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CLASSIFICATION OF NON-ISOMORPHIC REGULAR TOURNAMENTS IN S^6 SIMPLEX

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Abstract: In this paper, we explore the classification of regular tournaments of order 7. A tournament is a directed graph where each pair of vertices is connected by a single directed edge. A regular tournament is one where each vertex has the same number of outgoing edges. We focus on identifying all non-isomorphic regular tournaments of this order and analyzing which of them are homogeneous. We also provide the skewsymmetric matrices of all 26 non-isomorphic regular tournaments. Additionally, we discuss the relevance of such structures in the context of the Lotka-Volterra map, a well-known model in population dynamics.

1 Introduction

Tournaments are a fundamental concept in graph theory, often used to model competitions where every pair of players (vertices) competes in one direction. A tournament is called regular if every player wins and loses the same number of matches. For an odd number n, a tournament is regular if each vertex has out-degree $(n-1)/2$.

2 Classification

Skew-symmetric matrices play an essential role in various areas of mathematics and physics. A real-valued matrix A is called skew-symmetric if it satisfies $A=-A^T$, where A^T denotes the transpose of A. This implies that $a_{ii}=0$ and $a_{ij}=-a_{ji}$ for all $i\neq j$.

Such matrices are inherently square and have a structure that directly influences their determinant and applications in graph theory.

Consider the mapping $V:\mathbb{R}^m\to\mathbb{R}^m$, defined by the equations

$$x_k'=x_k\left(1+\sum_{i=1}^{m} a_{ki}x_i\right),\ k=1,\dots,m,$$

where $Vx=\left(x_1',x_2',\dots,x_m'\right)$.

This mapping, under the condition $|a_{ki}|\leq 1$, is called the Lotka-Volterra mapping. The paper considers the mapping $V:S^{m-1}\to S^{m-1}$, where

$$S^{m-1}=\left\{x=(x_1,\dots,x_m):\sum_{i=1}^{m} x_i=1, x_i\geq 0\right\}.$$

A (round-robin) tournament T_n consists of n nodes $p_1,p_2,\dots,p_n$ such that each pair of distinct nodes p_i and p_j is joined by one and only one of the oriented arcs $p_i\to p_j$ or $p_j\to p_i$. If the arc $p_i\to p_j$ is in T_n, then we say that p_i dominates p_j. The relation of dominance thus defined is a complete, irreflexive, antisymmetric, binary relation. A graph G is a finite, nonempty set Y containing p vertices, together with a specified set E containing q unordered pairs of distinct vertices from Y. A directed graph or digraph D is a finite, non-empty set of vertices, together with a set of ordered pairs associated with them. Let $A=(a_{ki})$ be a skew-symmetric matrix in the general form of the Lotka-Volterra operator. Assume that for $k\neq i$, we have $a_{ki}\neq 0$. Now,

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take m points on the plane and label them with the numbers $1,2,\ldots,m$. Next, connect point k to point i as follows: if $a_{ki}<0$, draw an arrow from k to i; otherwise, draw an arrow from i to k. The resulting graph is called the tournament associated with the skew-symmetric matrix $A=(a_{ki})$ of the dynamical system, and is denoted by T_m. Let x_1,x_2 be vertices of a tournament. The notation $x_1\to x_2$ means that the edge connecting x_1 and x_2 is directed from x_1 to x_2. A finite sequence of vertices $x_1\to x_2\to\cdots\to x_p$ is called a path if $x_i\neq x_j$ for all $i\neq j$. A cycle is a closed path, i.e., $x_p=x_1$. A tournament is called strong if, for any vertices $x,y\in Y$, there exists a path from x to y. A tournament that contains no cycles is called transitive. A tournament is called homogeneous if every subtournament is either strong or transitive. Let $G=(V,E)$ be a graph, where V is the set of vertices and E is the set of edges.
Two tournaments are isomorphic if there is a relabelling of the vertices that preserves the directed edges. That is, if there exists a bijection (one-to-one mapping) $f:V_1\to V_2$ between the vertex sets of two tournaments T_1 and T_2 such that:

$$x\to y \text{ in } T_1 \Leftrightarrow f(x)\to f(y) \text{ in } T_2$$

If no such mapping exists, the tournaments are non-isomorphic.
We consider a general skew-symmetric matrix of order m:

$$A=[a_{ij}] \text{ such that } a_{ij}=-a_{ji},\ a_{ii}=0$$

From this condition, it follows that all the diagonal elements of the matrix must be zero, and the matrix takes the following form:

$$\begin{bmatrix} 0 & a_{12} & a_{13} & \ldots & a_{1m} \\ -a_{12} & 0 & a_{23} & \ldots & a_{2m} \\ -a_{13} & -a_{23} & 0 & \ldots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{1m} & -a_{2m} & -a_{3m} & \ldots & 0 \end{bmatrix}$$

The determinant of a skew-symmetric matrix of odd order is always equal to zero.

3 Regular Tournaments of Order 7

For $n=7$, there are a total of $2^{\binom{7}{2}}=2^{21}=2{,}097{,}152$ possible tournaments. However, only 4560 of these are regular tournaments, where every vertex has out-degree 3.
Among these, there are only 16 non-isomorphic regular tournaments. Two tournaments are isomorphic if one can be transformed into the other by renaming vertices.

3.1 Homogeneous Tournaments

The following 8 tournaments are homogeneous (marked with $\star$):
Homogeneous: $A_1,A_2,A_3,A_4,A_5,A_6,A_7,A_8(\star)$

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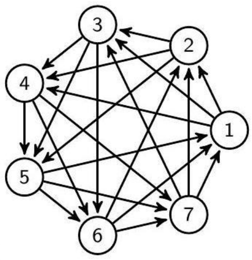
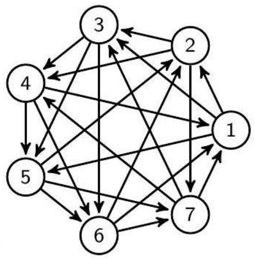


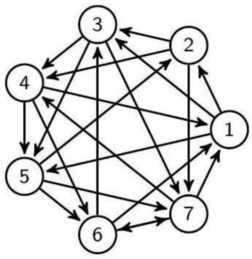
$A_1^\star$

$$\begin{pmatrix} 0 & -a_{12} & -a_{13} & -a_{14} & a_{15} & a_{16} & a_{17} \\ a_{21} & 0 & -a_{23} & -a_{24} & -a_{25} & a_{26} & a_{27} \\ a_{31} & a_{32} & 0 & -a_{34} & -a_{35} & -a_{36} & a_{37} \\ a_{41} & a_{42} & a_{43} & 0 & -a_{45} & -a_{46} & -a_{47} \\ -a_{51} & a_{52} & a_{53} & a_{54} & 0 & -a_{56} & -a_{57} \\ -a_{61} & -a_{62} & a_{63} & a_{64} & a_{65} & 0 & -a_{67} \\ -a_{71} & -a_{72} & -a_{73} & a_{74} & a_{75} & a_{76} & 0 \end{pmatrix}$$



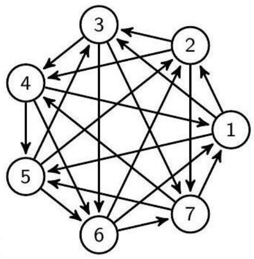
$A_3^\star$

$$\begin{pmatrix} 0 & -a_{12} & -a_{13} & a_{14} & -a_{15} & a_{16} & a_{17} \\ a_{21} & 0 & -a_{23} & -a_{24} & a_{25} & a_{26} & -a_{27} \\ a_{31} & a_{32} & 0 & -a_{34} & -a_{35} & -a_{36} & a_{37} \\ -a_{41} & a_{42} & a_{43} & 0 & -a_{45} & -a_{46} & a_{47} \\ a_{51} & -a_{52} & a_{53} & a_{54} & 0 & -a_{56} & -a_{57} \\ -a_{61} & -a_{62} & a_{63} & a_{64} & a_{65} & 0 & -a_{67} \\ -a_{71} & a_{72} & -a_{73} & -a_{74} & a_{75} & a_{76} & 0 \end{pmatrix}$$



