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Hypercomplex Representations of the Affine Group

Amjad Saleh Alghamdi*

Department of Mathematics, Jamoum University Collage, Umm Al-Qura University, P.O. Box 24382, Makkah, Saudi Arabia

**Corresponding Author:* asmghamdi@uqu.edu.sa

Abstract. In this paper, we consider representations of the affine group that are induced by hypercomplex characters specifically, double and dual characters instead of complex characters. The double and dual number systems provide a rich algebraic structure that allows for the development of generalized representations, extending beyond the complex Hilbert spaces. We derive the hypercomplex representations by starting from certain subgroups of the affine group. Then, we investigate how these representations act on vector spaces equipped with indefinite inner products, naturally leading to the framework of Krein spaces.

1. Introduction

The affine group, denoted by Aff, is a non-commutative, locally compact Lie group of smallest dimensionality. It consists of transformations of the real line of the form $x \to ax + b$ where $a \in \mathbb{R}^+, b \in \mathbb{R}$. The affine group arises naturally in mathematics and physics, as it describes transformations involving both scaling and translation. Understanding affine group representations is crucial for various branches of analysis, geometry, and mathematical physics. The unitary representations of the affine group were first introduced by Gelfand and Naimark [10].The induced representations of the affine group from a complex character were later developed in works such as [4,7].

Hypercomplex numbers are two-dimensional real, associative, and commutative algebras that are isomorphic to one of three systems: the complex numbers, dual numbers, or double (split-complex) numbers. The development of hypercomplex systems(including dual numbers, double numbers, quaternions, and Clifford algebras) has significantly enriched various areas of mathematics and physics. Double numbers were first introduced in 1848 by James Cockle through his work on tessarines (an early form of split-complex numbers) [24]. Dual numbers were later

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introduced by Clifford in 1873 as a natural extension for describing infinitesimal transformations in what he termed motor algebra [25]. Hypercomplex numbers have found diverse applications in physics. Specifically, dual numbers offer an effective algebraic framework for classical mechanics and kinematics, where they model infinitesimal displacements and derivatives. In contrast, double numbers underpin the structure of hyperbolic quantum mechanics, an alternative to standard quantum theory that uses an indefinite metric [3,8,12].

In this paper, we focus on representations of the affine group using only dual and double numbers. The geometric spaces associated with these number systems are often categorized as elliptic, parabolic, and hyperbolic. The elliptic case corresponds to the upper half-plane model based on complex numbers. In parallel, the parabolic and hyperbolic cases correspond to the upper half-planes modeled by dual and double numbers, respectively.

2. Double and Dual Numbers

Definition 2.1. *[6,16] Double numbers is given by $\mathbb{O} = \{a + jb : j^2 = 1 \text{ and } a, b \in \mathbb{R}\}$. The triple $(\mathbb{O}, +.\times)$ is a commutative ring with identity where addition and multiplication are defined as follows:*

(1) $(a + jb) + (a_1 + jb_1) = (a + a_1) + j(b + b_1)$,
(2) $(a + jb) \times (a_1 + jb_1) = (aa_1 + bb_1) + j(ab_1 + a_1 b)$.

For $w = a + jb \in \mathbb{O}$, the conjugate is given by $\bar{w} = a - jb$, then the modulus is $|w|^2 = w\bar{w} = a^2 - b^2$. In geometry the double numbers plane is a hyperbolic such that $e^{jt} = \cosh t + j \sinh t$, where the unit hyperbola is given by $|w|^2 = 1$.

Remark 2.1. *A distinctive feature of double numbers algebra is the existence of zero divisors:*

$$j_{\pm} = \frac{1}{\sqrt{2}}(1 \pm j), \tag{2.1}$$

which satisfy $j_+ j_- = 0$ and $j_{\pm}^2 = j_{\pm}$. This implies that the double numbers are algebraically isomorphic to $\mathbb{R} \times \mathbb{R}$, with the idempotents $j_{\pm}$.

Definition 2.2. *[11,16] Dual numbers is given by $\mathbb{D} = \{a + \epsilon b : \epsilon^2 = 0 \text{ and } a, b \in \mathbb{R}\}$, and the triple $(\mathbb{D}, +, \times)$ is a commutative ring with identity where the addition and multiplication is defined as follows:*

(1) $(a + \epsilon b) + (a_1 + \epsilon b_1) = (a + a_1) + \epsilon(b + b_1)$,
(2) $(a + \epsilon b) \times (a_1 + \epsilon b_1) = aa_1 + \epsilon(ab_1 + a_1 b)$.

