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Applications of Borel Distribution and Mittag-Leffler Function on a Class of Bi-Univalent Functions

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Abstract. In this study, we present a novel class of bi-univalent functions that incorporates the Borel distribution and the Mittag-Leffler function within the open unit disk $\mathbb{D}$. This is achieved by employing the q-analog of the hyperbolic tangent function in conjunction with the Hadamard product. The primary goal is determining the initial coefficient bounds for functions that fall within this newly defined class. Additionally, we explore the classical Fekete-Szegö functional problem as it pertains to these functions. Moreover, we highlight several known corollaries that arise from specific selections of the parameters associated with this class.

1. Introduction

The concept of probability distributions related to the potential outcomes of a random variable is a fundamental concept in the fields of statistics and probability theory. This concept is widely applied to describe and model various real-life events. The significance of specific distributions, along with the random experiments they pertain to, is underscored by the practice of giving them unique designations. Within probability theory, the geometric distribution is particularly useful for determining the number of trials needed to achieve a successful result in a random experiment that presents two possible outcomes.

Moreover, discrete probability distributions are crucial for analyzing outcomes in countable sample spaces and are widely used in fields like statistics, mathematics, and computer science. The Borel distribution, named after Emile Borel, is particularly important in reliability theory and lifetime analysis, providing a framework for modeling rare events. By utilizing the Borel

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distribution, researchers can effectively study discrete events, improving decision-making and insights based on probability principles.

In addition, a discrete random variable λ is said to adhere to a Borel distribution with the parameter $\lambda \in [0,1]$ if its probability mass function can be represented in the following manner

$$P(\lambda = n) = \frac{(\lambda n)^{n-1} e^{-\lambda n}}{n!}, \text{ where } n = 0, 1, 2, 3, \cdots$$

The Mittag-Leffler functions find widespread applications across numerous fields, such as fractional differential equations, stochastic systems, dynamical systems, statistical distributions, and chaotic systems. Moreover, the Mittag-Leffler function naturally appears in the solutions to fractional differential and integral equations. It plays a significant role in exploring the fractional generalization of the kinetic equation, as well as in the analysis of random walks, super-diffusive transport, and the examination of complex systems.

In this paper, we have implored the use of convolution of the Taylor-Macluarin series representation of Borel distribution and Mittag-Leffler function to establish our class. Let $\mathcal{H}$ be the collection of all functions $f(z)$ that are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. In this context, these functions are subject to the normalization conditions $f(0) = 1 - f'(0) = 0$. The study of such functions contributes significantly to a deeper comprehension of complex analysis and its various applications. Moreover, any function f that is a member of the set $\mathcal{H}$ can be written in the following specific form

$$f(\zeta) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \text{where } z \in \mathbb{D}. \tag{1.1}$$

Let f and g be analytic functions within the open unit disk $\mathbb{D}$. We say that f is subordinated to g in the open unit disk $\mathbb{D}$, denoted as $f(z) \prec g(z)$ for all $z \in \mathbb{D}$, if we can find a Schwarz function w such that $h(0) = 0$ and $|h(z)| < 1$ for every $z \in \mathbb{D}$, fulfilling the condition $f(z) = g(h(z))$ for all $z \in \mathbb{D}$. This relationship between f and g is a crucial concept in complex analysis, offering a means to compare the behaviors of two analytic functions within the unit disk. Importantly, when the function g is univalent on $\mathbb{D}$, the condition $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. For additional insights and in-depth discussions regarding the Subordination Principle, readers are encouraged to consult the monographs [12], [13], [29], [31] and [35]. These references offer thorough explanations and applications of this principle within the realms of complex analysis and geometric function theory.

The Hadamard product, also referred to as convolution, of two analytic functions $f(z)$ as described in Equation (1.1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is expressed as follows:

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Moreover, the convolution operation provides a deeper mathematical exploration and enhances our understanding of the geometric and symmetric properties of functions within the space $\mathcal{H}$. Its significance in operator theory and geometric function theory is well-established and thoroughly discussed in the available literature. For those seeking further insights into convolution within geometric function theory, we recommend consulting the monographs [7] and [12], as well as the articles [34], [37], and the associated references therein.

