# CHAPTER 16

## SPECTRA AND BISPECTRA OF OCEAN WAVES

by

OLE GUNNAR HOUMB

Division of Port and Ocean Engineering
The University of Trondheim
The Norwegian Institute of Technology

## INTRODUCTION

The two dimensional (directional) power spectrum gives an adequate description of water waves that may be regarded as a linear superposition of statistically independent waves. In such cases the sea surface is linear to the first order and the surface displacement is represented by

$$\eta(t) = \sum_{n=1}^{\infty} a_n \sin(\omega_n t + \phi_n)$$

where $a_n$ are the amplitudes of individual waves and $\phi_n$ is a radomly distributed phase angle, and the process is stationary.

Under such circumstances the wave surface is Gaussian, which means that ordinates measured from MWL are normally distributed if they are sampled at constant intervals of time. It is equally important that the wave heights are Rayleigh distributed.

This formulation of the wave surface is widely used e.g. in wave forcasting.

There are, however, phenomena such as wave breaking, energy transfer between wave components and surf beat which can only be described by higher order effects of wave motion (1, 2, 3, 4). In this case the two dimensional power spectrum fails to give an accurate description of the wave surface. This means that the first and second order moments (mean and covariance) no longer give all the probability information, and we have to consider higher order moments (5, 6, 7).

Third order moments of non-zero value indicate positive values of the bispectrum.

This paper gives an introduction to the bispectrum and some examples of bispectra calculated from wave records obtained during storm situations.

## THE BISPECTRUM

The bispectrum, $B(f_1, f_2)$, of a random stationary wave record is defined as the Fourier transform of the mean third order products

$$B(f_1, f_2) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} S(\tau_1, \tau_2) e^{-if_1\tau_1 - if_2\tau_2} d\tau_1 d\tau_2 \tag{2.1}$$

where

$$S(\tau_1, \tau_2) = \overline{\eta(t)\ \eta(t+\tau_1)\ \eta(t+\tau_2)} \tag{2.2}$$

and the overbar denotes ensemble means.

If $\eta(t)$ is real and stationary we have the following relations

$$B(f_1, f_2) = B(-f_1, -f_2)^* \tag{2.3}$$

and

$$\begin{aligned} B(f_1, f_2) &= B(f_2, f_1) = B(-f_1, -f_1-f_2) = B(-f_1-f_2, f_1) \\ &= B(f_2, -f_1-f_2) = B(-f_1-f_2, f_2) \end{aligned} \tag{2.4}$$

where $*$ denotes complex conjugate. As a consequense of equations 2.3 and 2.4 the bispectral values fall within an octant.

For a purely stationary Gaussian process the bispectrum has the value of zero.

It is of interest also to note that information on the relative directional spread of waves is derivable from the bispectrum which is calculated from one single wave record.

If, however, the wave surface has a non-Gaussian distribution the bispectrum has a non-zero value, which is also the case for the third order moments. It then follows that the third order products $\eta(t)^3$ are also non-zero. This can physically be interpreted as the peaking of waves with harmonics that are in phase with the fundamental. In other words, significant non-zero values in the bispectrum suggest that the wave surface can be approximated with an expression like

$$\eta(t) = a_1 \sin 2\pi ft + a_2 \sin 4\pi ft + \text{------} \tag{2.5}$$

which is a surface as given by the Stokes wave theory. The bispectrum thus indicates whether harmonic couplings between wave components are significant.

The possibility of triple interactions could be evaluated by a trispectral analysis. Trispectral interactions contribute nothing to $\eta(t)^3$ but somewhat to $\eta(t)^4$ which means, physically, that the waves tend to be assymetric about the crest. Such waves can be said to be close to breaking, and they are therefore in an irreversible state. There is much evidence for assuming that assymetric waves are liable to formation of shock pressures.

Trispectral calculations may therefore have practical consequences in that the probability of obtaining shock pressures on marine structures may be evaluated. To the author's knowledge, such calculations based on ocean wave data have not yet been made. For confidence to be placed in the trispectral estimates the calculations would require such long wave records that problems pertaining to stationarity might occur.

From elementary statistics the skewness g is known as third order moment that is used to estimate to which degree data have a Gaussian distribution.

Fisher (8) has developed a test for normality that is widely used in statistics. The test can be summarized in terms of the inquality

$$\frac{g}{\sqrt{\text{var } g}} > g_{1-\alpha/2} \tag{2.6}$$

where

$$g = \frac{\overline{\eta(t)^3}}{\overline{\eta(t)^2}^{3/2}} \qquad (2.7)$$

$$\text{var } g = \frac{6N(N-1)}{(N-2)(N+1)(N+3)}$$

N = number of values where $\eta(t)$ is known

If the inequality 2.6 holds, the hypothesis of a Gaussian distribution is rejected at the level $\alpha$. In other words, $\alpha$ is the probability that the hypothesis is rejected even if the distribution is Gaussian.

## BISPECTRA OF HARMONIC WAVES

Bispectra were calculated for the following artifical waves

$$\eta_1(t) = 10 \cos 0.08t$$

$$\eta_2(t) = 10 \cos 0.08t + 1.25 \cos 0.16t$$

$$\eta_3(t) = 10 \cos 0.08t + 2.5 \cos 0.16t$$

$$\eta_4(t) = 10 \cos 0.08t + 5 \cos 0.16t$$

$$\eta_5(t) = 10 \cos 0.04t + 5 \cos 0.08t$$

$$\eta_6(t) = 10 \cos 0.04t + 5 \cos 0.08t + 2.5 \cos 0.12t$$

$$\eta_7(t) = 10 \cos 0.04t + 5 \cos 0.08t + 2.5 \cos 0.12t + 1.25 \cos 0.16t$$

$$\eta_8(t) = 10 \cos 0.04t + 5 \cos 0.08t + 2.5 \cos 0.12t + 1.25 \cos 0.16t + 0.625 \cos 0.20t$$

These are all harmonic waves with a fundamental frequency of 0.04 Hz. $\eta_2(t)$ to $\eta_7(t)$ are all of the Stokes type with peaked wave crests and we should expect significant skewness as well as bispectral values.

