CHAPTER 25

EXCITATION OF LOW FREQUENCY TRAPPED WAVES

by

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Weak nonlinear interactions in water of non-constant depth between an incident wave, a side-band incident wave and a relatively low frequency trapped wave are shown to lead to the generation of the trapped wave. Three situations are considered in detail: edge waves in a wide rectangular basin, progressive edge waves on a straight beach, and standing waves in a narrow wave tank.

1. Introduction

The formation of crescent-shaped sand-bars along laboratory beaches was shown by Bowen and Inman (1971) to be due to the presence of standing edge waves. For their explanation to be relevant to naturally occurring crescentic bars it would be necessary for the edge waves to have periods of between 30 and 60 seconds. It is thus of interest to consider how such standing edge waves are generated. A possible mechanism would be via resonant interactions between two edge waves and an incident untrapped wave (Guza and Davis, 1974). However, for this to produce edge waves of the required period the necessary periods of the incident waves would be between 15 and 30 seconds, but, typically, there is not much surface wave energy at such long periods (Sonu, 1972).

In this paper we consider a wavefield consisting of a relatively short scale incident wave, the dominant Fourier component of which is characterised by (ω,k), and a long scale trapped wave characterised by (σ,κ). The wavelength of the long scale motion is taken to be much greater than that of the short scale motion, i.e. $|k| >> |\kappa|$. This means that, except in the immediate vicinity of the shoreline, we can assume the water to be "deep" with respect to (ω,k) and "shallow" with respect to (σ,κ). Since the system is non-linear, weakly non-linear interactions give rise to the harmonics $(\omega \pm \sigma, k \pm \kappa)$ being present.

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Now, if $\omega >> \sigma$, then the harmonics $(\omega \pm \sigma, k \pm \kappa)$ are very close to (ω,k) and therefore almost satisfy the dispersion relation for incident waves. Thus, there is an almost-resonant triad consisting of the short scale wave (ω,k), one of the side-band waves $(\omega \pm \sigma, k \pm \kappa)$ and the long scale wave (σ,κ). Indeed, if the strong inequalities are relaxed then there can be exact resonance between two incident and one trapped wave (Gallagher, 1971). The existence of such a triad of waves makes it feasible for there to be a significant energy transfer from (ω,k) to (σ,κ) (Phillips, 1974). Numerical estimates made in §2.4 suggests that edge waves in the sea can be generated by the side-band mechanism.

As has been pointed out by Phillips (1974), resonant interactions between waves of different length scales can be interpreted as short waves propagating on a non-uniform current. It was shown by Longuet-Higgins and Stewart (1961) that there is not a partition between wave energy and stream energy. However, in the absence of dissipation there is a conserved property of the short waves - wave action - which can be identified with wave energy when there is no current (Bretherton and Garrett, 1968). In the present problem the current is of limited extent and from wave action considerations it can be concluded that, in the absence of dissipation, there is no net energy transfer from the short waves to the trapped waves. Specifically, the energy feed from the incident wavetrain is exactly negated by the energy extracted by the reflected waves. Through ignoring the presence of any reflected waves, the mechanism being studied in this paper relies crucially upon there being dissipation, particularly near the shoreline.

Although the generation of standing edge waves is the primary concern of this paper, the side-band mechanism need not be restricted to standing waves nor to edge waves. In §§3 and 4 respectively, we briefly consider the generation of progressive edge waves on an open beach and the generation of longitudinal standing waves in a narrow wave tank. It is hoped that this latter situation might permit laboratory testing of the side-band mechanism using apparatus of modest dimensions.

2. The generation of standing edge waves

2.1 Equations of motion

Consider a body of water, the free surface of which is at rest, $z=0$, where z is measured vertically upwards. The equations for small amplitude oscillations of an ideal fluid can be written in the form

$$\nabla^2\Phi + \Phi_{zz} = 0,$$

$$\Phi_{tt} + g\Phi_z = R \quad \text{at} \quad z = 0,$$

$$\Phi_z + \nabla h.\nabla\Phi = 0 \quad \text{at} \quad z = -h(\underset{\sim}{x}),$$

where Φ denotes the velocity potential, $h(\underset{\sim}{x})$ the depth, ∇ the horizontal gradient operator and R represents complicated nonlinear terms which are, correct to second order in the wave amplitude,

$$R = -\nabla.(\Phi_t \nabla\Phi) - \partial_t\left[\tfrac{1}{2}(\nabla\Phi)^2 + \tfrac{1}{2}\Phi_z^2 + g^{-2}\,\Phi_t\,\Phi_{ttt}\right].$$

If $W(\underset{\sim}{x},z;\,\sigma)$ represents a free standing wave mode of the undamped linear system with frequency σ, then W satisfies the equations

$$\nabla^2 W + W_{zz} = 0,\quad -\sigma^2 W + gW_z = 0 \text{ at } z = 0,\quad W_z + \nabla h.\nabla W = 0 \text{ at } z = -h(\underset{\sim}{x}).$$