For $w = a + \epsilon b \in \mathbb{D}$, the conjugate is given by $\bar{w} = a - \epsilon b$ and the modulus is $|w|^2 = w\bar{w} = a^2$. Also, in geometry dual numbers plane is parabolic such that $e^{\epsilon t} = 1 + \epsilon t$, where the unit strip is given by $a = \pm 1$.

2.1. Inner product Structures over Double and Dual Numbers.

The dobule number algebra $\mathbb{O}$ contains idempotent elements $j_{\pm}$ (2.1). Also, the dual number algebra $\mathbb{D}$ contains nilpotent elements, specifically ϵ, satisfying $\epsilon^2 = 0$. Due to the presence of zero divisors and idempotents in $\mathbb{O}$, and nilpotents in $\mathbb{D}$, any inner product defined on these algebras is indefinite. As a result, the appropriate framework for defining inner product structures on $\mathbb{O}$ and $\mathbb{D}$ is within the theory of Krein spaces, which generalize Hilbert spaces to allow for indefinite inner products.

Definition 2.3. *[20,23] A Krein space $\mathcal{K}$ is a vector space equipped with indefinite inner product $[.,.]$, such that there exists a direct sum decomposition $\mathcal{K} = \mathcal{K}_+ \bigoplus \mathcal{K}_-$, where $(\mathcal{K}_+, [.,.]_+)$ is a Hilbert spaces with positive definite inner product and $(\mathcal{K}_-, -[.,.]_-)$ is a Hilbert space with negative definite inner product. The indefinite inner product on $\mathcal{K}$ is given as follows:*

$$[x,y] = [x,y]_+ - [x,y]_-, \tag{2.2}$$

for $\quad x = x_+ + x_-, \quad y = y_+ + y_-, \quad x_\pm, y_\pm \in \mathcal{K}_\pm.$

Definition 2.4. *[20,21] Let $(\mathcal{K}, [.,.])$ be a Krein space,with fundamental decomposition $\mathcal{K} = \mathcal{K}_+ \bigoplus \mathcal{K}_-$, then there exist a unique operators*

$$P^+ : (\mathcal{K}, [.,.]) \to (\mathcal{K}_+, [.,.]), \quad P^- : (\mathcal{K}, [.,.]) \to (\mathcal{K}_-, [.,.]),$$

defined as follows: $P^+(k) = K^+$ and $P^-(k) = k^-$ for all $k \in \mathcal{K}$, where $k^+ \in \mathcal{K}_+$,$k^- \in \mathcal{K}_-$ and $k = k^+ + k^-$. The operators P^+ and P^- are known as fundamental projectors. The operator $J : (\mathcal{K}, [.,.]) \to (\mathcal{K}, [.,.])$ defined by $J = P^+ - P^-$. That is, $Jk = P^+k - P^-k = k^+ - K^-$. The operator J is called the fundamental symmetry of the Krein space $\mathcal{K}$ associated with the fundamental decomposition.

Definition 2.5. *[20,21] Let $(\mathcal{K}, [.,.)$ be a Krein space and J the fundamental symmetry associated with the decomposition $\mathcal{K} = \mathcal{K}_+ \bigoplus \mathcal{K}_-$. The function $[.,.]_J : \mathcal{K} \times \mathcal{K} \to \mathbb{C}$ is defined as follows:*

$$[x,y]_J = [Jx,y], \quad x,y \in \mathcal{K}.$$

This function is referred to as the J-inner product and is usual positive definite inner product.

Theorem 2.1. *[19,20] Let $(\mathcal{K}, [.,.])$ be a Krein space , and consider J as the fundamental symmetry linked to the decomposition $\mathcal{K} = \mathcal{K}_+ \bigoplus \mathcal{K}_-$. Then the following holds:*

$$|[x,y]| \leq ||x||_J ||y||_J, \quad x,y \in \mathcal{K}.$$

Proposition 2.1. *[19–21] In the Krein space $(\mathcal{K}, [.,.])$, the fundamental symmetry J determines a norm on $\mathcal{K}$, given by*

$$||x||_J^2 = [x,x]_J, \quad \forall x \in \mathcal{K}. \tag{2.3}$$

Then, the Hilbert spaces $(\mathcal{K}_+, [.,.]_+)$ and $(\mathcal{K}_-, [.,.]_-)$, have the following norm respectively:

$$||x^+||^2 = [x^+, x^+], \quad ||x^-||^2 = [x^-, x^-], \quad \forall x^+ \in \mathcal{K}_+, x^- \in \mathcal{K}_-.$$

Therefore,

$$||x||_J^2 = ||x^+||_+^2 + ||x^-||_-^2, \quad x = x^+ + x^- \in \mathcal{K}.$$

2.2. $\mathcal{L}_2$ **Spaces over Double and Dual Numbers.** We define square-integrable function spaces over the of double and dual algebras as follows.