Figs. 1 to 7 show bispectra calculated for $\eta_1(t)$ to $\eta_7(t)$ respectively. The upper part in these Figs. shows the bispectrum, whereas the central part indicates the spectrum and the lower parts show one wavelength of the wave surface. The bispectral value of 1.9 in Fig. 1 means $10^{1.9}$ $m^3$ $Hz^{-2}$. $B(0.08, 0.08) = 10^{1.9}$ indicates a weak interaction of the frequency 0.08 Hz with itself. In fig. 2 we find $B(0.08, 0.08) = 10^{5.4}$ $m^3$ $Hz^{-2}$ and a lower value at $B(0.08, 0.16)$, from which we can conclude that the dominant component of $\eta_2(t)$ is that of f = 0.08 Hz. Proceeding to Fig. 4 we can see that the dominant component is still f = 0.08 Hz, but the interacting frequencies are now (0.08, 0,08), (0.08, 0.16) and (0.16, 0.16) Hz. From the lower part of Fig. 4 we can also see that the deviation from cosine surface is very significant.

The surface represented by $\eta_8(t)$ is the result of 5 harmonic components. It is clearly seen from Fig. 8 that the interacting frequences are (0.04, 0.04), (0.08, 0.04), (0.08, 0.08), (0.12, 0.04), (0.12, 0.08), (0.16, 0.04). The skewness is also computed for the 8 surfaces and the results are summarized below.

| Surface | $\eta_1(t)$ | $\eta_2(t)$ | $\eta_3(t)$ | $\eta_4(t)$ | $\eta_5(t)$ | $\eta_6(t)$ | $\eta_7(t)$ | $\eta_8(t)$ |
|---|---|---|---|---|---|---|---|---|
| Skewness | $0.92 \cdot 10^{-4}$ | 0.259 | 0.485 | 0.760 | 0.860 | 1.06 | 1.17 | 1.26 |

The skewness increases with the number of harmonic components, that is with the peakedness of the surface.

Kinsman (9) concludes from studies of the skewness of waves that values between 0.090 and 0.336 give rise to significant corrections of the Gaussian distribution.

## BISPECTRA OF WAVE RECORDS

The data were recorded off the coast of Northern Norway at a depth of 80 meters, using a Waverider. The sampling interval was 0.5 sec corresponding to a Nyquist frequency of 1 Hz.

Aliasing should therefore appear at frequencies higher than 1 Hz, a situation whichis satisfactory because shorter waves are not measured properly by a Waverider and they are also without interest in this case.

Three typical examples of storm wave situations were selected for bispectrum analysis. The results are shown in Figs. 9, 10 and 11, along with the spectra of the same wave records.

In Fig. 9 there are four ridges of positive bispectral values, which indicate interactions of the spectral peak with itself and higher frequencies. In the same way the negative bispectral value is interpreted as interactions between the spectral peak and lower frequencies.

Peaks of the energy spectrum are associated with ridges in the bispectrum.

As expected, there is a positive bispectral ridge at the spectral peak which indicates interactions within the peak in Fig. 9. $B(0.125, 0.125) = 10^{11}\ m^3\ Hz^{-2}$ indicates that the secondary peak of the spectrum at ~ 0.125 Hz is real, whereas $B(0.094, 0.031) = -10\ m^3\ Hz^2$ can be interpreted as an interaction which produces parts of the secondary peak.

Fig. 10 reveals three main ridges in the bispectrum and a singly peaked spectrum. The strongest interactions appear at $f_1 = 0.188$ Hz where 0.188 and 0.094 Hz contribute to the peak in producing the difference frequency 0.09 Hz. As a result of the bispectral ridges in Fig. 11 one may raise the question as to whether real peaks in the spectrum have been smoothed out.

In Fig. 11 the three peaks in the spectrum are beautifully accompanied by three distinct bispectral ridges, indicating that all three peaks are real. The peak interacts strongly with itself which gives the sum frequency of ~ 0.156 Hz i.e. the second spectral peak. The third peak is believed to be caused by interaction between 0.125 and 0.094 Hz, as indicated by the middle bispectral ridge.

The test on normality, as described in the text, was applied to estimate the deviations from normality of the three records, and the results are summarised in the table below.

| Situation | $\alpha$ | G | $g_1-\alpha/2$ | g |
|---|---|---|---|---|
| Fig. 10 | 0.20 | 1.4734 | 1.2816 | 0.075 |
| | 0.10 | | 1.6449 | |
| Fig. 11 | 0.20 | 1.9245 | 1.2816 | 0.102 |
| | 0.10 | | 1.6449 | |
| Fig. 12 | 0.20 | 1.1132 | 1.2816 | 0.059 |
| | 0.10 | | 1.6419 | |

From these results it is clear that wave records represented by Fig. 9 and 11 are closely normally distributed, but there are sizeable deviations from normality in the record of which Fig. 10 shows the spectrum.

## CONCLUSION

The bispectrum can be used to estimate the extent of harmonic couplings between wave components in an irregular wave surface. Significant bispectral values thus indicate some evidence for using the Stokes wave theory in describing waves on intermediate and deep water.

Secondary spectral peaks are found to be real, and not introduced by the analysis methods, when they are associated with significant bispectral values at the same frequency. In cases where bispectral ridges are not accompanied with spectral peaks, it may be questioned whether such peaks are smoothed out in the analysis. Finally, it is shown that both the bispectrum and skewness are good measures of the wave surface deviation from the Gaussion distribution.

The wave records from which significant nonzero bispectra were calculated, were all obtained during relatively extreme wave conditions.