For each eigenfunction W an application of Green's Theorem leads to the evolution equation

$$\left(\frac{d^2}{dt^2} + \sigma^2\right)\int_\Omega [W\Phi]_{z=0} + g\int_{\partial\Omega}\int_{-h}^{0}\left(\Phi\frac{\partial W}{\partial n} - W\frac{\partial\Phi}{\partial n}\right) = \int_\Omega [RW]_{z=0} \qquad (2.1)$$

where Ω denotes the water surface, $\partial\Omega$ the boundary of Ω and $\partial/\partial n$ denotes differentiation along the outward normal to $\partial\Omega$. If Ω is of infinite extent, then, to assure the convergence of the integrals, it may be necessary to take W to be that leaky trapped wave which decays exponentially at infinity (Longuet-Higgins, 1967). Equation (2.1) enables us to determine the effect of the nonlinearity upon the evolution of the standing waves without the necessity of explicitly evaluating the correction terms due to the nonlinearity. It corresponds to the non-secularity or integrability condition which arises in a direct solution of the governing differential equations (Bretherton and Garrett, 1968).

Let $w(z;\omega,h)$ be a local solution to the progressive wave eigenvalue problem for waves of frequency ω:

$$-(\nabla\theta)^2 w + w_{zz} = 0,$$

$$w = 1 \text{ and } w_z = \omega^2/g \text{ at } z = 0,$$

$$w_z = 0 \text{ at } z = -h(x),$$

where $|\nabla\theta|$ is the local wave-number (eigenvalue). The equation which enables us to focus our attention on the effect of nonlinearity on the ω-component of the wavefield is

$$(\partial_t^2 + \omega^2)\Phi_{z=0} - g\int_{-h}^{0} w(\nabla^2 + (\nabla\theta)^2)\Phi dz - g\nabla h.[w\nabla\Phi]_{z=-h} = R. \qquad (2.2)$$

Just as it is not essential to explicitly evaluate the nonlinear

corrections to Φ, in both equations (2.1) and (2.2) the vertical structure of Φ need not be represented exactly. This fact will be used subsequently when $w(z; \omega \pm \sigma, h)$ are approximated firstly by $w(z; \omega, h)$ and then by $w(z; \omega, \infty)$.

The model equations presented above are inadequate to describe the complicated structure of the surf zone. This problem is most easily overcome by making the following assumptions: (a) The incident short waves are not reflected at the beach, yet the trapped waves are totally reflected. (b) The wavelength of the incident waves does not change as they shoal. The consequence of these assumptions is that our approximation for Φ, and more particularly R, is grossly in error in the immediate vicinity of the shoreline. As can be seen from equation (2.1), a local error in R need not greatly effect our estimate of the growth rate of the standing wave. In the calculations given below, assumption (a) is made at the onset but the use of assumption (b) is delayed.

The assumptions (a) and (b) were also made by Gallagher (1971). For his analysis it is extremely difficult to provide a justification because there is not a marked disparity in length scales between the incident and trapped waves. Guza and Davis (1974) chose the opposite extreme of perfect reflection of the incident waves. The no-reflection boundary condition is also appropriate to situations where there is no coastal boundary, such as in the excitation of ducted waves on an undersea ridge (Buchwald, 1969).

On vertical sections of the boundary we impose the normal velocity conditions

$$\partial W/\partial n = 0, \qquad \partial \Phi/\partial n = Q,$$

where Q is non-zero on those sections of the boundary which are being used (or interpreted) as being wavemakers for the generation of the incident waves.

2.2 Side-band representation

For the interaction mechanism under consideration, the wavefield is dominated by the incident wave of frequency ω and the standing trapped wave of frequency σ. A vitally important role is also played by the side-band incident waves of frequencies $\omega \pm \sigma$. Accordingly, we write the velocity potential as

$$\Phi = \exp(i\omega t - i\theta(\underset{\sim}{x}))\Big[\Phi_o(\underset{\sim}{x},t)w(z;\omega,h) + \Phi_+(\underset{\sim}{x},t)w(z;\omega+\sigma,h)e^{i\sigma t} + \Phi_-(\underset{\sim}{x},t)w(z;\omega-\sigma,h)e^{-i\sigma t}\Big] + W(\underset{\sim}{x},z;\sigma)\eta(t)^{i\sigma t} + * + \Phi'. \tag{2.3}$$

Here Φ_o, Φ_+ and η are the slowly-varying amplitudes of the incident, side-band and the trapped waves respectively. All other contributions to the wavefield are assumed to play no significant role in the generation of the trapped waves, and are collectively represented by Φ'. The

formally largest of such neglected terms are virtual modes with frequencies 2ω, 2σ and the wave set-up which has zero frequency. Reflections on the vertical sections of the boundary $\partial\Omega$ could be represented by permitting Φ_o, $\Phi_\pm$ and θ to be multiple-valued.

Substituting for Φ in (2.2) and equating $\exp(i\omega t - i\theta)$ terms gives

$$2i\omega\Phi_{o_t} + \Phi_{o_{tt}} + ig\int_{-h}^{o} w\{2\nabla\theta.\nabla(w\Phi_o) + W\Phi_o\nabla^2\theta\}\,dz - \int_{-h}^{o} w\nabla^2(w\Phi_o)\,dz$$

$$-g\nabla h.\left[w\nabla(\Phi_o w)\right]_{z=-h} = R_o,$$

where w stands for $w(z;\omega,h)$ and R_o is the $\exp(i\omega t - i\theta)$ term in the side-band representation for the nonlinear terms. For the side-band amplitudes $\Phi_\pm$ similar equations involving $w(z;\omega\ \sigma,h)$ are obtained. We note that even though we shall only represent R correct to quadratic terms, the explicit representation for R_o in terms of $\Phi_\pm$, η is quite unwieldy.