In this study, the notation $\mathcal{S}$ denotes the collection of functions that are univalent within the open unit disk $\mathbb{D}$ and are members of the set $\mathcal{H}$. It is well-known that univalent functions are injective, which implies their invertibility. However, The inverse functions might not be valid across the whole unit disk $\mathbb{D}$. In particular, the Koebe one-quarter Theorem highlights that the image of $\mathbb{D}$ through any function $f \in \mathcal{S}$ includes the disk $D(0, 1/4)$, which is centered at the origin and has a radius of $1/4$. As a result, for every function $f \in \mathcal{S}$, there is an inverse function $f^{-1} = g$ that can be defined as follows

$$g(f(\zeta)) = \zeta, \quad \zeta \in \mathbb{D}$$

$$f(g(\eta)) = \eta, \quad |\eta| < r(f); \;\; r(f) \geq 1/4.$$

Moreover, the inverse function is given by

$$g(\eta) = \eta - a_2\eta^2 + (2a_2^2 - a_3)\eta^3 - (5a_2^3 - 5a_2a_3 + a_4)\eta^4 + \cdots. \tag{1.2}$$

A function $f \in \mathcal{H}$ is called bi-univalent if it maintains univalence in the unit disk $\mathbb{D}$, along with its inverse f^{-1}. Therefore, Σ is identified as the set of all bi-univalent functions within $\mathcal{H}$, as outlined in Equation (1.1). The class Σ includes, for instance, the following functions:

$$z(1-z)^{-1}, \quad -\log(1-z), \quad \sqrt{\log(1+z) - \log(1-z)}.$$

The Koebe function, $\dfrac{2z - z^2}{2}$ and $\dfrac{z}{1-z^2}$, are not part of the class Σ. For those interested in learning more about univalent and bi-univalent functions, we recommend checking out the articles [24], [26], [32], as well as the monographs [12] and [15], along with the references included in those works.

Research in geometric function theory reveals complex relationships between function coefficients and their geometric properties. By examining constraints on the modulus of these coefficients, scholars enhance their understanding of function behavior within the mathematical framework. This approach not only deepens comprehension of geometric function theory but also encourages further exploration in the field. For instance, in the class $\mathcal{S}$, the modulus of the coefficient a_n is limited by the integer n, providing valuable insights into the geometric characteristics of these functions. Specifically, restrictions on the second coefficients in class $\mathcal{S}$ yield important information about growth and distortion bounds.

The investigation into the coefficient-related characteristics of functions within the bi-univalent class Σ began in the 1970s. A pivotal contribution was made by Lewin in 1967 [24], who analyzed the bi-univalent function class and established a limit for the coefficient $|a_2|$. Subsequently, in 1969, Netanyahu's research [32] identified that the maximum value of $|a_2|$ is $\frac{4}{3}$ for functions classified under Σ. Additionally, Brannan and Clunie, in 1979 [9], proved that for functions in this category, the inequality $|a_2| \leq \sqrt{2}$ is valid. This research has established a foundation for studies on coefficient bounds of bi-univalent function subclasses. However, there remains a significant gap in understanding the general coefficients $|a_2|$ for $n \geq 4$. Estimating these coefficients, particularly $|a_n|$, remains unresolved in geometric function theory, indicating the complexity of bi-univalent functions and the need for further investigation into their behavior in higher dimensions.

In 1933, Fekete and Szegö advanced the study of univalent functions by establishing the maximum value of $|a_3 - \lambda a_2^2|$ for λ between 0 and 1. This led to the Fekete-Szegö problem, which aims to maximize the functional $\Psi_\lambda(f) = a_3 - \lambda a_2^2$ for functions in the class $\mathcal{H}$, with λ as any complex number. The Fekete-Szegö functional and its related coefficient estimations have since attracted considerable attention from numerous researchers in the field. Notable contributions can be found in articles such as [3], [4], [5], [10], [11], [14], [16], [18], [26], [28], [36], along with the references cited therein. These investigations have significantly enhanced the comprehension of the Fekete-Szegö problem and its relevance within the domain of geometric function theory.

2. Preliminaries and Lemmas

The information presented in this section are essential for understanding the key conclusions of this paper. In a recent study, Wanas and Khuttar [39] put forth a power series defined by coefficients that reflect the probabilities associated with the Borel distribution, which can be articulated in the following manner:

$$\mathcal{B}_\lambda(z) = z + \sum_{n=2}^{\infty} \frac{(\lambda(n-1))^{n-2} e^{-\lambda(n-1)}}{\Gamma(n)} z^n,$$

where the parameter $0 \leq \lambda \leq 1$ and $z \in \mathbb{D}$. This power series converges throughout the whole complex plane, as demonstrated by the well-known ratio test. For additional details regarding Borel distribution and its applications in geometric function theory, interested readers are encouraged to refer to the articles [2], [6], [23], [38] and the related references mentioned within these articles.