ACKNOWLEDGEMENTS

The work reported here was sponsored by The Royal Norwegian Council for Technical and Scientific Research.

The author wants to record his thanks to Nils Utne for permission to use his computer programs for the numerical calculations.

REFERENCES

1. Munk, W.H.: Surf beats. Trans. Amer. Geophys. Union, 30, 6, pp 849-854, 1949.

2. Longuet-Higgins, M.S. and R.W. Stewart: Radiation stress and mass transport in gravity waves, with application to surf beats. J. Fluid Mech. 1962.

3. Hasselmann, K.: On the non linear energy transfer in a gravity-wave spectrum, Part I. General theory.

4. Phillips, O.M.: On the dynamics of unsteady gravity waves of finite amplitude, Part I. The elementary interactions. J. Fluid Mech., 9. 2, pp 192-217. 1960

5. Hasselmann, K. et al.: Measurements of Wind-Wave Growth and Swell Decay during the Joint North Sea Wave Project (JONSWAP). Deutsches Hydrographisches Institut, Hamburg. Reihe A, Nr. 12, 1973.

6. Barnett, T.: Observations of Wind Wave Generation and Dissipation in the North Sea: Implications for the Offshore Industry. Offshore Technology Conference, 1972.

7. Garrett, J.: Field Observations of Frequency Domain Statistics and Nonlinear Effects in Wind-Generated Ocean Waves. The University of British Columbia, Phd. thesis 1970.

8. Fisher, R.A.: Statistical methods for research workers. Oliver & Boyd, 1950.

9. Kinsman, B.: Surface waves at short fetches and low wind speeds - a field study. Chesapeake Bay Institute technical report XIX. 1960.