We now make the following simplifying assumptions

$$\omega >> \sigma\ ,\quad |\nabla\theta|L >> 1\ ,\quad \omega/(L|\nabla\theta|) >> \eta_t/\eta\ ,$$

where L is a length scale appropriate to either the topographic variations or to the trapped waves. Physically the assumptions mean that the incident wave is very short and that an individual wave packet is "unaware" that the standing wave is growing in amplitude. Mathematically the assumptions mean that in the expression for R_o, and the corresponding expressions in the equation for the side-bands, derivatives of the short wave exponent $(\omega t - \theta(x))$ dominate derivatives either of the wave amplitude Φ_o, $\Phi_\pm$ η or of the mode shapes w, W. Also, we can justify approximating $w(z;\omega \pm \sigma,h)$ by $w(z;\omega,h)$. Retaining only the largest real and imaginary terms, we can simplify the equations for the progressive waves to

$$\underset{\sim}{c}.\nabla\Phi_o + (\nu + \tfrac{1}{2}\nabla.\underset{\sim}{c})\Phi_o = \left[W\,\frac{\omega^2\sigma^2}{g^2} + i\nabla\theta.\nabla W\right](\eta^*\Phi_+ + \eta\Phi_-)$$

$$+ \left[\frac{\omega^4}{g^2} - (\nabla\theta)^2\right]\frac{\sigma W}{2\omega}\,(\eta^*\Phi_+ - \eta\Phi_-), \tag{2.4a}$$

$$\underset{\sim}{c}.\nabla\Phi_+ + i\sigma\Phi_+ + (\nu + \tfrac{1}{2}\nabla.\underset{\sim}{c})\Phi_+ = \left[W\frac{\omega^2\sigma^2}{g^2} + i\nabla\theta.\nabla W\right]\eta\Phi_o$$

$$- \left[\frac{\omega^4}{g^2} - (\nabla\theta)^2\right]\frac{\sigma W}{2\omega}\eta\Phi, \tag{2.4b}$$

$$\underset{\sim}{c}.\nabla\Phi_- - i\sigma\Phi_- + (\nu + \tfrac{1}{2}\nabla.\underset{\sim}{c})\Phi = \left[W\frac{\omega^2\sigma^2}{g^2} + i\nabla\theta.\nabla W\right]\eta^*\Phi_o$$

$$+ \left[\frac{\omega^4}{g^2} - (\nabla\theta)^2\right]\frac{\sigma W}{2\omega}\eta^*\Phi_o, \tag{2.4c}$$

where $\underset{\sim}{c}$ is the group velocity of the incident waves of frequency ω, and the ν terms are an empirical representation of the damping. The left-hand side terms could have been anticipated from the fact that in the absence of currents and damping, the wave energy of the short waves propagates at the group velocity.

If the low frequency component of the wavemaker motion is neglected and if W is normalised:

$$\int_\Omega \left[W^2\right]_{z=0} = \ell^{-2},$$

then substituting (2.3) into (2.1), and equating coefficients of $e^{i\sigma t}$, we obtain the following simplification of the standing wave equation

$$\eta_t + \nu'\eta = \int_{\Omega_1} (\Phi_o^*\Phi_+ + \Phi_o\Phi_-^*)\left[\tfrac{1}{2}W\left(\frac{\omega^4}{g^2} - (\nabla\theta)^2\right) + \frac{i\omega}{\sigma}\nabla\theta.\nabla W\right]\ell^2$$

$$- \frac{i\omega\ell^2}{\sigma}\int_{\partial\Omega_1} (\Phi_o^*\Phi_+ + \Phi_o\Phi_-^*)\frac{\partial\theta}{\partial n}W\Big|_{z=0}$$

$$+ \int_{\Omega_1} (\Phi_o^* \Phi_+ - \Phi_o \Phi^*) \; 2 \frac{\omega^3 \sigma}{g^2} W|_{z=0} \; \ell^2 - \frac{i\ell^2}{2\sigma} \int_{\Omega-\Omega_1} |RW|_{z=0} \qquad (2.5)$$

Here, ν' is an empirical representation of damping, Ω_1 denotes that region of the water surface in which (2.3) is thought to be an adequate description of the waves, and in Ω_1 we have only retained the largest real and largest imaginary contributions to R. If θ were multiple-valued (due to there being reflections off vertical portions of $\partial\Omega$) then different branches of θ would contribute separately to the nonlinear terms.

To make further progress we must either give a description of the waves which is valid in the surf zone $\Omega - \Omega_1$ or we must be able to neglect the $\Omega - \Omega_1$ term in (2.5). Here we choose the latter alternative. It is reasonable to assume that the effects of both reflection and strong nonlinearities is to keep the interaction term R bounded. Thus to ignore the effect of the $\Omega - \Omega_1$ region it suffices that it has small extent relative to the interaction region. The previously assumed disparity of length scales between the incident and trapped waves adds credence to this further assumption.