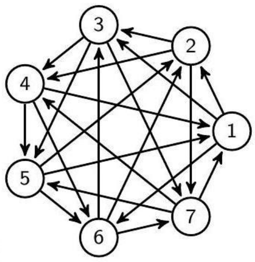
$A_2^\star$

$$\begin{pmatrix} 0 & -a_{12} & -a_{13} & a_{14} & -a_{15} & a_{16} & a_{17} \\ a_{21} & 0 & -a_{23} & -a_{24} & a_{25} & -a_{26} & a_{27} \\ a_{31} & a_{32} & 0 & -a_{34} & -a_{35} & a_{36} & -a_{37} \\ -a_{41} & a_{42} & a_{43} & 0 & -a_{45} & -a_{46} & a_{47} \\ a_{51} & -a_{52} & a_{53} & a_{54} & 0 & -a_{56} & -a_{57} \\ -a_{61} & a_{62} & -a_{63} & a_{64} & a_{65} & 0 & -a_{67} \\ -a_{71} & -a_{72} & a_{73} & -a_{74} & a_{75} & a_{76} & 0 \end{pmatrix}$$



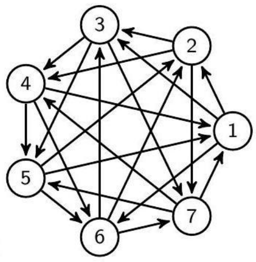
$A_4^\star$

$$\begin{pmatrix} 0 & -a_{12} & -a_{13} & a_{14} & -a_{15} & a_{16} & a_{17} \\ a_{21} & 0 & -a_{23} & -a_{24} & a_{25} & a_{26} & -a_{27} \\ a_{31} & a_{32} & 0 & -a_{34} & a_{35} & -a_{36} & -a_{37} \\ -a_{41} & a_{42} & a_{43} & 0 & -a_{45} & -a_{46} & a_{47} \\ a_{51} & -a_{52} & -a_{53} & a_{54} & 0 & -a_{56} & a_{57} \\ -a_{61} & -a_{62} & a_{63} & a_{64} & a_{65} & 0 & -a_{67} \\ -a_{71} & a_{72} & a_{73} & -a_{74} & -a_{75} & a_{76} & 0 \end{pmatrix}$$



$A_5^\star$

$$\begin{pmatrix} 0 & -a_{12} & -a_{13} & a_{14} & a_{15} & -a_{16} & a_{17} \\ a_{21} & 0 & -a_{23} & -a_{24} & a_{25} & a_{26} & -a_{27} \\ a_{31} & a_{32} & 0 & -a_{34} & -a_{35} & a_{36} & -a_{37} \\ -a_{41} & a_{42} & a_{43} & 0 & -a_{45} & -a_{46} & a_{47} \\ -a_{51} & -a_{52} & a_{53} & a_{54} & 0 & -a_{56} & a_{57} \\ a_{61} & -a_{62} & -a_{63} & a_{64} & a_{65} & 0 & -a_{67} \\ -a_{71} & a_{72} & a_{73} & -a_{74} & -a_{75} & a_{76} & 0 \end{pmatrix}$$



$A_7^\star$

$$\begin{pmatrix} 0 & -a_{12} & -a_{13} & a_{14} & a_{15} & -a_{16} & a_{17} \\ a_{21} & 0 & -a_{23} & -a_{24} & a_{25} & a_{26} & -a_{27} \\ a_{31} & a_{32} & 0 & -a_{34} & -a_{35} & a_{36} & -a_{37} \\ -a_{41} & a_{42} & a_{43} & 0 & a_{45} & -a_{46} & -a_{47} \\ -a_{51} & -a_{52} & a_{53} & -a_{54} & 0 & -a_{56} & a_{57} \\ a_{61} & -a_{62} & -a_{63} & a_{64} & a_{65} & 0 & -a_{67} \\ -a_{71} & a_{72} & a_{73} & a_{74} & -a_{75} & a_{76} & 0 \end{pmatrix}$$

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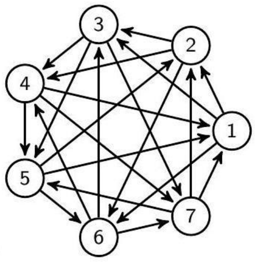
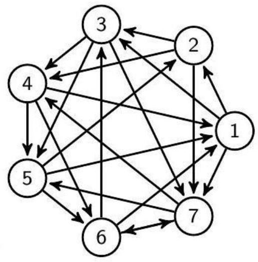