A function $f : \mathbb{R}^2 \to \mathbb{O}$ is written as

$$f(x) = f_1(x) + jf_2(x), \quad \text{with } f_1, f_2 : \mathbb{R}^2 \to \mathbb{R},$$

where $j^2 = 1$. The double numbers form a real commutative algebra with an indefinite inner product:

$$\begin{aligned} [f, f] = \|f\|^2 &= \int_{\mathbb{R}^2} f(x)\overline{f(x)}dx \\ &= \int_{\mathbb{R}^2} f_1(x)^2 - f_2(x)^2 dx \\ &= \int_{\mathbb{R}^2} |f(x)|^2 dx \end{aligned} \tag{2.4}$$

We define the $\mathcal{L}_2$-space with respect to this indefinite inner product as a Krein space:

$$\mathcal{L}_2(\mathbb{R}^2, \mathbb{O}) := \left\{ f = f_1 + jf_2 \mid f_1, f_2 \in \mathcal{L}_2(\mathbb{R}^2) \right\}.$$

A function $f : \mathbb{R}^2 \to \mathbb{D}$ is said to be *square-integrable* if it can be written as

$$f(x) = f_1(x) + \epsilon f_2(x), \quad \text{with } f_1, f_2 : \mathbb{R}^2 \to \mathbb{R},$$

and

$$[f, f] = \|f\|^2 \int_{\mathbb{R}^2} |f_1(x)|^2 \, dx < \infty.$$

Since $\epsilon^2 = 0$, the dual component does not contribute to the norm. The space of such functions is denoted by

$$\mathcal{L}_2(\mathbb{R}^2, \mathbb{D}) := \left\{ f = f_1 + \epsilon f_2 \mid f_1, f_2 \in \mathcal{L}_2(\mathbb{R}^2) \right\}.$$

These spaces generalize the classical $\mathcal{L}_2$ space to hypercomplex-valued functions, with double numbers to indefinite norms and dual numbers leading to degenerate norms.

3. The Affine Group

An element of the affine group Aff [4,7] is denoted by (a, b) where $a \in \mathbb{R}_+$ and $b \in \mathbb{R}$. The group law on Aff is defined by

$$(a, b) * (a', b') = (aa', ab' + b), \tag{3.1}$$

where $e = (1, 0)$ is the identity element and the inverse of (a, b) is given by $(a, b)^{-1} = (a^{-1}, -ba^{-1})$.

We can decompose the affine group as a semi-direct product Aff $= A \ltimes N$. The subgroup N is a closed normal subgroup defined by $\{(1, b) : b \in \mathbb{R}\}$, and identified with $\mathbb{R}$ through the mapping $(1, b) \leftrightarrow b$. The subgroup $A = \{(a, 0) : a > 0\}$ is identified with $\mathbb{R}_+$ where $(a, 0) \leftrightarrow a$, [5].

The affine group is a locally compact group thus it has a left Haar measure which is given as follow:

$$d\nu(a, b) = a^{-2} da db, \tag{3.2}$$

and it is left invariant measure that is $dv((a',b') * (a,b)) = dv(a,b)$. In addition, we can obtain a right Haar measure

$$d\mu(a,b) = a^{-1} da db, \tag{3.3}$$

which is right invariant.

Therefore, the affine group is a non-unimodular group, and the modular function of the group is given by $\triangle(a,b) = a^{-1}$ [5]. The measure on the subgroup A is the Haar measure $\frac{da}{a}$ and on the subgroup N is the Lebesgue measure db.

4. Induced Representations

In this section, we describe the construction of induced representations [2,5,7]. Let G be a group H be a closed subgroup of G; then $X = G/H$ is the left coset space. For a character $\chi : H \to \mathbb{T}$, where $\chi(h_1h_2) = \chi(h_1)\chi(h_2)$ and $|\chi(h)| = 1$, let V_χ be the vector space of functions $F : G \to \mathbb{C}$ having the property:

$$F(gh) = \chi(h)F(g), \quad \forall g \in G, h \in H. \tag{4.1}$$

The space V_χ is invariant under the left action of G, that is

$$\Lambda(g) : V_\chi \to V_\chi, \quad [\Lambda(g)F](g') = F(g^{-1}g'), \quad g, g' \in G. \tag{4.2}$$

The restriction of the left action of G on the space V_χ is called the induced representation.

An equivalent realization of the above induced representation can be defined on the homogeneous space $X = G/H$. Let $\mathsf{s} : X \to G$, be a section map that is a right inverse of the natural projection map $\mathsf{p} : G \to X$, that is $\mathsf{p} \circ \mathsf{s} = \mathbb{I}_X$. Then the left action of G on the homogeneous space X is given by:

$$g \cdot x = \mathsf{p}(g\mathsf{s}(x)),$$

where $g \in G$ and $x \in X$.

