A new theoretical envelope framework in the Hamiltonian theory of nonlinear deep- and finite-water surface gravity waves

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An extensive body of literature has addressed the question that how we can accurately and efficiently predict the spatial-temporal evolution of surface gravity waves in the open ocean and coastal waters. The main methods for an answer to it can be partitioned into two groups: (i) theoretical frameworks which make use of Fourier transforms such as a nonlinear Schrödinger (NLS) equation-based model [1], the High-Order Spectral method (HOS) [2,3], and Zakharov integral equations[4,5], and (ii) direct numerical solvers which propose to obtain the numerical solution of the fully nonlinear potential flow equations like *OceanWave3D*[6], *Reef3D*[7], and a harmonic polynomial cell method [8]. Generally, the first group has been known to excel at numerical efficiency[9].

If numerical efficiency in wave steepness is given the highest priority, it is understood that a NLS equation-based model has clear advantages[10]. Compared to the HOS method, it does not require an expensive numerical cost and it is an excellent analytical tool for elucidating physics, as is widely known. The former is especially important in problems where surface waves vary rapidly and have a shorter length scale compared to their ambient environment, e.g., the coupling between surface waves and subsurface flows. However, a NLS equation-based model is only accurate up to third order in wave steepness and hence, has limited applicability.

Following the discussion above, a question is whether it is possible to derive a new framework that combines the main advantages of both a NLS-based model and the HOS method, in a sense that it has the same order of numerical efficiency as a NLS equation-based model while keeping the arbitrary accuracy of the HOS method for the spatial-temporal evolution of surface waves in the open ocean and coastal waters. In this work, the main objective is to present such a framework, based on (a) a perturbation expansion like the framework used in the HOS method, (b) the new introduction of the envelope of the surface elevation and velocity potential on a free water surface, and (c) the main features of Fourier transform. It can be shown that the newly introduced envelopes are a pair of canonical variables and therefore, the equations for the evolution of the envelopes are in the Hamiltonian theory of nonlinear surface gravity waves. We show how the new framework can recover to the framework of the HOS method. Moreover, we propose to solve the evolution equations for the envelopes by exponential integrator. The new framework allows us to clearly separate the rapidly varying wave phase and the slowly varying wave amplitude (and hence, energy). Moreover, it permits the free choice of a coarse spatial resolution and a large time interval for numerical computations without compromising the numerical accuracy. Thereby, the numerical efficiency is significantly improved by the new framework compared to a HOS method, see, e.g., [11].

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