In the year 1903, Mittag-Leffler [30] defined a special function named after him as follows

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+n\alpha)}, \text{ where } z \in \mathbb{C}, \text{ and } \Re\{\alpha\} > 0.$$

This last series converges in the whole complex plane for all values of $\Re(\alpha) > 0$ and diverges for $z \neq 0$ when $\Re\{\alpha\} < 0$. Additionally, when $\Re\{\alpha\} = 0$, the radius of convergence is given by $e^{\pi/2|Im(z)|}$. In the year 1905, Wiman ([40], [41]) introduced and studied the Mittag-Leffler function of two parameters as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta+n\alpha)}, \text{ where } z \in \mathbb{C}, \ \Re\{\alpha\} > 0, \Re\{\beta\} > 0.$$

The Mittag-Leffler function of two parameters was later studied by Agarwal and Humbert, see for example [1], [21] and [22]. It is important to note that Mittag-Leffler functions represent fractional extensions of fundamental functions. The function $E_{\alpha,\beta}$ includes a variety of well-known functions as particular instances, among others are the following

$$E_{1,1}(\zeta) = e^{\zeta},\ E_{1,2}(\zeta) = \frac{e^{\zeta}-1}{\zeta},\ E_{2,1}(\zeta) = \zeta\cosh\sqrt{\zeta},$$

$$E_{2,2}(\zeta) = \sqrt{\zeta}\sinh\sqrt{\zeta},\ \text{and}\ E_{2,3}(\zeta) = 2\left(\cosh\sqrt{\zeta}-1\right).$$

It is important to highlight that the two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ is not included in the set $\mathcal{H}$. In the year 2016, Bansal and Prajapat [8] introduced the normalized Mittag-Leffler function $M_{\alpha,\beta}$, which is defined in the following manner:

$$M_{\alpha,\beta}(z) = \Gamma(\beta)zE_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha(n-1))}z^n,$$

where z, α, β are complex numbers, $\beta \neq 0, -1, -2, -3, \cdots$, $\Re\{\alpha\} > 0$, $\Re\{\beta\} > 0$. For additional insights into Mittag-Leffler functions and their various applications, we recommend that readers consult the works of [17], [19], [20], [25], [27], [33], [42] and the references provided therein.

By employing the convolution, also known as the Hadamard product, of two analytic functions, we introduce an analytic function denoted as $\mathcal{H}^{\lambda}_{\alpha,\beta}(z) = \mathcal{M}_{\alpha,\beta}(z) * \mathcal{B}_{\lambda}(z)$. The power series representation of this function is expressed as follows:

$$\mathcal{H}^{\lambda}_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty}\frac{\Gamma(\beta)\left(\lambda(n-1)\right)^{n-2}e^{-\lambda(n-1)}}{\Gamma(n)\Gamma(\alpha(n-1)+\beta)}z^n.$$

Moreover, we introduce the linear operator $\mathcal{H}^{\lambda}_{\alpha,\beta} : \mathcal{H} \to \mathcal{H}$ which we define as follows. For any $f(z) \in \mathcal{H}$ this linear operator is defined as $\mathcal{K}^{\lambda}_{\alpha,\beta}f(z) = \mathcal{H}^{\lambda}_{\alpha,\beta}(z) * f(z)$. More precisely, it can be expressed as follows:

$$\mathcal{K}^{\lambda}_{\alpha,\beta}f(z) = z + \sum_{n=2}^{\infty}\frac{\Gamma(\beta)\left(\lambda(n-1)\right)^{n-2}e^{-\lambda(n-1)}}{\Gamma(n)\Gamma(\alpha(n-1)+\beta)}a_n z^n.$$

Now, utilizing the aforementioned linear operator, we present a novel class comprising bi-univalent functions characterized by Borel distribution and Mittag-Leffler functions. This class is associated with the q-analogue of the hyperbolic tangent function, which we denote as $\mathcal{W}^{\gamma}(\alpha, \beta, \lambda)$. The formal definition of this class is provided as follows.

Definition 2.1. *A function $f(z)$ belongs to the family Σ is considered to be part of the class $\mathcal{W}^{\gamma}(\alpha, \beta, \lambda)$ if it obeys the following subordination conditions:*

$$(1-\gamma)\left(\frac{\mathcal{K}^{\lambda}_{\alpha,\beta}f(z)}{z}\right) + \gamma\left(\mathcal{K}^{\lambda}_{\alpha,\beta}f(z)\right)' \prec 1 + \tanh(qz)$$

and

$$(1-\gamma)\left(\frac{\mathcal{K}^{\lambda}_{\alpha,\beta}g(w)}{w}\right) + \gamma\left(\mathcal{K}^{\lambda}_{\alpha,\beta}g(w)\right)' \prec 1 + \tanh(qw),$$

where the function $g(w) = f^{-1}(w)$ *is given by the Equation (1.2), the parameters* $0 < \lambda \leq 1$, $\gamma \geq 0$, $\alpha, \beta \in \mathbb{C}$ *with* $\Re(\alpha) > 0$, $\Re(\beta) > 0$ *and* $\beta \neq 0, -1, -2, -3, \cdots$.