The map $\mathsf{r} : G \to H$ is defined by $\mathsf{r}(g) = \mathsf{s}(\mathsf{p}(g))^{-1}g$. Then each element $g \in G$ has a unique decomposition of the form $g = \mathsf{s}(\mathsf{p}(g))\mathsf{r}(g)$ [2].

Now, for a character χ of the subgroup H, introduce the lifting map $\mathcal{L}_\chi : W(X) \to V_\chi$, as follows:

$$[\mathcal{L}_\chi f](g) = \overline{\chi(\mathsf{r}(g))} f(\mathsf{p}(g)), \quad f \in W(X),$$

where $W(X) := \{f : X \to \mathbb{C}\}$ is the vector space of all complex functions on the homogeneous space $X = G/H$. Let the pulling map $\mathcal{P} : V_\chi \to W(X)$, given by:

$$[\mathcal{P}F](x) = F(\mathsf{s}(x)).$$

Next, the operator $\pi_\chi(g)$ on $W(X)$ is given as follows:

$$\pi_\chi(g) := \mathcal{P} \circ \Lambda(g) \circ \mathcal{L}_\chi. \tag{4.3}$$

Thus, the representation π_χ acts on $W(X)$ via the following explicit formula:

$$[\pi_\chi(g)f](x) = \chi(\mathsf{r}(g^{-1} * \mathsf{s}(x)))f(g^{-1} \cdot x). \tag{4.4}$$

5. Elliptic Representations of the Affine Group

By elliptic representations we mean the representation that is induced by a complex character.

The affine group has three non-conjugated subgroups $\{e\}$, N and A. Hence, we can obtain the following representations [4]:

- the left regular representation which induced from the subgroup $H = \{e\}$,

$$\Big[\Lambda(a,b)F\Big](x,y) := F((a,b)^{-1} * (x,y)) = F\left(\frac{x}{a}, \frac{y-b}{a}\right), \tag{5.1}$$

  where $(x,y) \in \text{Aff}$.
- the co-adjoint representation which induced from the subgroup N,

$$[\rho_{\chi_\omega}(a,b)f](x) = \sqrt{a}e^{2\pi i b\omega t} f(ax), \tag{5.2}$$

  where $f \in L_2(\mathbb{R}, da)$.
- the quasi-regular representation which induced from the subgroup A,

$$[\pi_{\chi_\tau}(a,b)f](x) = \left(\frac{1}{a}\right)^{-i\tau+\frac{1}{2}} f\left(\frac{x-b}{a}\right), \tag{5.3}$$

  where $f \in L_2(\mathbb{R})$.

6. Hyperbolic and Parabolic Representations of the Affine Group

In this section, we construct hyperbolic and parabolic representations of the group Aff. We consider the inductions from double character of the subgroups for the hyperbolic case and dual character for the parabolic case.

6.1. **Induced representation from the trivial subgroup.**

For the trivial subgroup $H = \{e\}$, the homogeneous space is given by $X = \text{Aff}/H \sim \text{Aff}$. Let $\mathcal{L}_2(\text{Aff}, d\nu)$, be the space of double-valued functions (for the hyperbolic case) or dual-valued functions (for the parabolic case) on Aff with respect to the left Haar measure $d\nu$. The induced representation of the affine group on $\mathcal{L}_2(\text{Aff}, d\nu)$, is called the left regular representation and defined by the operator (5.1).

6.2. **Induced representations from the subgroup N.**

In accordance with the general construction in 4, we will obtain an induced representation constructed in the homogeneous space left $X = \text{Aff}/N$ using lifting and pulling maps. First, because the affine group is a semi-direct product of subgroups N and A, there is a natural section map s for $\text{Aff}/N \cong \mathbb{R}_+$ in Aff. The map s is given as follows:

$$\mathsf{s} : \mathbb{R}_+ \to \text{Aff}, \quad \text{where} \quad \mathsf{s}(a) = (a, 0). \tag{6.1}$$

This is the right inverse of the following natural projection map:

$$\mathsf{p} : \text{Aff} \to \mathbb{R}_+, \quad \text{where} \quad \mathsf{p}(a,b) = a. \tag{6.2}$$

Then, the unique decomposition of any $(a,b) \in \text{Aff}$, takes the following form:

$$(a,b) = (a,0) * \left(1, \frac{b}{a}\right), \tag{6.3}$$

and the map $\mathsf{r} : \text{Aff} \to N$ is given by

$$\mathsf{r}(a,b) = \mathsf{s}(a)^{-1} * (a,b) = \left(1, \frac{b}{a}\right). \tag{6.4}$$