We observe that all the retained quadratic terms in equations (2.4) and (2.5) involve derivatives of the short wave exponent $(\omega t - \theta)$. Thus of the formally largest neglected terms in (2.3) it is only the $2(\omega t - \theta)$ virtual modes which would lead to any change in equations (2.4) or (2.5). A calculation akin to that of Stokes (1849) leads to the extra terms

$$i\lambda\Phi_o^2\Phi_o^* \;, \quad i\lambda\Phi_o^2\Phi_-^*, \quad i\lambda\Phi_o^2\Phi_+^*$$

on the right-hand sides of equations (2.4a,b,c), where

$$\lambda = \tfrac{1}{8}\omega^3 k^2 g^{-2} \; (1 + \tfrac{1}{8} \operatorname{Coth} 4\,kh).$$

These terms play a very minor role in the energy transfer mechanism and for simplicity we continue to ignore them.

2.3 Onset of instability

To explain the onset of the instability we assume that the system is started from a state of near rest and we can thus regard the sidebands Φ_+ and the standing wave η as being of very small amplitude. Thus the incident wave amplitude Φ_o approximately satisfies the linear equation

$$\underset{\sim}{c}\cdot\nabla\Phi_0 + (\nu + \tfrac{1}{2}\nabla\cdot\underset{\sim}{c})\Phi_0 = 0, \tag{2.6}$$

and the perturbation quantities Φ_+, Φ_-, η satisfy couple linear equations. The form of equations (2.5) leads us to define the sum and difference terms:

$$\eta|\Phi_0|^2 S = \Phi_0^*\Phi_+ + \Phi_0\Phi_-^*, \quad \eta|\Phi_0|^2 D = \Phi_0^*\Phi_+ - \Phi_0\Phi_-^*,$$

where during the initial stages of the instability S and D satisfy the first order differential equations

$$\begin{aligned} \underset{\sim}{c}\cdot\nabla S + i\sigma S &= 2\,\omega^2\sigma^2 W|_{z=0}, \\ \underset{\sim}{c}\cdot\nabla D + i\sigma D &= 2i\nabla\theta\cdot\nabla W|_{z=0} - (\frac{\omega^4}{g^2} - (\nabla\theta)^2)\,\frac{\sigma}{\omega}W|_{z=0} \end{aligned} \tag{2.7}$$

The boundary conditions for these equations are that S = D = 0 far away from the interaction region. It now follows from equation (2.5), with the $\Omega - \Omega_1$ term neglected, that η grows exponentially with time, the exponent being

$$\begin{aligned} -\nu' &+ \int_{\Omega_1} |\Phi_0|^2\, S\left[\tfrac{1}{2}W(\frac{\omega^4}{g^2} - (\nabla\theta)^2) + i\,\frac{\omega}{\sigma}\,\nabla\theta\cdot\nabla W\right]\ell^2_{z=0} \\ &+ \int_{\Omega_1} |\Phi_0|^2\, D\; 2\,\frac{\omega^3\sigma}{g^2}\,W|\ell^2_{z=0} - \int_{\partial\Omega_1} |\Phi_0|^2\,\frac{i\omega}{\sigma}\,SW|_{z=0}\,\frac{\partial\theta}{\partial n}\,\ell^2. \end{aligned} \tag{2.8}$$

The condition for there to be an instability is that the time exponent (2.8) should have a positive real part.

2.4 Uniformly sloping beach

In the representation (2.3) no account is taken of any reflection of the short waves. As a consequence, even if there is no wave-breaking the description is inappropriate near the shoreline and predicts spurious singularities. For our present purposes these singularities are particularly unfortunate in that the expression (2.8) is sensitive to our choice of Ω_1. It is to avoid this dilemma that we invoke the assumption (b) referred to in §2.1. Thus we assume that the

beach angle is so steep that the deep water description for the incident waves is appropriate in a region Ω_1 which extends to within the order of one wavelength of the shoreline. Since the modified representation for Φ is free from singularities (though still not valid in $\Omega - \Omega_1$) we can also approximate Ω_1 by the entire water region Ω in the integrals (2.8).

The only geometry for which an exact solution for a linear trapped wave is known is a beach of constant slope in a channel with parallel vertical walls which are orthogonal to the beach. The exact solutions involve a superposition of exponentials (Ursell, 1952), the normalised solution for the lowest mode being

$$W = \left[2(\cos\alpha)^{\frac{1}{2}}\,\pi^{\frac{1}{2}}\right] \cos \ell y \exp(-\ell x\cos\alpha + \ell z\sin\alpha),\quad \sigma^2 = g\ell\sin\alpha,$$

where α is the beach angle and π/ℓ is the distance between the vertical walls. If we are to retain the convenience of working only with exponential functions, then it is necessary that the coefficients c, ν in equations (2.6-2.7) are constant. Fortunately, this is indeed the case when the beach angle is sufficiently steep (i.e. $(\sigma/\omega)^2 \ll \tan^2\alpha$) that the deep water approximation (b) can be used. For incident waves normal to the beach the appropriate solutions to equations (2.6-2.8) for the lowest mode edge waves are

$$\Phi_0 = A \exp(\nu x/c) \quad \text{with} \quad \underset{\sim}{c} = (-c,\ 0,\ 0),$$

$$S = \frac{2\omega^2\ell \sin\alpha\, W|_{z=0}}{g(c\ell\cos\alpha + i\sigma)} \qquad D = \frac{i2\omega^2\ell\cos\alpha\, W|_{z=0}}{g(c\ell\cos\alpha + i\sigma)}$$

$$\text{exponent} \doteq \nu' + |A|^2 \frac{2\omega^5\ell\cos\alpha}{g^3}\left[1 + i(c\ell\cos\alpha/\sigma)\right]\left[\frac{\ell\cos\alpha + 2(\nu/c)}{\ell\cos\alpha - (\nu/c)}\right]$$