Fig. 1. Bispectrum of $\eta_1(t) = 10 \cos 0.08t$

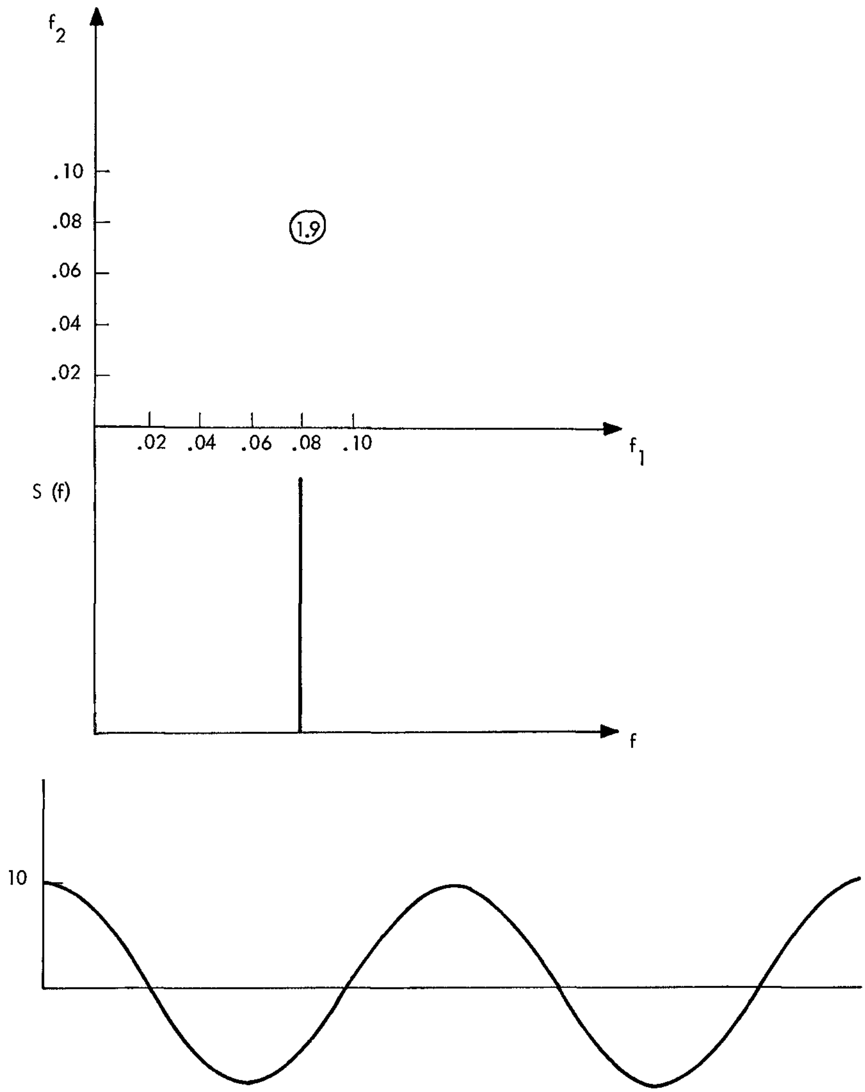


Fig.2. Bispectrum of $\eta_2(t) = 10 \cos 0.08t + 1.25 \cos 0.16t$

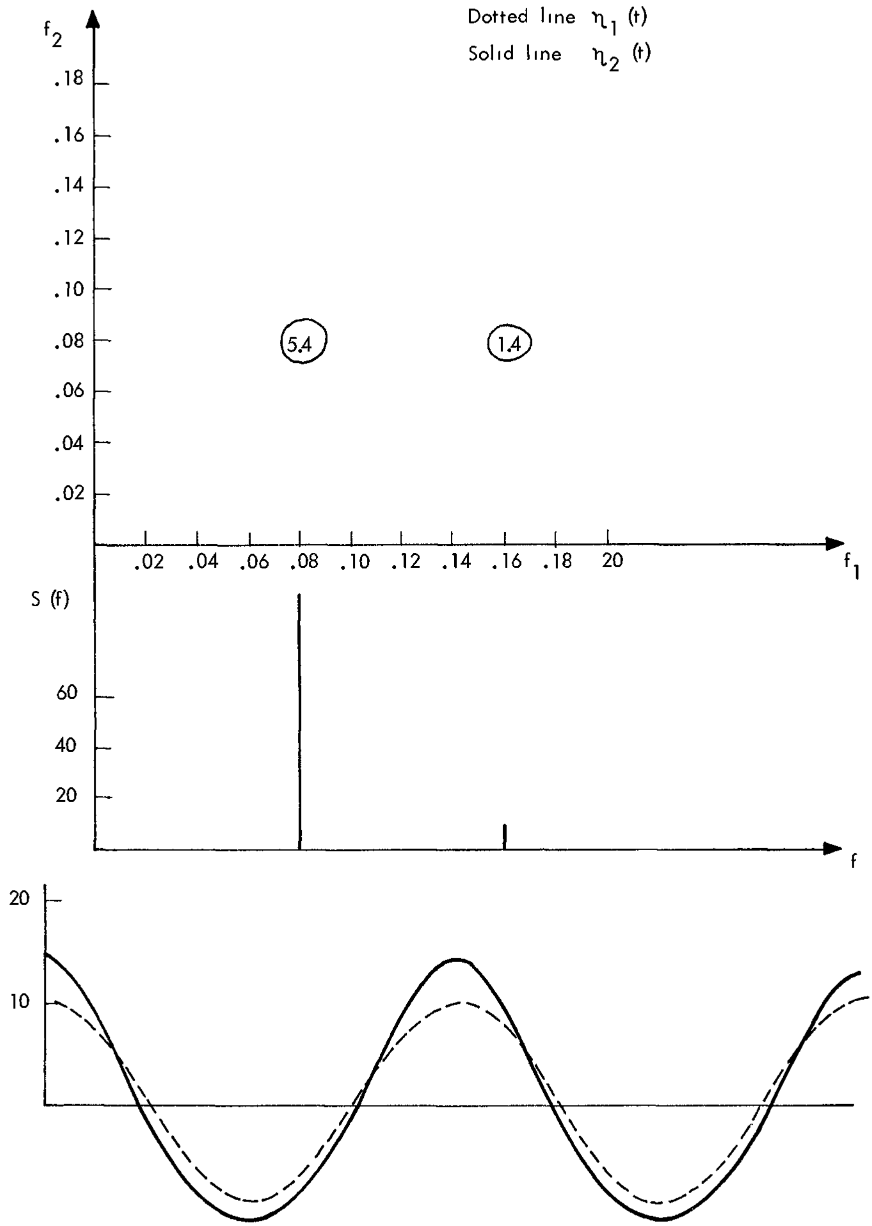


Fig. 3 Bispectrum $\eta_3(t) = 10 \cos 0.08\, t + 2.5 \cos 0.16\, t$