The lemma presented below is extensively documented in the literature (see, for example, [18]) and is regarded as a fundamental principle that significantly contributes to the research we are conducting.

Lemma 2.1. *Let* $p(z)$ *be a function in the Caratheodory class* $\mathcal{P}$. *Then for any* $z \in \mathbb{D}$ *the function* p *can be written as* $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. *Moreover,* $|p_n| \leq 2$ *for each natural number* n. *In addition, for any complex number* ζ, *we have*

$$|p_2 - \zeta p_1^2| \leq 2 \max\{1, |2\zeta - 1|\}.$$

The purpose of this article is to explore a novel class of bi-univalent functions that are defined through the Borel distribution and the Mittag-Leffler function, utilizing the Hadamard product in relation to the q-analogue hyperbolic tangent function. The main goal is to obtain estimates for the moduli of the initial coefficients in the Taylor series expansion of functions within this category. Additionally, the paper delves into the Fekete-Szegö functional problem related to this class of functions, which enhances our comprehension of their essential properties.

3. Coefficient Estimates of the Class $\mathcal{W}^{\gamma}(\alpha, \beta, \lambda)$

This part of the paper focuses on examining the bounds for the modulus of the initial coefficients of functions belonging to the class $\mathcal{W}^{\gamma}(\alpha, \beta, \lambda)$, as indicated in Equation (1.1). Furthermore, we seek to determine the coefficient bounds for several subclasses that fall under our established class. Additionally, we aim to establish the coefficient bounds for some of the subclasses within our defined class.

Theorem 3.1. *Let a function* f *be a bi-univalent function that belongs to the class* $\mathcal{W}^{\gamma}(\alpha, \beta, \lambda)$ *and is represented by the Equation (1.1). The following inequalities hold:*

$$|a_2| \leq \frac{qe^{\lambda}\Gamma(\alpha+\beta)\sqrt{2\Gamma(2\alpha+\beta)}}{\sqrt{4q\lambda(1+2\gamma)\Gamma(\beta)\Gamma^2(\alpha+\beta) + (1+\gamma)^2\Gamma^2(\beta)\Gamma(2\alpha+\beta)}}, \tag{3.1}$$

and

$$|a_3| \leq \begin{cases} \frac{2qAe^{2\lambda}}{\lambda}, & \text{if } \left|\frac{q\lambda B^2}{A}\right| \leq 1 \\ 2q^2e^{2\lambda}B^2, & \text{if } 1 < \left|\frac{q\lambda B^2}{A}\right| < 2 \\ \frac{4qAe^{2\lambda}}{\lambda}, & \text{if } \left|\frac{q\lambda B^2}{A}\right| \geq 2 \end{cases} \tag{3.2}$$

where

$$A = \frac{\Gamma(2\alpha+\beta)}{(1+2\gamma)\Gamma(\beta)} \quad \text{and} \quad B = \frac{\Gamma(\alpha+\beta)}{(1+\gamma)\Gamma(\beta)}.$$

Proof. Let f be a function that is part of the class $\mathcal{W}^{\gamma}(\alpha,\beta,\lambda)$. Based on Definition 2.1 and the Subordination Principle, it is feasible to recognize two Schwarz functions, $\eta(z)$ and $\zeta(w)$, that are defined within the open unit disk $\mathbb{D}$ such that

$$(1-\gamma)\left(\frac{\mathcal{K}^{\lambda}_{\alpha,\beta}f(z)}{z}\right)+\gamma\left(\mathcal{K}^{\lambda}_{\alpha,\beta}f(z)\right)'=1+\tanh(q\eta(z)), \tag{3.3}$$

and

$$(1-\gamma)\left(\frac{\mathcal{K}^{\lambda}_{\alpha,\beta}g(w)}{w}\right)+\gamma\left(\mathcal{K}^{\lambda}_{\alpha,\beta}g(w)\right)'=1+\tanh(q\zeta(w)). \tag{3.4}$$

Now, using these Schwarz functions, we define two analytic functions $h(z)$ and $P(w)$ as follow:

$$h(z)=\frac{1+\eta(z)}{1-\eta(z)} \quad \text{and} \quad P(w)=\frac{1+\zeta(w)}{1-\zeta(w)}.$$

It is obvious that the functions $h(z)$ and $P(w)$ are analytic within the open unit disk $\mathbb{D}$ and are classified under the Caratheodory class. Therefore, we can express them in the following manner:

$$h(z)=\frac{1+\eta(z)}{1-\eta(z)}=1+h_1z+h_2z^2+\cdots$$

and

$$P(w)=\frac{1+\zeta(w)}{1-\zeta(w)}=1+p_1w+p_2w^2+\cdots$$

Moreover, $h(0)=1=P(0)$, they have positive real parts, $|h_j|\le 2$ and $|p_j|\le 2$ for all $j\in\mathbb{N}$.

