ORIGINAL ARTICLE

INTERNATIONAL JOURNAL OF ARTIFICIAL INTELLIGENCE

ISSN: 2692-5206, Impact Factor: 12,23

American Academic publishers, volume 05, issue 04,2025

Journal: https://www.academicpublishers.org/journals/index.php/ijai

ANALYSIS OF THE FUNCTION AT INFINITY

Risolat Esomurodova,Atajanova Ruxiya, Rajapov Xursand

Abstract: This article describes methods for solving integrals using Function deduction with respect to a point at infinity. This article can be a great guide for independent learners

Keywords: Complex area, deduction ,continuous, integral, curved line.

Let the point at infinity $z =$ be an isolated singular point of the function $f(z)$. Let $U(\ \ ) = \{z : |z| > R\}$ be a neighborhood of a point at infinity and let $f(z)$ be analytic at $U(\ \ )$. Denote by C the closed contour that lies entirely in $U(\ \ )$.

Definition. The residue of the function $f(z)$ with respect to the infinitely distant point is the value of the integral $\frac{1}{2\pi i}\ _{C^-}\ f(z)dz$, where the integration along the contour C occurs in the negative direction.

The Laurent series expansion in $U(\ \ )$ for the function $f(z)$ has the form

$$f(z) = c_0 + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \quad + c_1 z + c_2 z^2 +$$

Since this series converges uniformly on the contour C , we can integrate it term by term. Noticing that

$$_{C^-}\ c_n z^n dz = 0 \text{, если } n \quad 1; \quad _{C^-}\ c_{-1}\frac{dz}{z} = c_{-1}2\pi i$$

we get after integration

$$\frac{1}{2\pi i}\ _{C^-}\ f(z)dz = -c_{-1}$$

thus, the residue of a function with respect to a point at infinity $res f(\ \ ) = -c_{-1}$.

Comment. In the case of a removable singular point lying at a finite distance, the residue is always zero. This may not be the case for a point at infinity.

ISSN: 2692-5206, Impact Factor: 12,23

American Academic publishers, volume 05, issue 04,2025

Journal: https://www.academicpublishers.org/journals/index.php/ijai

For example, the function $\frac{1}{z}$ at infinity has a removable singularity, and the corresponding residue is -1.

Theorem. If $f(z)$ is analytic at any point of the extended complex plane except for a finite number of singular points, then the sum of the residues with respect to all its singular points (including the point at infinity) is always equal to zero.

Proof. Let us describe a circle of finite radius such that all singular points fall into this circle. According to the main residue theorem, the value of $\frac{1}{2\pi i}{}_{C} f(z)dz$ is equal to the sum of residues with respect to all singular points lying inside C.

C on the other hand, the value $\frac{1}{2\pi i}{}_{C^-} f(z)dz$ is equal to the residue of the function $f(z)$ with respect to the point at infinity. Therefore, the sum of all calculations is equal to

$$\frac{1}{2\pi i}{}_{C} f(z)dz + \frac{1}{2\pi i}{}_{C^-} f(z)dz = 0.$$

Examples with solutions.

Example 1. Find the residue with respect to the point at infinity for a function $f(z) = \frac{z+1}{z}$.

Decision. $\lim\limits_{z\to} \frac{z+1}{z} = 1$ point $z =$ is a removable singular point. Expression $f(z) = 1 + \frac{1}{z}$ can be considered as its Laurent expansion in the neighborhood of an infinitely distant point. So $c_{-1} = 1$ and therefore $res f(\ \ ) = -1$.

Example 2. Calculate the integral $\frac{dz}{1+z^4}$ $_{|z|=2}$

Decision. The roots $z_k\ (k = 1,2,3,4)$ of the equation $1 + z^4 = 0$ are the poles (finite) of the function . Note that all these roots lie inside the circle $|z| = 2$.

ORIGINAL ARTICLE

INTERNATIONAL JOURNAL OF ARTIFICIAL INTELLIGENCE

ISSN: 2692-5206, Impact Factor: 12,23

American Academic publishers, volume 05, issue 04,2025

Journal: https://www.academicpublishers.org/journals/index.php/ijai

Function $f(z)=\dfrac{1}{1+z^4}$ in a neighborhood of a point at infinity has a decomposition

$$f(z)=\frac{1}{1+z^4}=\frac{1}{z^4}\;\frac{1}{1+\frac{1}{z^4}}=\frac{1}{z^4}-\frac{1}{z^8}+\frac{1}{z^{12}}-\quad ,$$

that's why $resf(\quad)=-c_{-1}=0.$

according to the above theorem, we have

$$\overset{4}{\underset{k=1}{\;}} resf(z_k)+resf(\quad)=0$$

this implies

$$\underset{|z|=2}{\;}\frac{dz}{1+z^4}=2\pi i\overset{4}{\underset{k=1}{\;}} resf(z_k)=-2\pi i resf(\quad)=0.$$

Example 3. Calculate the integral

$$\underset{|z|=3}{\;}\frac{z^{17}}{(z^2+2)^3(z^3+3)^4}dz.$$

Decision. The function $f(z)=\dfrac{z^{17}}{(z^2+2)^3(z^3+3)^4}$ has five singular points $|z|=3$ inside the circle z_k, which are multiple poles. According to him, it is convenient to use the above theorem to calculate this integral. $\overset{5}{\underset{k=1}{\;}} resf(z_k)+resf(\quad)=0$

hence,

$$\underset{|z|=3}{\;}\frac{z^{17}}{(z^2+2)^3(z^3+3)^4}dz=-resf(\quad)$$

compute $resf(\quad)$:

$$f(z)=\frac{z^{17}}{(z^2+2)^3(z^3+3)^4}=\frac{z^{17}}{z^6(1+\frac{2}{z^2})^2z^{12}(1+\frac{3}{z^3})^4}=\frac{1}{z}\;\frac{1}{(1+\frac{2}{z^2})^3(1+\frac{3}{z^3})^4}.$$

INTERNATIONAL JOURNAL OF ARTIFICIAL INTELLIGENCE

ISSN: 2692-5206, Impact Factor: 12,23

American Academic publishers, volume 05, issue 04,2025

Journal: https://www.academicpublishers.org/journals/index.php/ijai

This shows that the correct part of the Laurent expansion of this function in the vicinity of the point at infinity begins with the term $\frac{1}{z}$.

Hence, $resf(\quad) = -1$. Thus,

$$\oint_{|z|=3} \frac{z^{17}}{(z^2+2)^3(z^3+3)^4} dz = 2\pi i$$

Conclusion

It is shown to give a complete idea of the deduction of a function to a point in infinity, which is one of the basic concepts of the theory of complex variable functions. The deduction of a function with respect to an infinite point is explained, and simple methods of solving examples are explained.

References:

1. V.T. Dubrovin. Theory of functions of a complex variable theory of practice, Kazan, 2010.
2. Jumarie G (2010). Cauchy's integral formula via the modified Riemann–Liouville derivative for analytic functions of fractional order. Applied Mathematical Letters, 23(12) 1444-1450.
3. Blaya R.A., Reyes J. B., Brackx F, Schepper H.D. and Sommen F. 2012, Cauchy Integral Formulae in Quaternionic Hermitean Clifford Analysis, Complex Analysis and Operator Theory, 6(5) 971-985.
4. Estrada R, Vindas J. A general integral. Disertationes Mathematicae (Rozprawy Mat). 2012;483. 49 pages.
5. Vindas J, Estrada R. A tauberian theorem for distributional point values. Arch Math (Basel). 2008;91:247–253.