Thus, if the wave steepness $\omega^3|A|g^{-2}$ exceeds the critical value

$$\left[\frac{\nu'\omega}{2g\ell\cos\alpha}\left\{\frac{\ell\cos\alpha - (\nu/c)}{\ell\cos\alpha + 2(\nu/c)}\right\}\right]^{\frac{1}{2}} \qquad (2.9)$$

then the edge wave will grow in amplitude.

As a numerical example of the estimate (2.9), to a moderately large laboratory size situation we specify the values

$$\alpha = \pi/6, \quad \ell = \pi m^{-1}, \quad \sigma \sim 4s^{-1}, \quad \omega \sim 20\ s^{-1}, \quad \nu/c \sim 10^{-1}m^{-1}.$$

Although the value of ν/c corresponds to a dirty surface, we note that $(\nu c/\ell)$ is essentially negligible. Guza and Davis (1974) show that a laminar model of edge wave damping gives

$$\nu' = \bar{\nu}^{\frac{1}{2}}\ell\sigma^{\frac{1}{2}} \cot\alpha \quad 2^{-\frac{1}{2}},$$

where $\bar{\nu}$ is the kinematic viscosity. Thus we can estimate that $\nu' \sim 7 \times 10^{-3}s^{-1}$, critical steepness ~ 0.05.

Hence the instability would appear to be realisable in a laboratory situation.

If we regard σ/ω, $\bar{\nu}$ as being fixed, then we can use the above results to determine the dependence of the critical steepness upon the beach angle α and wavenumber ℓ:

$$\text{critical steepness} \quad \sim \ell^{\frac{3}{8}}\alpha^{\frac{1}{8}}.$$

Thus an increase of the length scale, i.e. a decrease of ℓ, lowers the critical steepness. Regrettably, the deep water approximation is not applicable to naturally occuring beaches where $(\tan\alpha)^2$ is typically 10^{-3} or less. In such cases there seems no alternative but to allow for shoaling. Within the framework of the present calculations, this would entail choosing Ω_1 as to avoid the spurious singularities. (The authors hope to present at a later date a more complicated calculation procedure which avoids the singularities).

3. Progressive edge waves.

For progressive trapped waves the longshore direction plays the role of a second time-coordinate. Although the mathematical consequences of this change of role are numerous, they are individually quite minor and we can follow the pattern of calculations developed in §2.

On the assumptions that the coastal curvature is small relative to the wavenumber of the trapped waves and that the seaward depth topography varies very slowly in the longshore y-direction, the local eigenvalue problem for trapped waves of frequency σ and longshore wavenumber ψ_y is

$$W_{xx} + W_{zz} - \psi_y^2 W = 0, \qquad -\sigma^2 W + gW_z = 0 \text{ at } z = 0,$$

$$W_z + h_x W_x = 0 \text{ at } z = -h.$$

The corresponding integral form of the equations of motion is

$$(\partial_t^2 + \sigma^2) \int_0^\infty \left[W\Phi\right] dx - g \int_0^\infty \int_{-h}^\infty W(\partial_y^2 + \psi_y^2)\Phi dz dx + g \int_0^\infty h_y \left[W\Phi\right]_{z=-h} dx \tag{3.1}$$

$$+ \left[\int_{-h}^0 (\Phi \frac{\partial W}{\partial x} - W\partial\Phi \frac{\partial \Phi}{\partial x} \, dz \right]_0^\infty = \int_0^\infty \left[RW\right]_{z=0} dx.$$

For the progressive waves the eigenvalue problem and integral version of the nonlinear equations remain as in §2.1, but it is natural to regard the longshore dependence of the phase measurement θ as being imposed by conditions far from the shore.

The time-like character of the y-coordinate leads us to modify the side-band representation of the wavefield:

$$\Phi = e^{i(\omega t-\theta)} \left[\Phi_0(x,y,t)w(z;\omega,h)+\Phi_+(x,y,t)w(z;\omega+\sigma,h)e^{i(\sigma t-\psi)}\right.$$
$$\left.+\Phi_-(x,y,t)w(z;\omega-\sigma,h)e^{-i(\sigma t-\psi)}\right]+W(x,z;\sigma,y)\eta(y,t)e^{i(\sigma t-\psi)}+*+\Phi'. \tag{3.2}$$

Thus, the y-dependence on the length scale of ψ_y^{-1} is represented explicitly by the ψ exponents, and possible very slow y-dependence associated with any instability is represented via the amplitude factors Φ_0, Φ_t, η. This representation is assumed to be accurate in a region $(a, \infty)^t$ which excludes the shoreline.