Equivalently, we get the following representations of $\eta(z)$ and $\zeta(w)$

$$\zeta(z)=\frac{h(z)-1}{h(z)+1}=\frac{h_1}{2}z+\left(\frac{h_2}{2}-\frac{h_1^2}{4}\right)z^2+\cdots, \tag{3.5}$$

and

$$\zeta(w)=\frac{P(w)-1}{P(w)+1}=\frac{p_1}{2}w+\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)w^2+\cdots. \tag{3.6}$$

By referring to Equation (3.5), we can express the right-hand sides of Equation (3.3) in the following manner:

$$\begin{aligned}1+\tanh(q\eta(z))=1+\frac{qh_1}{2}z+q\left(\frac{h_2}{2}-\frac{h_1^2}{4}\right)z^2&\\ +q\left(\frac{h_2}{2}-\frac{h_1h_2}{2}+\frac{(3-2q^2)h_1^3}{24}\right)z^3+\cdots&\end{aligned} \tag{3.7}$$

Moreover, the left-hand side Equation (3.3) can be written as:

$$\begin{aligned}&(1-\gamma)\left(\frac{\mathcal{K}^{\lambda}_{\alpha,\beta}f(z)}{z}\right)+\gamma\left(\mathcal{K}^{\lambda}_{\alpha,\beta}f(z)\right)'\\ &=1+\frac{(1+\gamma)\Gamma(\beta)}{e^{\lambda}\Gamma(\alpha+\beta)}a_2z+\frac{\lambda(1+2\gamma)\Gamma(\beta)}{2e^{2\lambda}\Gamma(2\alpha+\beta)}a_3z^2+\cdots\end{aligned} \tag{3.8}$$

Hence, considering Equation (3.3), then comparing coefficients on both sides of Equation (3.7) and Equation(3.8) we get the following two equations:

$$2(1+\gamma)\Gamma(\beta)a_2 = qe^{\lambda}\Gamma(\alpha+\beta)h_1, \tag{3.9}$$

$$2\lambda(1+2\gamma)\Gamma(\beta)a_3 = qe^{2\lambda}\Gamma(2\alpha+\beta)\left(2h_2 - h_1^2\right), \tag{3.10}$$

Similarly, by consulting Equation (3.6), the right-hand sides of Equation (3.4) can be written as:

$$\begin{aligned}1+\tanh(q\zeta(w)) &= 1+\frac{qp_1}{2}w + q\left(\frac{p_2}{2}-\frac{p_1^2}{4}\right)w^2 \\ &\quad + q\left(\frac{p_2}{2}-\frac{p_1p_2}{2}+\frac{(3-2q^2)p_1^3}{24}\right)z^3+\cdots\end{aligned} \tag{3.11}$$

Moreover, the left-hand side Equation (3.4) can be written as:

$$\begin{aligned}&(1-\gamma)\left(\frac{\mathcal{K}^{\lambda}_{\alpha,\beta}g(w)}{w}\right)+\gamma\left(\mathcal{K}^{\lambda}_{\alpha,\beta}g(w)\right)' \\ &= 1+\frac{(1+\gamma)\Gamma(\beta)}{e^{\lambda}\Gamma(\alpha+\beta)}a_2w+\frac{\lambda(1+2\gamma)\Gamma(\beta)}{2e^{2\lambda}\Gamma(2\alpha+\beta)}(2a_2^2-a_3)w^2+\cdots\end{aligned} \tag{3.12}$$

To proceed, we take into account Equation (3.4) and then analyze the coefficients from both sides of Equations (3.11) and (3.12), which leads us to derive the following two equations.

$$-2(1+\gamma)\Gamma(\beta)a_2 = qe^{\lambda}\Gamma(\alpha+\beta)p_1, \tag{3.13}$$

and

$$2\lambda(1+2\gamma)\Gamma(\beta)(2a_2^2-a_3) = qe^{2\lambda}\Gamma(2\alpha+\beta)\left(2p_2-p_1^2\right). \tag{3.14}$$

Now, on one hand, using Equation (3.9) and Equation (3.13) we get the following equation

$$\left(\frac{2(1+\gamma)\Gamma(\beta)}{qe^{\lambda}\Gamma(\alpha+\beta)}\right)^2 a_2^2 = h_1^2+p_1^2.$$