The action of the group Aff on the homogeneous space $X = \text{Aff}/N$ from the left is expressed in in terms of p and s as follows:

$$(a,b) : w \mapsto (a,b) \cdot w = \mathsf{p}((a,b) * \mathsf{s}(w)) = aw, \tag{6.5}$$

where $(a,b) \in \text{Aff}$, $w \in X$

- For the hyperbolic representation, let $\eta_\tau : N \to \mathbb{O}$, be the double character of the subgroup N, defined by the following:

$$\eta_\tau(1,b) = e^{jb\tau}, \tag{6.6}$$

where $j^2 = 1$, $|\eta_\tau(1,b)| = 1$, and $\tau, b \in \mathbb{R}$. This character induces a representation of the affine group based in the space $\mathcal{L}_2^{\eta_\tau}(\text{Aff}, N)$, which consists of the $\mathbb{O}$- valued functions $F_\tau : \text{Aff} \to \mathbb{O}$, with the properties:

$$F_\tau(a,b) = \overline{\eta_\tau}\Big(1, \frac{b}{a}\Big)F(a,0).$$

The norm is given by:

$$\|F_\tau\|_A^2 = \int_{\mathbb{R}_+} \left|F(a,0)\right|^2 \frac{da}{a},$$

where $F(a,0) = f(a) \in \mathcal{L}_2(\mathbb{R}_+, \frac{da}{a})$ the space of $\mathbb{O}$-valued functions on X. The space $\mathcal{L}_2^{\eta_\tau}(\text{Aff}, N)$, is invariant under the left Aff-shift . Then, we can obtain the hyperbolic representation constructed in the left homogeneous space $X = \text{Aff}/N$, by using the formula (4.4) as follows:

$$[\rho_{\eta_\tau}(a,b)f](x) = e^{j\frac{b}{x}\tau} f(\frac{x}{a}), \tag{6.7}$$

where $(a,b) \in \text{Aff}$, $t \in X$ and $f \in \mathcal{L}_2(\mathbb{R}, \mathbb{O})$.

Theorem 6.1. *The operator ρ_{η_τ} (6.7) is isometry on the Krein space $\mathcal{L}_2(\mathbb{R}, \mathbb{O})$*

Proof. Let $(a,b) \in \text{Aff}$ and $f \in \mathcal{L}_2(\mathbb{R}, \mathbb{O})$. Then

$$\begin{aligned}
\|\rho_{\eta_\tau}(a,b)f\|_A^2 &= \int_{\mathbb{R}_+} \left|\rho_{\eta_\tau}(a,b)f(x)\right|^2 \frac{dx}{x} \\
&= \int_{\mathbb{R}_+} \left| e^{j\frac{b}{x}\tau} f\left(\frac{x}{a}\right)\right|^2 \frac{dx}{x} \\
&= \int_{\mathbb{R}_+} \left| f(t)\right|^2 dt \\
&= \|f\|_A^2,
\end{aligned}$$

where $t = \frac{x}{a}$ and $dt = \frac{dx}{a}$. □

Performing the changing of variables $t = x^{-1}$ in (6.7), we obtain the following expression:

$$[\rho_{\eta_\tau}(a,b)f](t) = \sqrt{a}e^{jb\tau t}f(at).$$

where $(a,b) \in \mathrm{Aff}$, $t \in X$ and $f \in \mathcal{L}_2(\mathbb{R},\mathbb{O})$.

- For the parabolic representation, let $\sigma_\tau : N \to \mathbb{D}$, be the dual character of the subgroup N, expressed as:

$$\sigma_\tau(1,b) = e^{\varepsilon b\tau} = 1 + \varepsilon b\tau, \tag{6.8}$$

where $\varepsilon^2 = 0$, $|\sigma_\tau(1,b)| = 1$, and $\tau, b \in \mathbb{R}$. This character induces a representation of the group Aff constructed in the space $\mathcal{L}_2^{\sigma_\tau}(\mathrm{Aff}, N)$, which consists of the $\mathbb{D}$-valued functions $F_\tau : \mathrm{Aff} \to \mathbb{D}$, with the properties:

$$F_\tau(a,b) = \overline{\sigma_\tau}\Big(1, \frac{b}{a}\Big)F(a,0).$$

The measure on the homogeneous space $X = \mathrm{Aff}/N \cong A$ is $\frac{da}{a}$. Then, the norm is given as follows:

$$||F_\tau||_A^2 = \int_{\mathbb{R}_+} \Big|F(a,0)\Big|^2 \frac{da}{a},$$

where $F \in \mathcal{L}_2(\mathbb{R}_+, \frac{da}{a})$ the space of $\mathbb{D}$-valued function in X . The space $\mathcal{L}_2^{\sigma_\tau}(\mathrm{Aff}, N)$, is invariant under the left Aff-shifts (5.1). By using the formula (4.4) the parabolic representation acts on the space $X = \mathrm{Aff}/N$ via the following operators:

$$[\rho_{\sigma_\tau}(a,b)g](x) = (1 + \varepsilon b\tau\frac{x}{a})g(x), \tag{6.9}$$

where $(a,b) \in \mathrm{Aff}$, $x \in X$ and $g \in \mathcal{L}_2(\mathbb{R},\mathbb{D})$.