Making the same simplifying assumptions as in §2.2, i.e.

$$\omega >> \sigma, \quad |\nabla\theta| L >> 1, \quad \omega/(L|\nabla\theta|) >> \eta_t/\eta,$$

we can simplify the equations for the incident waves to

$$c_1\Phi_{0_x} + (\nu + \tfrac{1}{2}c_{1_x})\Phi_0 = \left[W \frac{\omega^2\sigma^2}{g^2} + i\theta_x W_x\right](\eta^*\Phi_+ + \eta\Phi_-)$$

$$+\left[\left(\frac{\omega^4}{g^2} - (\nabla\theta)^2\right)\frac{\sigma W}{2\omega} - \theta_y\psi_y W\right](\eta^*\Phi_+ - \eta\Phi_-), \tag{3.3a}$$

$$c_1\Phi_{+x} + i(\sigma - c_2\psi_y)\Phi_+ + (\nu + \tfrac{1}{2}c_{1x})\Phi_+ = \left[W\frac{\omega^2\sigma^2}{g^2} + i\theta_x W_x\right]\eta\Phi_o$$

$$-\left[\left(\frac{\omega^4}{g^2} - (\nabla\theta)^2\right)\frac{\sigma W}{2\omega} - \theta_y\psi_y W\right]\eta\Phi_o, \tag{3.3b}$$

$$c_1\Phi_{-x} - i(\sigma - c_2\psi_y)\Phi_- + (\nu + \tfrac{1}{2}c_{1x})\Phi_- = \left[W\frac{\omega^2\sigma^2}{g^2} + i\theta_x W_x\right]\eta^*\Phi_o$$

$$+\left[\left(\frac{\omega^4}{g^2} - (\nabla\theta)^2\right)\frac{\sigma W}{2\omega} - \theta_y\psi_y W\right]\eta^*\Phi_o, \tag{3.3c}$$

where $(c_1, c_2, 0)$ is the group velocity of the incident waves of frequency ω and the ν terms are an empirical allowance for the damping. The corresponding equation for the trapped wave is

$$\eta_t + c'\eta_y + (\nu' + \tfrac{1}{2}c'_y)\eta = \frac{\sigma^2}{g}\int_a^\infty (\Phi_o^*\Phi_+ + \Phi_o\Phi_-^*)\left[\tfrac{1}{2}W\left(\frac{\omega^4}{g^2} - (\nabla\theta)^2\right)\right.$$

$$\left. + i\frac{\omega\theta_x W_x}{\sigma} - \frac{\omega\theta_y W\psi_y}{\sigma}\right]dx \quad + \frac{\sigma^2}{g}\int_a^\infty (\Phi_o^*\Phi_+ - \Phi_o\Phi_-^*)\frac{2\omega^3\sigma}{g^2}W\Big|_{z=0}dx$$

$$+ \frac{i\sigma\omega}{g}\left[(\Phi_o^*\Phi_+ + \Phi_o\Phi_-^*)W\Big|_{z=0}\right]\theta_x - \frac{i\sigma}{g}\int_o^a [RW]_{z=0}\,dx. \tag{3.4}$$

where c' is the longshore group velocity of the trapped waves and W is normalised:

$$\int_o^\infty |W^2|_{z=0}\,dx = g\sigma^{-2}.$$

We shall assume that the representation (3.2) applies sufficiently close to the shore that the (0,a) term in equation (3.4) can be neglected. As noted in §2.2 the particularly simple form of the left-hand-side terms is related to the fact that in the absence of interactions or damping the wave energy of the incident and trapped waves propagate at their respective group velocities.

In considering the onset of instability it is again convenient to introduce the sum and difference terms S and D. To determine the development of η it is convenient to use axes moving with the local group velocity c'. The evolution can be assumed to be exponential only if the topography and the incident wave amplitude Φ_0 are both independent of y. For the special case of edge waves on a uniformly sloping beach of large angle (i.e. $\tan^2\alpha >> (\alpha/\omega)^2$) it is possible to obtain an analytic description of the onset condition

$$\text{critical steepness} \doteq \left(1 - \frac{\ell c_2}{\sigma}\right)^{\frac{1}{2}} \left[\frac{\nu'\omega}{2g\ell\cos\alpha} \left\{\frac{\ell\cos\alpha - \nu/c_1}{\ell\cos\alpha + 2\nu/c_1}\right\}\right]^{\frac{1}{2}} \tag{3.5}$$

The group velocity c_2 of the incident waves is less than the phase velocity σ/ℓ of the trapped wave. Thus, the extra factor in (3.5) as opposed to (2.9) does not make a substantial reduction in the numerical estimates.

4. Standing waves in a narrow tank

Laboratory experiments permit much more stringent testing of theories than field measurments, but edge wave experiments can make severe demands upon facilities due to the large area of water involved (Bowen and Inman, 1969). With this in mind, we now show the side-band theory can be used to predict the growth of a standing wave in a narrow channel. Numerical estimates of the onset conditions are obtained for the particular case of a tank with a sloping bottom and a plane wave-maker positioned at one end. As with §3, the pattern of calculations developed in §2 can again be followed with only a few minor alterations.