On the other hand, adding Equation (3.10) to Equation (3.14), we obtain the following equation

$$\left(\frac{4(1+2\gamma)\Gamma(\beta)}{qe^{2\lambda}\Gamma(2\alpha+\beta)}\right)a_2^2 = 2(h_2+p_2)-(h_1^2+p_1^2).$$

Now, consulting the last two equations, we obtain the following equation

$$a_2^2 = \frac{2q^2e^{2\lambda}\Gamma(2\alpha+\beta)\Gamma^2(\alpha+\beta)(h_2+p_2)}{2\Gamma(\beta)[4q\lambda\Gamma^2(\alpha+\beta)+(1+\gamma)^2\Gamma(\beta)\Gamma(2\alpha+\beta)]}. \tag{3.15}$$

Therefore, considering equation (3.15) and using constraints $|h_2| \le 2$ and $|p_2| \le 2$, the simple calculations gives the desired inequality (3.1).

Secondly, our objective is to ascertain the coefficient estimate for $|a_3|$. By referring to Equation (3.10), we can derive the subsequent equation

$$a_3 = \frac{qe^{2\lambda}\Gamma(2\alpha+\beta)}{2\lambda(1+2\gamma)\Gamma(\beta)}(2h_2 - h_1^2).$$

Thus, by applying the constraints $|h_1| \leq 2$ and $|h_2| \leq 2$, we can derive the following inequality from the previous equation

$$|a_3| \leq \frac{4qe^{2\lambda}\Gamma(2\alpha+\beta)}{\lambda(1+2\gamma)\Gamma(\beta)}. \tag{3.16}$$

By utilizing Equation (3.14), we can derive the subsequent equation

$$a_3 = \frac{qAe^{2\lambda}}{2\lambda}(p_1^2 - 2p_2) + 2a_2^2. \tag{3.17}$$

Furthermore, by applying Equation (3.13), the last equation can be expressed in the following manner

$$\begin{aligned} a_3 =& \frac{qe^{2\lambda}}{2\lambda}\left\{-2Ap_2 + Ap_1^2 + q\lambda B^2 p_1^2\right\} \\ =& \frac{-qAe^{2\lambda}}{\lambda}\left\{p_2 - \left(\frac{A+q\lambda B^2}{2A}\right)p_1^2\right\}. \end{aligned}$$

Now, by referring to Lemma 2.1 along with the preceding equation, we can derive the subsequent inequality

$$|a_3| \leq \frac{2qAe^{2\lambda}}{\lambda}\max\left\{1, \left|\frac{q\lambda B^2}{A}\right|\right\}. \tag{3.18}$$

By applying Equation (3.16) in conjunction with Equation (3.18), we can establish the subsequent inequality.

$$|a_3| \leq \frac{2qAe^{2\lambda}}{\lambda}\min\left\{2, \max\left\{1, \left|\frac{q\lambda B^2}{A}\right|\right\}\right\}. \tag{3.19}$$

Finally, the straightforward computations derived from Equation (3.19) yield the necessary estimation of $|a_3|$ that is represented by Inequality (3.2). Therefore, the demonstration of Theorem 3.1 is now complete. □

In our analysis, we introduce a parameter γ that plays a crucial role in categorizing our class $\mathcal{W}^{\gamma}(\alpha,\beta,\lambda)$. The choice of γ can significantly influence the properties and behaviors of this class, leading us to identify distinct subclasses based on its values.

Example 3.1. *Let the function f be in the class Σ and represented by the Equation (1.1). We say f belongs to the subclass $\mathcal{W}^1(\alpha,\beta,\lambda)$ if the following conditions hold:*

$$\left(\mathcal{K}_{\alpha,\beta}^{\lambda} f(z)\right)' \prec 1 + \tanh(qz), \tag{3.20}$$

and

$$\left(\mathcal{K}_{\alpha,\beta}^{\lambda} g(w)\right)' \prec 1 + \tanh(qw), \tag{3.21}$$

where $g(w) = f^{-1}(w)$ is expressed by the Equation (1.2), the parameters $0 < \lambda \leq 1$, $\alpha, \beta \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and $\beta \neq 0, -1, -2, -3, \cdots$.

Example 3.2. *Let the function f be in the class Σ and represented by the Equation (1.1). We say f belongs to the subclass $\mathcal{W}^0(\alpha,\beta,\lambda)$ if the following conditions hold:*

$$\left(\frac{\mathcal{K}_{\alpha,\beta}^{\lambda}f(z)}{z}\right) \prec 1+\tanh(qz), \tag{3.22}$$

and

$$\left(\frac{\mathcal{K}_{\alpha,\beta}^{\lambda}g(w)}{w}\right) \prec 1+\tanh(qw), \tag{3.23}$$

where $g(w) = f^{-1}(w)$ is expressed by the Equation (1.2), the parameters $0 < \lambda \le 1$, $\alpha,\beta \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and $\beta \neq 0,-1,-2,-3,\cdots$.

