, the rotational characteristics of the flow motion play an essential role in affecting the wave-current interaction process. Analytical Stokes wave solutions on linear shear currents provide necessary information on wave properties and velocity fields for ocean and coastal engineering applications (see Fig.1). However, existing high-order (above third-order) Stokes wave solutions are not valid. Thus, in this paper, following the methodology introduced by Zhao & Liu (2022), a new fifth-order solution for Stokes wave in a linear shear current is derived. The new analytical solutions are checked with the exact numerical solutions by Francius & Kharif (2017).

## THEORETICAL METHOD

The Stokes wave perturbation method developed in Zhao & Liu (2022) is extended to find perturbation solutions in the analytical form for fifth-order Stokes wave in a linearly sheared current. The perturbation procedure is formulated in terms of the potential function. To avoid the secular terms presented in the third- and fifth-order free surface boundary conditions, a perturbation expansion for wave angular frequency (or wave phase speed) is also introduced. The details of the solutions can be found in Fang et al. (2023). The present solutions can be reduced to the pure wave solutions of Zhao & Liu (2022) if the shear current vanishes.

## RESULTS AND DISCUSSIONS

Kishida & Sobey (1988) presented a third-order Stokes wave solution in a linear shear current, based on the stream function formulation. In their derivation the wave height is forced to be twice of the wave amplitude of the first-order wave (i.e.,  $H=2A$ ) as originally suggested in Fenton (1985), which is deemed to be non-physical (Zhao & Liu 2022). Thus, their solutions do not satisfy the free surface boundary conditions. The performance of the present solutions is examined by comparing against Kishida & Sobey (1988) and the exact numerical solutions by Francius & Kharif (2017). As illustrated in Fig. 2, the present solutions agree with the numerical solutions very well, but differ significantly from the solutions of Kishida & Sobey (1988) for larger nonlinearity. The present solutions also provide accurate phase speeds for highly nonlinear waves. Without adopting the artificial assumption that  $H=2A$ , in deriving the present solutions, the dispersion equation, wave height equation and surface elevation are solved simultaneously. As a result, the present derivations provide accurate analytical solutions for phase speed, velocity distribution and wave elevation even under the

strong nonlinearity conditions.

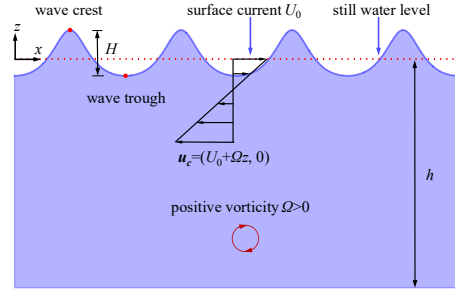


Figure 1 A sketch for a periodic Stokes wave propagation with a vertical linear shear current.

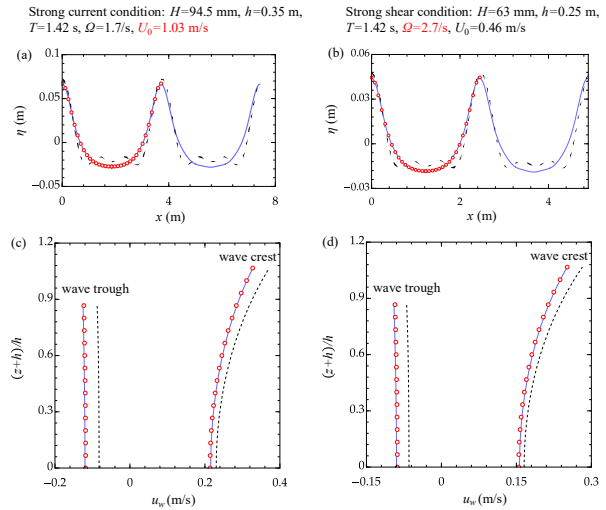


Figure 2 (a) and (c) are surface elevations and wave horizontal velocity profiles under wave crest and wave trough for strong current condition; (b) and (d) are for strong vorticity conditions. Here, blue solid lines are the present solutions, black dashed lines are the third-order solutions of Kishida & Sobey (1988) and red circles represent the numerical solutions calculated based on Francius & Kharif (2017).

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