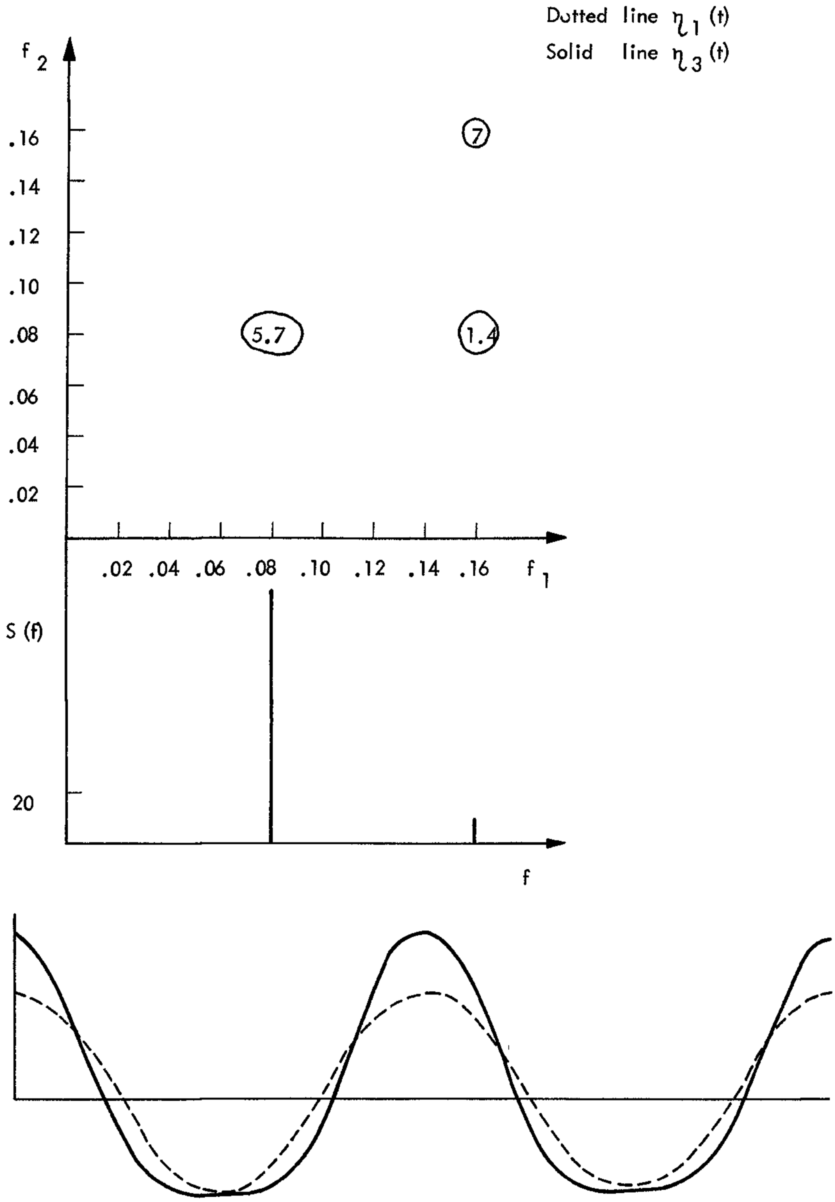


Fig.4. Bispectrum of $\eta_4(t) = 10\cos 0.08t + 5\cos 0.16t$

Dotted line $\eta_1(t)$

Solid line $\eta_4(t)$

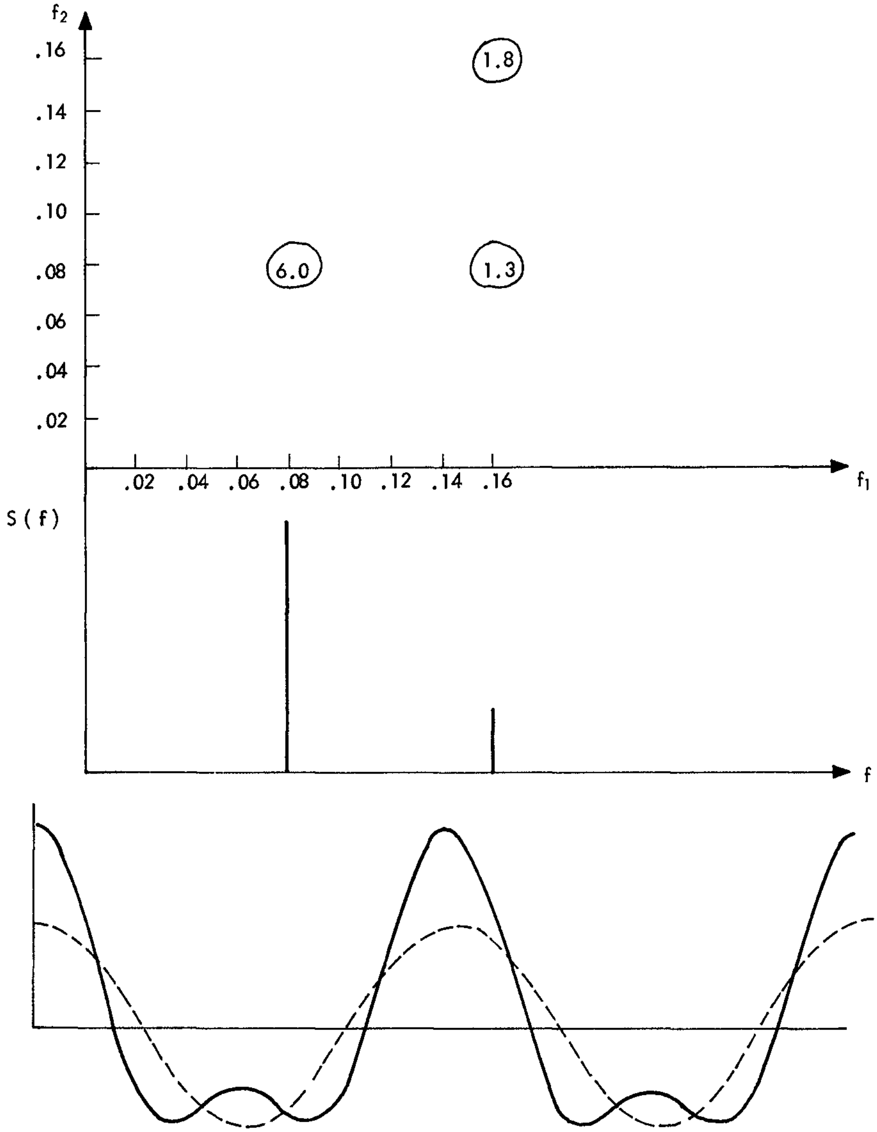


Fig. 5 Bispectrum of $\eta_5(t) = 10 \cos 0.04\,t + 5 \cos 0.08\,t$