We consider here a tank of length L and breadth B such that L >> B. Axes are chosen so that x is measured along the tank, y across the tank and z vertically upwards. In this case we have $w(y,z;\omega,h)$ as a local solution to the progressive wave eigenvalue problem:

$$-\theta_x^2 w + w_{yy} + w_{zz} = 0, \quad w_z = \frac{\omega^2}{g} w \quad \text{at } z = 0,$$

$$w_z = 0 \quad \text{at} \quad z = -h(x), \quad w_y = 0 \quad \text{at } y = 0, B,$$

with the normalisation

$$\int_0^B \left[w^2\right]_{z=0} dz = B.$$

For each position x, the local energy transport equation for the progressive waves is

$$(\partial_t^2 + \omega^2) \int_0^B \left[w\Phi\right]_{z=0} dy - \int_0^B \int_{-h}^0 w(\partial_x^2 + \theta_x^2)\Phi dz dy - g\int_0^B \left[wh_x\Phi_x\right]_{z=-h} dy$$

$$= \int_0^B \left[wR\right]_{z=0} dy. \qquad (4.1)$$

Following §2.2, the side-band representation of the wavefield is written as

$$\Phi = \exp(i\omega t - i\theta(x))\left[\Phi_0(x,t)w(y,z;\omega,h) + \Phi_+(x,t)w(y,z;\omega+\sigma,h)e^{i\sigma t}\right.$$

$$\left. + \Phi_-(x,t)w(y,z;\omega-\sigma,h)e^{-i\sigma t}\right] + W(x,z;\sigma)\eta(t)e^{i\sigma t} + *+\Phi'. \qquad (4.2)$$

We again make the assumptions

$$\omega >> \sigma \qquad |\theta_x|L >> 1, \qquad \omega >> L|\theta_x|\eta_t/\eta.$$

The resulting equations for the progressive waves are

$$c\Phi_{0_x} + (\nu + \tfrac{1}{2}c_x)\Phi_0 = R_0,$$

$$c\Phi_{+_x} + i\sigma\Phi_+ + (\nu + \tfrac{1}{2}c_x)\Phi_+ = R_+,$$

$$c\Phi_{-_x} - i\sigma\Phi_- + (\nu + \tfrac{1}{2}c_x)\Phi_- = R_-,$$

where

$$R_0 = B^{-1}\int_0^B w\left[w\left(W\frac{\omega^2\sigma^2}{g^2} + i\theta_x W_x\right)(\eta^*\Phi_+ + \eta\Phi_-)\right.$$

$$\left. + \frac{\sigma W}{2\omega}\left(\frac{\omega^4 w}{g^2} - \theta_x^2 w + w_{yy}\right)(\eta^*\Phi_+ - \eta\Phi_-)\right]_{z=0} dy,$$

$$R_+ = B^{-1}\int_0^B w\Big[w\left(W\frac{\omega^2\sigma^2}{g^2} + i\theta_x W_x\right)\eta\Phi_0 - \frac{\sigma W}{2\omega}\left(\frac{\omega^4 w}{g^2} - \theta_x^2 w + w_{yy}\right)\eta\,\Phi_0\Big]_{z=0} dy,$$

$$R_- = B^{-1}\int_0^B w\Big[w\left(W\frac{\omega^2\sigma^2}{g^2} + i\theta_x W_x\right)\eta^*\Phi_0 + \frac{\sigma W}{2\omega}\left(\frac{\omega^4 w}{g^2} - \theta_x^2 w + w_{yy}\right)\eta^*\Phi_0\Big]_{z=0} dy.$$

The corresponding equation for the standing wave is

$$\eta_t + \nu'\eta = \ell B^{-1}\int_0^B\int_a^L \Big[\tfrac{1}{2}W\left(\frac{\omega^4 w^2}{g^2} - \theta_x^2 w^2 + w_y^2\right) + i\frac{\omega}{\sigma}\theta_x W_x\Big]_{z=0}(\Phi_0^*\Phi_+ + \Phi_0\Phi_-^*)\,dxdy$$

$$+ \ell B^{-1}\int_0^B\int_a^L \frac{2\omega^3\sigma}{g^2} w^2 W\Big|_{z=0}(\Phi_0^*\Phi_+ - \Phi_0\Phi_-^*)\,dxdy$$

$$+ \ell B^{-1}\int_0^B \Big[\frac{i\omega}{\sigma}\theta_x W\Big|_{z=0} w^2(\Phi_0^*\Phi_+ + \Phi_0\Phi_-^*)\Big]_{x=a} dy - \frac{iB^{-1}}{2\sigma}\int_0^a [RW]_{z=0}\,dy,$$

where we have assumed the normalisation

$$\int_0^B\int_0^L [W^2]_{z=0}\,dxdy = B\ell^{-1}.$$

If the progressive and standing waves are plane, i.e. w and W are independent of y, then the above equations reduce to a one-dimensional form of equations (2.4) and (2.5). Thus, provided we can neglect the effect of the nearshore region (0,a), we can directly apply the analysis of §2.3.