Theorem 6.2. *The operator ρ_{σ_τ} (6.9) is isometry on the Krein space $\mathcal{L}_2(\mathbb{R},\mathbb{D})$.*

Proof. Applying the same technique used in proving Theorem6.1. □

Now, substituting $t = x^{-1}$ into equation (6.9) yields the following formula:

$$[\rho_{\sigma_\tau}(a,b)g](t) = \sqrt{a}(1 + \varepsilon b\tau t)g(at).$$

where $(a,b) \in \mathrm{Aff}$, $t \in X$ and $g \in \mathcal{L}_2(\mathbb{R},\mathbb{D})$.

6.3. **Induced representations from the subgroup** A**.** In the following, we will obtain an equivalent form of this induced representation constructed in the left homogeneous space $X = \mathrm{Aff}/A$. Let s be the section map from the homogeneous space $\mathrm{Aff}/A = \mathbb{R}$ to the affine group, given by

$$\mathsf{s} : \mathbb{R} \to \mathrm{Aff}, \quad \text{such that} \quad \mathsf{s}(b) = (1,b), \quad b \in \mathbb{R}. \tag{6.10}$$

The right inverse of s is the natural projection map, given as follows:

$$\mathsf{p} : \mathrm{Aff} \to \mathbb{R}, \quad \text{where} \quad \mathsf{p}(a,b) = b. \tag{6.11}$$

Therefore, the unique decomposition of any $(a,b) \in \text{Aff}$, takes the following form:

$$(a,b) = (1,b) * (a,0),$$

and the map $\mathsf{r} : \text{Aff} \to A$ is given by

$$\mathsf{r}(a,b) = \mathsf{s}(b)^{-1} * (a,b) = (a,0). \tag{6.12}$$

The Aff-action on the homogeneous space $X = \text{Aff}/N$ from the left is defined in terms of p and s as follows:

$$(a,b) : x \mapsto (a,b) \cdot x = ax + b, \tag{6.13}$$

where $(a,b) \in \text{Aff}$, $x \in X$.

- For the hyperbolic representation, consider the double character $\sigma_\omega : A \to \mathbb{O}$, be of the subgroup A defined as follows:

$$\eta_\omega(a,0) = a^{j\omega}, \qquad j^2 = 0, \quad \omega \in \mathbb{R}, \tag{6.14}$$

where $|\eta_\omega(a,0)| = 1$. This character induced a linear representation of the group Aff built in the space $\mathcal{L}_2^{\eta_\omega}(\text{Aff})$ containing the functions $F_\omega : \text{Aff} \to \mathbb{C}$ with the property:

$$F_\omega(a,b) = \overline{\eta_\omega(a,0)} F(1,b).$$

The norm expressed as:

$$\|F_\omega\|_N^2 = \int_{\mathbb{R}} |F(1,b)|^2 db,$$

where $F(1,b) = f(b) \in \mathcal{L}_2(\mathbb{R}, db)$. This space is invariant under the left Aff-shift (5.1). Then, the hyperbolic representation acts on the left homogeneous space $X = \text{Aff}/A$, via the following operator:

$$[\pi_{\eta_\omega}(a,b)f](x) = \left(\frac{1}{a}\right)^{-j\omega+\frac{1}{2}} f\left(\frac{x-b}{a}\right), \tag{6.15}$$

where $(a,b) \in \text{Aff}$, $x \in X = \text{Aff}/A$ and $f \in \mathcal{L}_2(\mathbb{R}, \mathbb{O})$.

