EXPECTED WAVE ENERGY FLUX OF PREDICTED SEA STATES

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INTRODUCTION

Whether coupled into an earth-system model or as separate tool, third-generation wave models (*e.g.* WAVEWATCHIII, WAM, SWAN) are the tools used by weather centers and researchers to describe the evolution of expected sea-states. Due to the complicated nature of the wave generation process, wave phases are unpredictable, thus the sea states are represented by the distribution of the wave action density over a finite frequency range.

The conservative part of the wave action equation, which is obtained disregarding sources and sinks, is the kinetic equation first introduced by Hasselmann (see Komen et al., 1996). It is understood that the term which is usually named as "non-linear source" is not a proper source term but constitutes the bulk of the dynamics, accounting for the energy transfer among the modes.

The Hasselmann kinetic equation can be derived starting from the four wave Zakharov equation (later ZE; see Zakharov, 1968; Krasitskii, 1994) after invoking a statistical closure which eliminates the phases (see *e.g.* Zakharov, 1999; Onorato and Dematteis, 2020). Therefore, the ZE is the backbone of the third-generation wave models.

The ZE resolves phases and amplitudes of a normal variable that can be interpreted as a representation of the free wave modes. Since the ZE is obtained from the Euler equation after a nonlinear canonical transformation, the latter describes the associated bound modes. In practice, the relevant dynamics are described by the ZE, but the physical variables must be recovered applying the canonical transformations, *i.e.* dressing the normal variable with the bound modes. We must also add that the wave action density is proportional to the square of Zakharov normal variable.

Clearly now, we see that the wave action propagated in spectral wave models does not coincide with the observable wave action. In order to recover the observable sea spectrum, one has to invoke the Zakharov canonical transformation, *i.e.* account for the contribution of the bound modes.

As shown by Janssen (2009) the zero-th moment of the spectrum, that is the significant wave height (H_S), is not affected by the bound modes' contribution, whereas any other integral is. Therefore, except for H_S, any other diagnostic quantity, such as mean spectral periods, mean squared slope, spectral width to name a few, should not be straightforwardly derived from the spectral wave model output, but computed after a nonlinear correction.

This operation is not always necessary. The modelled wave action is, in some sense, the small-steepness approximation of the observable one. Thus, for mild sea states the weight of the nonlinear correction is negligible. For relevant steepness, however, depending on the diagnostic quantity we are interested in, we need the above-described higher order corrections to get a proper representation of the sea states.

Other important quantities cannot be derived in a straightforward manner from the corrected spectrum, still they deserve the same kind of accuracy. For example, the radiation stresses, or similar quadratic quantities, are necessary for earth-system models because they are part of the information that is transferred from the wave model to the circulation model (see *e.g.* Arduhin et al. 2017). Similarly, the wave energy flux is useful to assess the severity of a sea state or to quantify the available wave power.

The evidence of the importance of high order corrections to these quantities is not new. Starting from fifth order Stokes solution obtained by Fenton (1985), Jonsson and Arneborg (1995) showed the importance of higher order corrections for determining the energy properties of long crested waves. For example, depending on the relative water depth and steepness, the first nonlinear correction to the mean wave energy flux can reach values which are on the order of 10% of the linear contribution. These results apply to regular long crested waves only.

To find the nonlinear corrections for a broad banded directional sea state one needs to start from a consistent third order two-dimensional theory. It has been lately reported that the Zakharov Hamiltonian formulation and its transformations, previously considered incomplete because of the presence of some singularities in intermediate waters, are instead self-sufficient (Pezzutto and Shira, 2023). We make use of these results to pursue our aim, that is to find an expression for the nonlinear correction of the simplest of the wave energy properties, *i.e.* the wave energy flux, in two dimensions for short crested waves. And then, introduce the formulation to be applied to the output of phase averaged models.

WAVE ENERGY FLUX

To obtain the general form of the wave energy flux we make no special assumptions about the shape of the spectrum. We directly proceed from Longuet-Higgins and Stewart (1960) and disregard the ambient current.

We then Taylor-expand the vertical integral for small surface elevation gradient. At this point we use Krasitskii (1994) canonical transformations, one after the other, applying proper symmetrization procedures and neglecting fifth order terms. The result is a fourth order (in amplitude) form of the rate of transfer of energy across a surface fixed in space.

We then assume that phases of the free modes are totally uncorrelated. Therefore, after phase averaging the deterministic equation previously obtained, we finally get the wanted average energy flux. If computed across a vertical surface of unit width, the flux **r** reads:

$$\mathbf{r} = \frac{1}{2}\rho \mathrm{g} \left(\int \boldsymbol{c}_g a^2 \mathrm{d}\boldsymbol{k} + \iint \boldsymbol{d}_{n,m} a_n^2 a_m^2 \mathrm{d}\boldsymbol{k}_n \mathrm{d}\boldsymbol{k}_m \right) \quad (1)$$

The first term is well known, it is an integral in R^2, with the integration variable **k** indicating the wavevector. The vector $\mathbf{c_g}$ stands for the group-celerity and the scalar a for the wave amplitude. The second term is a double integral in R^2, with n and m indicating the two separate wavenumber spaces. The main result is contained in the vector $\mathbf{d}_{n,m}$ which expresses the energy flux due to the "interaction" of two wave modes. The expression for $\mathbf{d}_{n,m}$ is too long to be presented in this abstract. Instead, we will give the audience some interesting results obtainable from Eq.1 for bi-chromatic waves and general broad banded sea conditions.

In this abstract we only wish to present the simplest situation of a long-crested sea state. For a monochromatic wave, that is considering a Dirac-delta in wavenumber space, from Eq. 1 we find:

$$\mathbf{r} = \frac{1}{2}\rho g \boldsymbol{c} a^2 \left(1 + \gamma\right) \quad (2)$$

The second term in brackets, which is nondimensional, derives from the second term of Eq. 1, and it is therefore proportional to $\mathbf{d}_{n,n}$, which is a function of kh only, and to the square of the wave steepness (ka). In practice γ measures the weight of the second order correction of the energy flux, with respect to first order. To understand its potential importance, in Figure 1., we have plotted its value for the maximum theoretical steepness, which is obtained from the known breaking conditions:

$$ka_{max} = \min\left(\pi \tanh kh/7,\, 0.39kh\right) \quad (3)$$

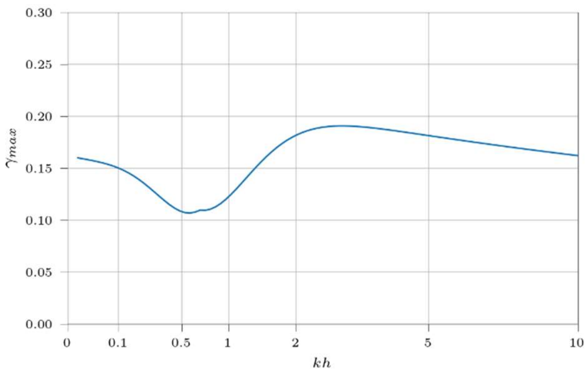

Figure 1 - Weight of second order energy flux, with respect to first order, for long crested regular waves at maximum theoretical steepness.

Therefore, for a monochromatic wave the resulting equations are compared with truncated expressions given by Jonsson and Arneborg (1995), confirming that second order (in wave energy) contributions can be relevant, depending on steepness, at any water depth. It is also important to note that Eq.2 does not blow up in the limit of shallow waters, it therefore provides meaningful results at any water depth.

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