For a shallow tank of depth αx an approximate normalised solution for the standing wave is

$$W = KJ_o(2\sqrt{\ell x}),$$

where

$$J_o'(2\sqrt{\ell L}) = 0, \quad \sigma^2 = \alpha g \ell, \quad K^2 = \frac{1}{\ell \int_a^L J_o^2(2\sqrt{\ell x})\,dx}$$

Following the onset-of-instability calculation of §§2.3,2.4 we make the deep water assumption i.e. $(\sigma/\omega)^2 << \tan^2\alpha$, and define the sum and difference terms to be

$$S = \frac{2\omega^2\sigma^2}{cg^2} Ke^{i\sigma x/c} \int_x^L J_o(2\sqrt{\ell x})e^{-i\sigma x/c}\,dx,$$

$$D = \frac{-2i\theta_x}{c} Ke^{i\sigma x/c} \int_x^L \ell^{\frac{1}{2}}x^{-\frac{1}{2}}J_1(2\sqrt{\ell x})e^{-i\sigma x/c}\,dx,$$

where $\underset{\sim}{c} = (-c,0,0)$. Hence an expression for the onset condition can be found, and the standing wave will grow if the steepness at the wavemaker exceeds the critical value

$$\left[\frac{\omega\nu' c \int_o^L J_o^2(2\sqrt{\ell x})\,dx}{2\sigma g\left[I_1 + 2I_2 + I_3\right]}\right]^{\frac{1}{2}},$$

where the real integrals I_1, I_2, I_3 are defined:

$$I_1 = -\int_o^L \int_x^L \ell^{\frac{1}{2}}x^{-\frac{1}{2}}e^{-2\nu(L-x)/c}J_1(2\sqrt{\ell x})J_o(2\sqrt{\ell \bar{x}})\sin(\sigma(\bar{x}-x)/c)\,d\bar{x}dx,$$

$$I_2 = -\int_o^L \int_x^L \ell^{\frac{1}{2}}\bar{x}^{-\frac{1}{2}}e^{-2\nu(L-x)/c}J_1(2\sqrt{\ell x})J_o(2\sqrt{\ell \bar{x}})\sin(\sigma(\bar{x}-x)/c)\,d\bar{x}dx,$$

$$I_3 = \int_o^L e^{-2\nu L/c}J_o(2\sqrt{\ell x})\sin(\sigma x/c)\,dx.$$

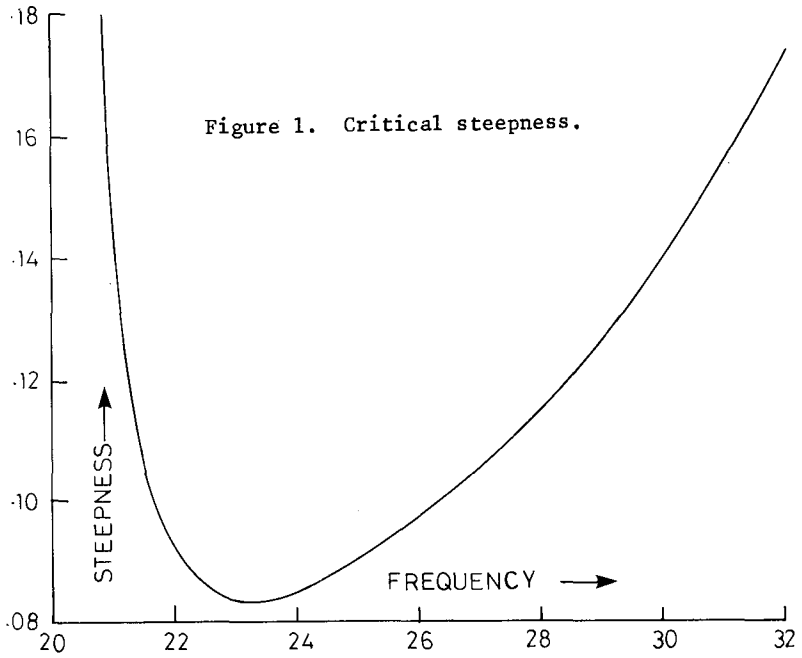

Figure 1. Critical steepness.

If we consider a particular situation of a narrow tank of length 4m, with a beach of slope $\alpha = 0.1$, the numerical integration of the above expression leads to the graph (fig. 1) of the wave steepness at the wavemaker. The damping for the standing wave was taken to be constant $\nu = 0.01 s^{-1}$, and the damping for the progressive waves was calculated using the formula

$$\nu = \tfrac{1}{2} \frac{\omega^2}{g^2} \left(\frac{\omega \times \text{viscosity}}{2} \right)^{\frac{1}{2}} , \text{ with viscosity} = 1.3 \times 10^{-6} m^2 s^{-1}.$$

This expression for ν corresponds to the "dirty water" limit of an inextensible surface film. Results for frequencies below $\omega = 20$ have not been presented because the results are dominated by the beach contribution I_3, and it may not be justifiable to neglect the effects of shoaling and wave reflèction. For example, at $\omega = 20.5$ the other two contributions are exactly cancelled out. However, by $\omega = 28$ the beach contribution is less than one percent. The narrowness of the frequency band in which the low frequency mode can be generated is due to the difficulties in simultaneously satisfying the requirements that the short waves penetrate far enough to be "aware" of the nonuniformity of the low frequency mode, and yet do not penetrate so far that there is significant cancelling of contributions from successive crests of the standing wave.

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