Theorem 6.3. *The operator π_{η_ω} (6.15) is isometry on the Krein space $\mathcal{L}_2(\mathbb{R}, \mathbb{O})$.*

Proof. Let $(a,b) \in \text{Aff}$ and $f \in \mathcal{L}_2(\mathbb{R}, \mathbb{O})$. Then

$$\begin{aligned}
\|\pi_{\eta_\omega} f\|_N^2 &= \int_{\mathbb{R}} |\pi_{\eta_\omega}(a,b) f(x)|^2 dx \\
&= \int_{\mathbb{R}} \left| \left(\frac{1}{a}\right)^{-j\omega+\frac{1}{2}} f\left(\frac{x-b}{a}\right) \right|^2 dx \\
&= \int_{\mathbb{R}} \frac{1}{a} \left| f\left(\frac{x-b}{a}\right) \right|^2 dx \\
&= \int_{\mathbb{R}} |f(t)|^2 dt = \|f\|_N^2,
\end{aligned}$$

where $t = \frac{x-b}{a}$ and $dt = \frac{dx}{a}$. $\square$

- For the parabolic representations,consider the dual character $\sigma_\omega : A \to \mathbb{D}$, of the subgroup A defined by:
$$\sigma_\omega(a,0) = a^{\varepsilon\omega}, \quad \varepsilon^2 = 0, \quad \omega \in \mathbb{R}, \tag{6.16}$$
where $|\sigma_\omega(a,0)| = 1$. This character induced a linear representation of the group Aff constructed on the space $\mathcal{L}_2^{\sigma_\omega}(\text{Aff})$ consisting of the functions $F_\omega : \text{Aff} \to \mathbb{D}$ with the property:
$$F_\omega(a,b) = \overline{\sigma_\omega(a,0)}F(1,b).$$
The norm is given by:
$$\|F_\omega\|_N^2 = \int_{\mathbb{R}} |F(1,b)|^2 db,$$
where $F(1,b) = f(b) \in \mathcal{L}_2(\mathbb{R}, db)$ the space of $\mathbb{D}$-valued on X. This space is invariant under the left Aff-shifts (5.1). Moreover, by using the formula (4.4) the parapolic representation constructed in the left homogeneous space $X = \text{Aff}/A$, can be obtained as follows:
$$[\pi_{\sigma_\omega}(a,b)f](x) = \left(\frac{1}{a}\right)^{-\varepsilon\omega+\frac{1}{2}} f\left(\frac{x-b}{a}\right), \tag{6.17}$$
where $(a,b) \in \text{Aff}$, $v \in X = \text{Aff}/A$ and $f \in \mathcal{L}_2(\mathbb{R}, \mathbb{D})$.

Theorem 6.4. *The operator π_{σ_ω} (6.17) isometry on the Krein space $\mathcal{L}_2(\mathbb{R}, \mathbb{D})$.*

Proof. Proceeding as in the proof of Theorem 6.3. □

7. Conclusions

In this paper, we study representations of the affine group associated with three types of hypercomplex units: complex, dual, and double numbers. In the complex case, these representations are unitary, acting as isometries on the Hilbert space L_2. In contrast, for dual and double numbers, the corresponding representations are isometric with respect to an indefinite inner product, and thus act on Krein spaces rather than Hilbert spaces.

A possible direction for future work is to investigate the extension of the complex representation framework to the dual and double number cases, aiming to establish a connection between the complex case and its hypercomplex counterparts. This approach is motivated by the fact that linear representations which commute with scalar multiplication can naturally interact with additional algebraic structures. In the complex case, the real vector space $\mathbb{R}^2$ acquires a complex structure via the linear transformation
$$I : \mathbb{R}^2 \to \mathbb{R}^2, \quad I^2 = -E,$$
where E is the identity transformation of $\mathbb{R}^2$. This allows us to define complex scalar multiplication by setting
$$iv = Iv, \quad \text{for } v \in \mathbb{R}^2.$$
thus turning $\mathbb{R}^2$ into a complex vector space isomorphic to $\mathbb{C}$. Similarly, since the algebras of dual and double numbers are also two-dimensional real associative commutative algebras (and

hence real vector spaces isomorphic to $\mathbb{R}^2$), they can be equipped with similar algebraic structures.This perspective provides a unified framework for studying hypercomplex representations of the affine group in relation to the classical complex case. Additionally, exploring the irreducibility of hypercomplex representations of the affine group and characterizing intertwining operators between hyperbolic (double number) and parabolic (dual number) representations present valuable avenues for research. Furthermore, given the deep connections between hypercomplex-valued affine group representations and quantum mechanics, it is worthwhile to investigate their physical applications, particularly in relation to coherent states and the harmonic oscillator.

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References

[1] A.A. Harkin, J.B. Harkin, Geometry of Generalized Complex Numbers, Math. Mag. 77 (2004), 118–129. https://doi.org/10.1080/0025570x.2004.11953236.

[2] A.A. Kirillov, Elements of the Theory of Representations, Springer, Berlin, 1976. https://doi.org/10.1007/978-3-642-66243-0.

[3] A. Khrennikov, Hyperbolic Quantum Mechanics, Adv. Appl. Clifford Algebr. 13 (2003), 1–9. https://doi.org/10.1007/s00006-003-0001-1.