The following corollaries are derived directly from Theorem 3.1 and are associated with Example 3.1 and Example 3.2, respectively. The techniques employed to prove these corollaries are quite similar to those used in the proof of Theorem 3.1, which is the reason we have chosen to leave out the detailed proofs.

Corollary 3.1. *If a function $f \in \Sigma$ is represented by the Equation (1.1) and belong to the class $\mathcal{W}^1(\alpha,\beta,\lambda)$, then the following hold*

$$|a_2| \le \frac{qe^{\lambda}\Gamma(\alpha+\beta)\sqrt{\Gamma(2\alpha+\beta)}}{\sqrt{6q\lambda\Gamma(\beta)\Gamma^2(\alpha+\beta)+2\Gamma^2(\beta)\Gamma(2\alpha+\beta)}},$$

and

$$|a_3| \le \begin{cases} \frac{2qA^*e^{2\lambda}}{3\lambda}, & \text{if } \left|\frac{q\lambda(B^*)^2}{A^*}\right| \le \frac{4}{3} \\ \frac{q^2e^{2\lambda}(B^*)^2}{2}, & \text{if } \frac{4}{3} < \left|\frac{q\lambda B_1^2}{A_1}\right| < \frac{8}{3} \\ \frac{4qA^*e^{2\lambda}}{3\lambda}, & \text{if } \left|\frac{q\lambda B_1^2}{A_1}\right| \ge \frac{8}{3} \end{cases}$$

where

$$A^* = \frac{\Gamma(2\alpha+\beta)}{\Gamma(\beta)} \quad \text{and} \quad B^* = \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}.$$

Corollary 3.2. *If a function $f \in \Sigma$ is represented by the Equation (1.1) and belong to the class $\mathcal{W}^0(\alpha,\beta,\lambda)$, then it can be concluded that*

$$|a_2| \le \frac{qe^{\lambda}\Gamma(\alpha+\beta)\sqrt{2\Gamma(2\alpha+\beta)}}{\sqrt{4q\lambda\Gamma(\beta)\Gamma^2(\alpha+\beta)+\Gamma^2(\beta)\Gamma(2\alpha+\beta)}},$$

and

$$|a_3| \le \begin{cases} \frac{2qA^*e^{2\lambda}}{\lambda}, & \text{if } \left|\frac{q\lambda(B^*)^2}{A^*}\right| \le 1 \\ 2q^2e^{2\lambda}(B^*)^2, & \text{if } 1 < \left|\frac{q\lambda(B^*)^2}{A^*}\right| < 2 \\ \frac{4qA^*e^{2\lambda}}{\lambda}, & \text{if } \left|\frac{q\lambda(B^*)^2}{A^*}\right| \ge 2 \end{cases}$$

where

$$A^* = \frac{\Gamma(2\alpha+\beta)}{\Gamma(\beta)} \quad \text{and} \quad B^* = \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}.$$

4. Fekete-Szegö Inequalities for the Class $\mathcal{W}^{\gamma}(\alpha,\beta,\lambda)$

This part of the paper will focus on the advancement of the Fekete-Szegö inequalities for functions that are part of the designated class $\mathcal{W}^{\gamma}(\alpha,\beta,\lambda)$. This class includes bi-univalent functions that are defined through the Borel distribution and the Mittag-Leffler function. Additionally, we aim to establish Fekete-Szegö inequalities for several subclasses that are included within the boundaries of our defined class.

Theorem 4.1. *Let a function f be a bi-univalent function that belongs to the class $\mathcal{W}^{\gamma}(\alpha,\beta,\lambda)$ and is represented by the Equation (1.1), then for a real number ζ the following inequality holds*

$$|a_3-\zeta a_2^2| \leq \frac{2qAe^{2\lambda}}{\lambda}\min\left\{\max\left\{1,\left|\frac{q\zeta\lambda B^2}{2A}\right|\right\},\max\left\{1,\left|\frac{q(2-\zeta)\lambda B^2}{2A}\right|\right\}\right\}, \tag{4.1}$$

where

$$A=\frac{\Gamma(2\alpha+\beta)}{(1+2\gamma)\Gamma(\beta)} \quad \text{and} \quad B=\frac{\Gamma(\alpha+\beta)}{(1+\gamma)\Gamma(\beta)}.$$