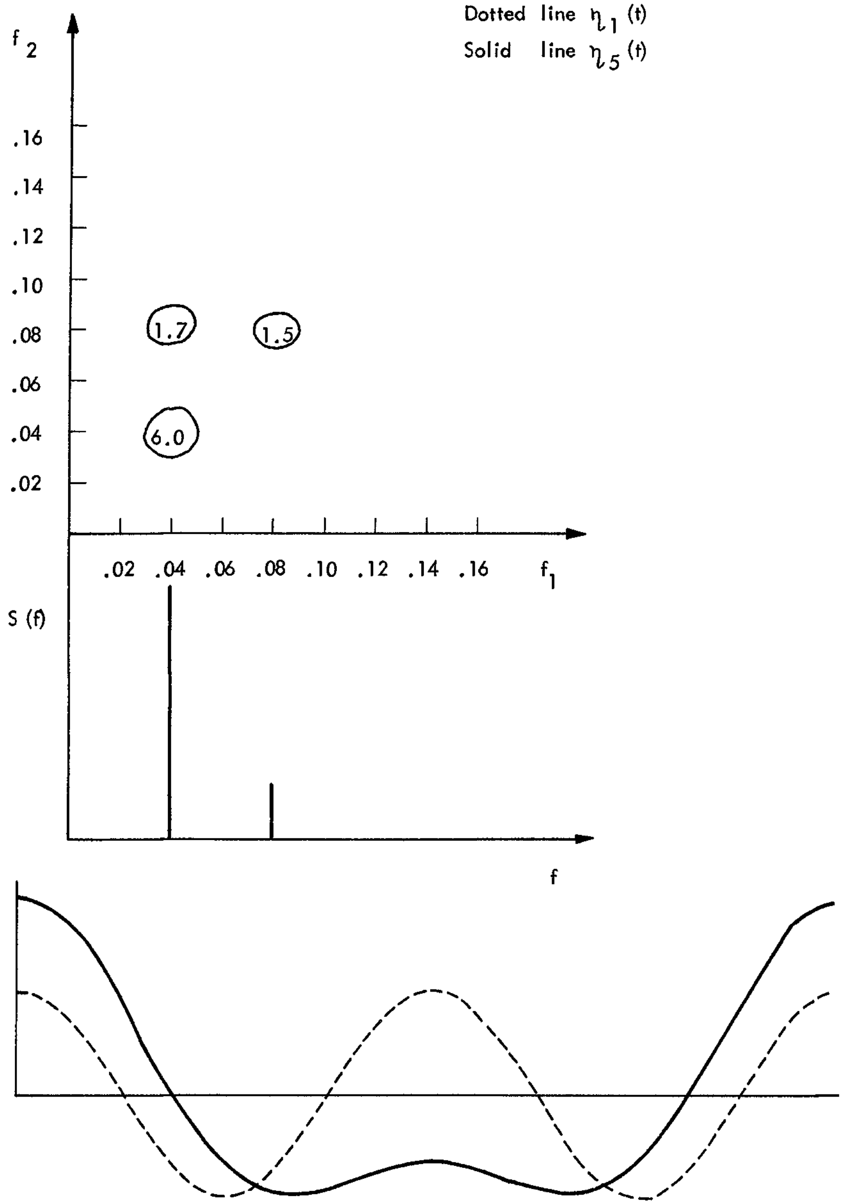


Fig. 6 Bispectrum of $\eta_6(t) = 10 \cos 0.04\,t + 5 \cos 0.08\,t + 2.5 \cos 0.12\,t$

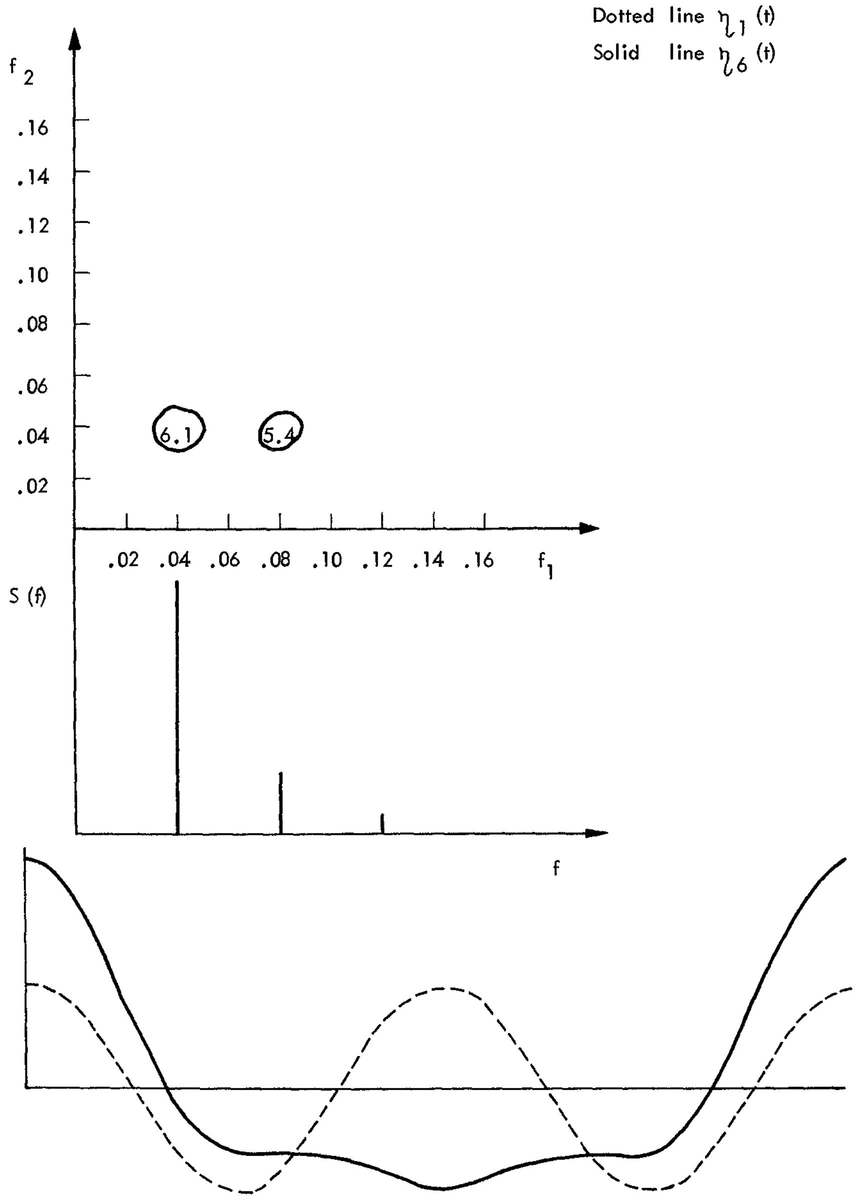


Fig. 7 Bispectrum of $\eta_7(t) = 10 \cos 0.04t + 5 \cos 0.08t + 2.5 \cos 0.12t + 1.25 \cos 0.16t$