[4] A.S. Elmabrok, O. Hutník, Induced Representations of the Affine Group and Intertwining Operators: I. Analytical Approach, J. Phys.: Math. Theor. 45 (2012), 244017. https://doi.org/10.1088/1751-8113/45/24/244017.

[5] E. Kaniuth, K.F. Taylor, Induced Representations of Locally Compact Groups, Cambridge University Press, 2013. https://doi.org/10.1017/cbo9781139045391.

[6] F. Catoni, D. Boccaletti, R. Cannata, V. Catoni, P. Zampetti, Geometry of Minkowski Space-Time, Springer, Berlin, 2011. https://doi.org/10.1007/978-3-642-17977-8.

[7] G.B. Folland, A Course in Abstract Harmonic Analysis, Chapman and Hall/CRC, 1995. https://doi.org/10.1201/b19172.

[8] G.F. Torres Del Castillo, Some Applications in Classical Mechanics of the Double and the Dual Numbers, Rev. Mex. Fís. E 65 (2019), 152–155. https://doi.org/10.31349/revmexfise.65.152.

[9] I.L. Kantor, A.S. Solodovnikov, Hypercomplex Numbers: An Elementary Introduction to Algebras, Springer, Berlin, 1989.

[10] I.M. Gelfand, M.A. Naimark, Unitary Representations of the Group of Linear Transformations of the Straight Line, Dokl. Akad. Nauk SSSR 55 (1947), 567–570.

[11] B. Gordon, ed., A Simple Non-Euclidean Geometry and Its Physical Basis, Springer, New York, 1979. https://doi.org/10.1007/978-1-4612-6135-3.

[12] Y. Gu, Clifford Algebra and Hypercomplex Number as Well as Their Applications in Physics, J. Appl. Math. Phys. 10 (2022), 1375–1393. https://doi.org/10.4236/jamp.2022.104097.

[13] Y. Gu, Miraculous Hypercomplex Numbers, Math. Syst. Sci. 1 (2023), 2258. https://doi.org/10.54517/mss.v1i1.2258.

[14] J. Fulman, R.M. Guralnick, Enumeration of Conjugacy Classes in Affine Groups, Algebr. Number Theory 18 (2024), 1189–1219. https://doi.org/10.2140/ant.2024.18.1189.

[15] V.V. Kisil, Hypercomplex Representations of the Heisenberg Group and Mechanics, Int. J. Theor. Phys. 51 (2011), 964–984. https://doi.org/10.1007/s10773-011-0970-0.

[16] V.V. Kisil, Geometry of Möbius Transformations: Elliptic, Parabolic and Hyperbolic Actions of $SL_2(R)$ (With DVD-ROM), Imperial College Press, 2012. https://doi.org/10.1142/p835.

[17] V.V. Kisil, Symmetry, Geometry and Quantization with Hypercomplex Numbers, Geom. Integrability Quantization 18 (2017), 11–76. https://doi.org/10.7546/giq-18-2017-11-76.

[18] A. Alghamdi, T. Alqurashi, Note on the Affine Group Representations, Eur. J. Pure Appl. Math. 18 (2025), 5735. https://doi.org/10.29020/nybg.ejpam.v18i1.5735.

[19] O. Ferrer, K. Ferrer, J. Cure, Construction of Spaces with an Indefinite Two-Metric and Applications, arXiv:2504.08239 (2025). https://doi.org/10.48550/arXiv.2504.08239.

[20] T.I. Azizov, I.S. Iokhvidov, Linear Operators in Spaces with an Indefinite Metric, Wiley, Chichester, 1989.

[21] O.F. VILLAR, E.A. ORTIZ, J.N. MARTÍNEZ, Atomic Systems in Krein Spaces, Turk. J. Math. 47 (2023), 1335–1349. https://doi.org/10.55730/1300-0098.3432.

[22] J. Bognár, Indefinite Inner Product Spaces, Springer, Berlin, (1974).

[23] T.Y. Azizov, Y.P. Ginsburg, H. Langer, On M. G. Krein's Papers in the Theory of Spaces with an Indefinite Metric, Ukr. Math. J. 46 (1995), 3–14. https://doi.org/10.1007/bf01056997.

[24] J. Cockle, III. On a New Imaginary in Algebra, Lond. Edinb. Dublin Philos. Mag. J. Sci. 34 (1849), 37–47. https://doi.org/10.1080/14786444908646169.

[25] W.K. Clifford, Preliminary Sketch of Biquaternions, Proc. Lond. Math. Soc. s1-4 (1871), 381–395. https://doi.org/10.1112/plms/s1-4.1.381.