Proof. By consulting Equation (3.10), we arrive at the subsequent equation, which can be expressed as follows

$$a_3=\frac{qAe^{2\lambda}}{2\lambda}(2h_2-h_1^2).$$

For any real number ζ, we can express the last equation using Equation (3.9) as follows

$$\begin{aligned} a_3-\zeta a_2^2 &= \frac{qAe^{2\lambda}}{2\lambda}(2h_2-h_1^2)-\frac{q^2\zeta e^{2\lambda}}{4}B^2h_1^2 \\ &= \frac{qAe^{2\lambda}}{\lambda}\left\{h_2-\left(\frac{1}{2}+\frac{q\zeta\lambda B^2}{4A}\right)h_1^2\right\}. \end{aligned}$$

By applying Lemma 2.1 to the last equation, we can derive the following inequality. This process involves substituting the relevant expressions or conditions outlined in the lemma into our equation, which allows us to manipulate the terms accordingly. As a result, we obtain an inequality that provides a clearer understanding of the relationship between the variables involved.

$$|a_3-\zeta a_2^2| \leq \frac{2qAe^{2\lambda}}{\lambda}\max\left\{1,\frac{q\zeta\lambda B^2}{2A}\right\}. \tag{4.2}$$

In contrast, for the real number ζ, we can derive the following result by applying Equation (3.14)

$$a_3-\zeta a_2^2=\frac{qAe^{2\lambda}}{2\lambda}(p_1^2-2p_2)+(2-\zeta)a_2^2.$$

Consequently, by applying Equation (3.13), the final equation can be expressed as follows

$$\begin{aligned} a_3-\zeta a_2^2 &= \frac{qAe^{2\lambda}}{2\lambda}(p_1^2-2p_2)+\frac{(2-\zeta)q^2e^{2\lambda}B^2}{4}p_1^2 \\ &= \frac{qAe^{2\lambda}}{2\lambda}\left(p_1^2-2p_2+\frac{q\lambda(2-\zeta)B^2}{2A}p_1^2\right) \\ &= \frac{-qAe^{2\lambda}}{\lambda}\left\{p_2-\left(\frac{1}{2}+\frac{q\lambda(2-\zeta)B^2}{4A}\right)p_1^2\right\}. \end{aligned}$$

By utilizing Lemma 2.1 in the previous equation, we can derive a new relationship that allows us to establish the subsequent inequality

$$|a_3 - \zeta a_2^2| \le \frac{2qAe^{2\lambda}}{\lambda} \max\left\{1, \frac{q\lambda(2-\zeta)B^2}{2A}\right\}. \tag{4.3}$$

Finally, using Equation (4.2) and Equation (4.3), we can straightforwardly derive the required inequality represented by (4.1). This concludes the proof. □

The subsequent corollaries arise directly from Theorem 4.1. The approach employed to establish these corollaries closely resembles that of the previous theorem, utilizing similar methodologies, logical reasoning, and mathematical techniques. Therefore, we have opted to omit the comprehensive proofs for these corollaries.

Corollary 4.1. *Let a function $f \in \Sigma$ be represented by equation (1.1) and be part of the subclass $\mathcal{W}^1(\alpha, \beta, \lambda)$, then for a real number ζ the following inequality holds*

$$|a_3 - \zeta a_2^2| \le \frac{2qA^*e^{2\lambda}}{3\lambda} \min\left\{\max\left\{1, \left|\frac{3q\zeta\lambda(B^*)^2}{8A^*}\right|\right\}, \max\left\{1, \left|\frac{3q(2-\zeta)\lambda(B^*)^2}{8A^*}\right|\right\}\right\}.$$

Corollary 4.2. *Let a function $f \in \Sigma$ be represented by equation (1.1) and be part of the subclass $\mathcal{W}^0(\alpha, \beta, \lambda)$, then for a real number ζ the following inequality holds*

$$|a_3 - \zeta a_2^2| \le \frac{2qA^*e^{2\lambda}}{\lambda} \min\left\{\max\left\{1, \left|\frac{q\zeta\lambda(B^*)^2}{2A^*}\right|\right\}, \max\left\{1, \left|\frac{q(2-\zeta)\lambda(B^*)^2}{2A^*}\right|\right\}\right\}.$$

5. Conclusion

This study explored a novel category of bi-univalent functions, defined through the convolution of the Borel distribution and the Mittag-Leffler function, which are linked to the q-analogue hyperbolic tangent function. We have effectively obtained estimates for the initial coefficients and has formulated the Fekete-Szegö inequalities that pertain to functions within this category and its different subclasses. The results of this investigation are expected to provide a wide range of insights for subclasses related to orthogonal polynomials, such as Legendre and Horadam polynomials. Additionally, the findings presented in this study are likely to inspire researchers to expand these concepts to include symmetric q-calculus and harmonic functions.

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