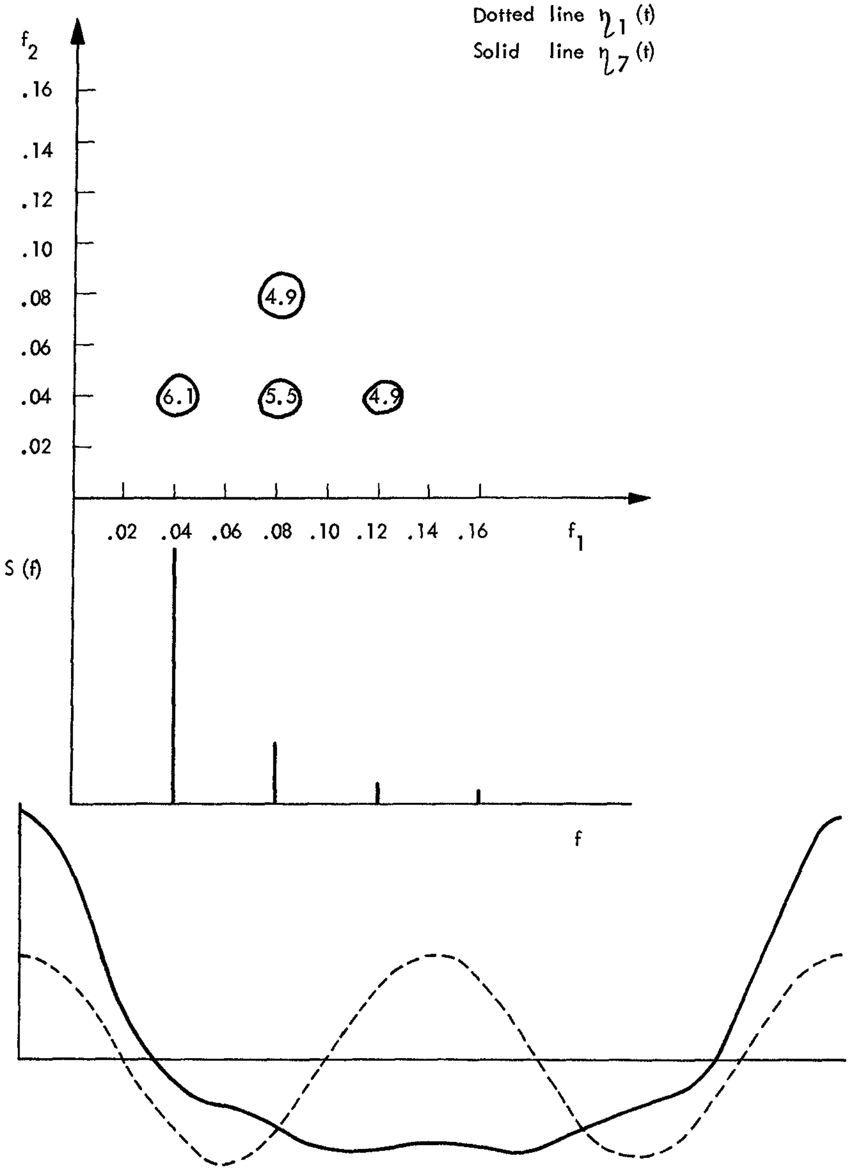


Fig. 8. Bispectrum of $\eta_8(t) = 10\cos 0.04t + 5\cos 0.08t + 2.5\cos 0.12t + 1.25\cos 0.16t + 0.625\cos 0.20t$

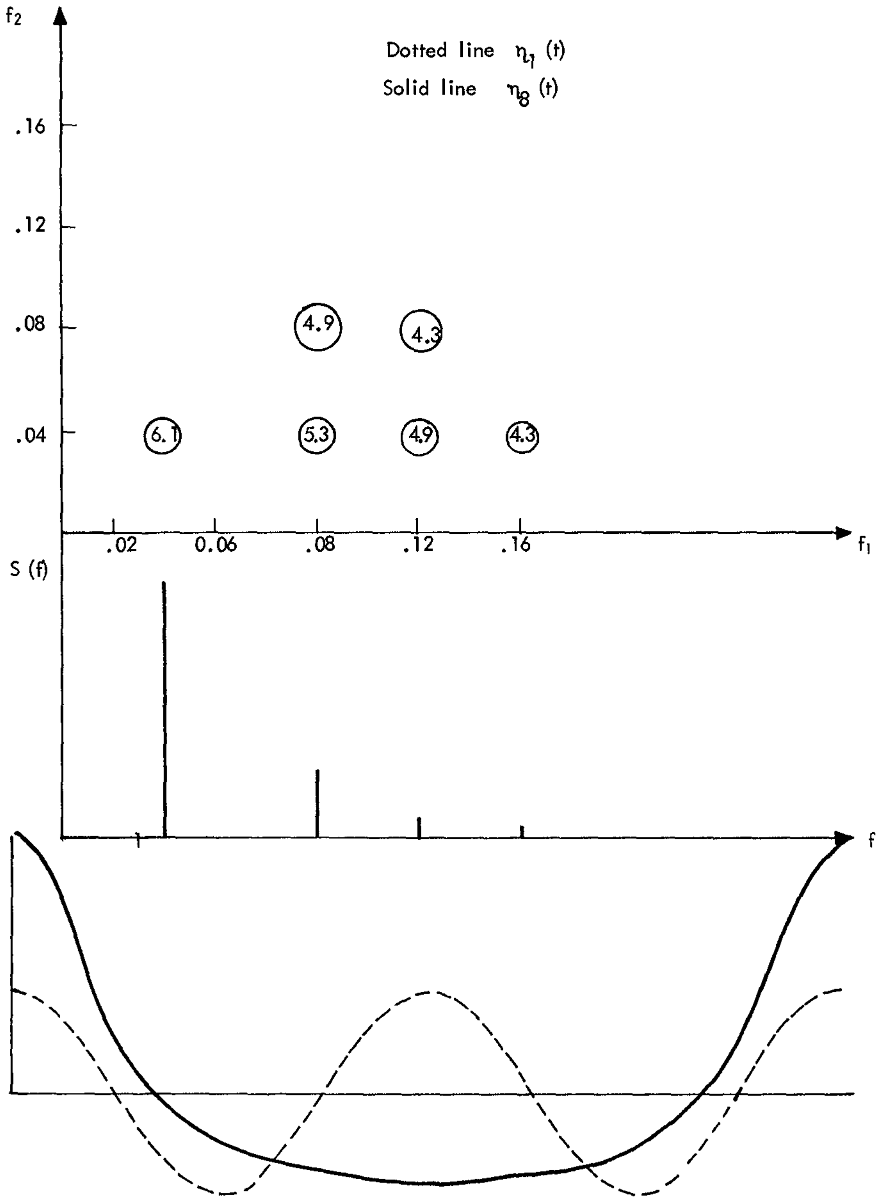


Fig. 9 Bispectrum of wave record 711219.03.

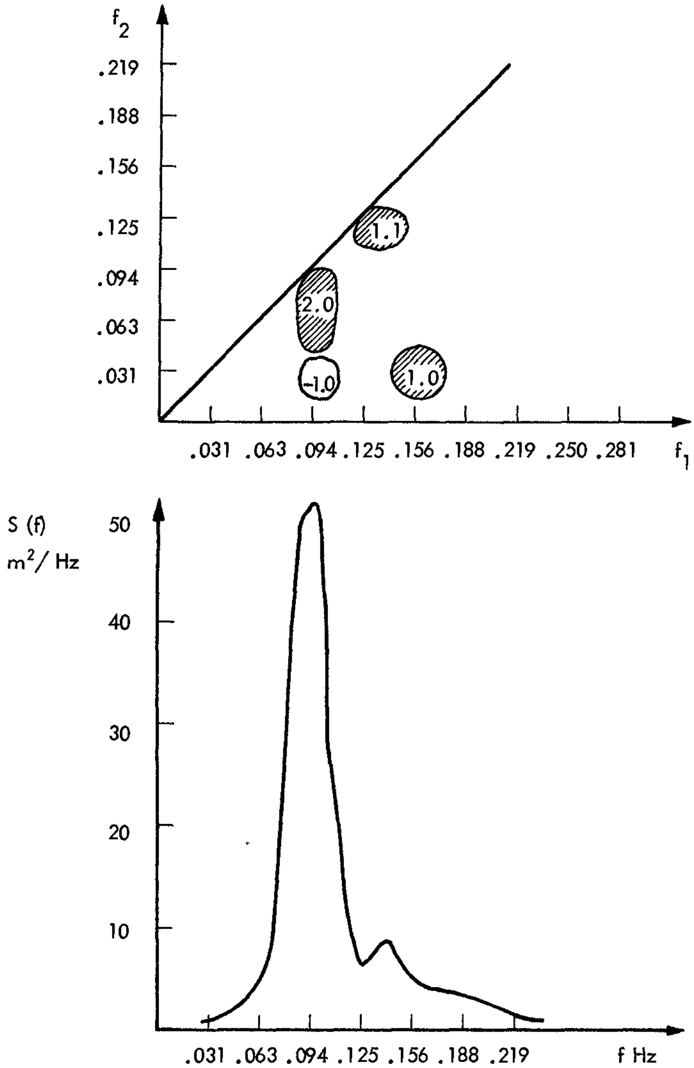
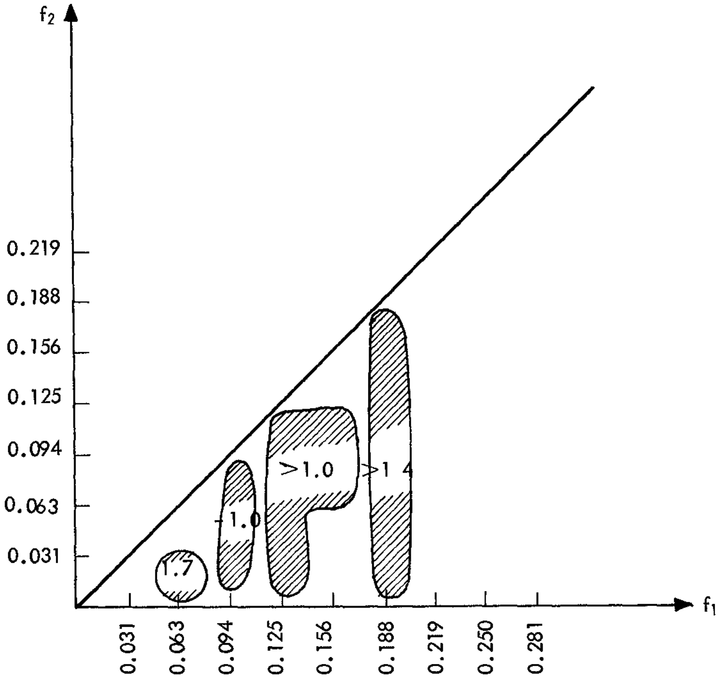


Fig. 10. Bispectrum of wave record 711218.21



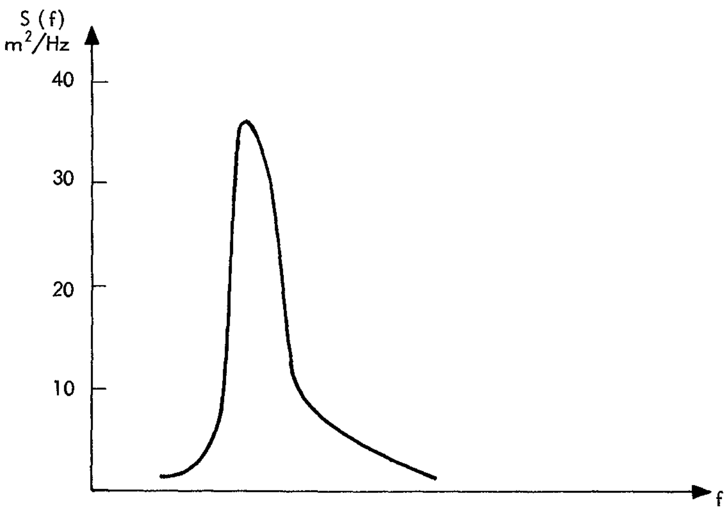



Fig 11. Bispectrum of wave record 711208.21.