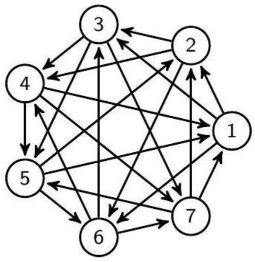
$A_6^{\star}$

$$\begin{pmatrix} 0 & -a_{12} & -a_{13} & a_{14} & a_{15} & -a_{16} & a_{17} \\ a_{21} & 0 & -a_{23} & -a_{24} & a_{25} & -a_{26} & a_{27} \\ a_{31} & a_{32} & 0 & -a_{34} & -a_{35} & a_{36} & -a_{37} \\ -a_{41} & a_{42} & a_{43} & 0 & -a_{45} & a_{46} & -a_{47} \\ -a_{51} & -a_{52} & a_{53} & a_{54} & 0 & -a_{56} & a_{57} \\ a_{61} & a_{62} & -a_{63} & -a_{64} & a_{65} & 0 & -a_{67} \\ -a_{71} & -a_{72} & a_{73} & a_{74} & -a_{75} & a_{76} & 0 \end{pmatrix}$$



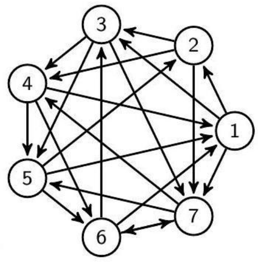
$A_8^{\star}$

$$\begin{pmatrix} 0 & -a_{12} & -a_{13} & a_{14} & a_{15} & a_{16} & -a_{17} \\ a_{21} & 0 & -a_{23} & -a_{24} & a_{25} & a_{26} & -a_{27} \\ a_{31} & a_{32} & 0 & -a_{34} & -a_{35} & a_{36} & -a_{37} \\ -a_{41} & a_{42} & a_{43} & 0 & a_{45} & -a_{46} & -a_{47} \\ -a_{51} & -a_{52} & a_{53} & -a_{54} & 0 & -a_{56} & a_{57} \\ -a_{61} & -a_{62} & -a_{63} & a_{64} & a_{65} & 0 & a_{67} \\ a_{71} & a_{72} & a_{73} & a_{74} & -a_{75} & -a_{76} & 0 \end{pmatrix}$$



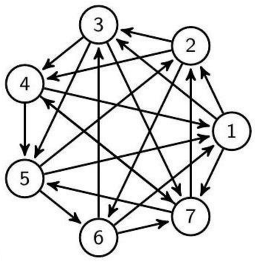
$A_6^{\star}$

$$\begin{pmatrix} 0 & -a_{12} & -a_{13} & a_{14} & a_{15} & -a_{16} & a_{17} \\ a_{21} & 0 & -a_{23} & -a_{24} & a_{25} & -a_{26} & a_{27} \\ a_{31} & a_{32} & 0 & -a_{34} & -a_{35} & a_{36} & -a_{37} \\ -a_{41} & a_{42} & a_{43} & 0 & -a_{45} & a_{46} & -a_{47} \\ -a_{51} & -a_{52} & a_{53} & a_{54} & 0 & -a_{56} & a_{57} \\ a_{61} & a_{62} & -a_{63} & -a_{64} & a_{65} & 0 & -a_{67} \\ -a_{71} & -a_{72} & a_{73} & a_{74} & -a_{75} & a_{76} & 0 \end{pmatrix}$$



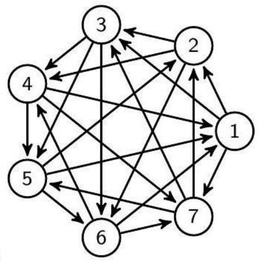
$A_8^{\star}$

$$\begin{pmatrix} 0 & -a_{12} & -a_{13} & a_{14} & a_{15} & a_{16} & -a_{17} \\ a_{21} & 0 & -a_{23} & -a_{24} & a_{25} & a_{26} & -a_{27} \\ a_{31} & a_{32} & 0 & -a_{34} & -a_{35} & a_{36} & -a_{37} \\ -a_{41} & a_{42} & a_{43} & 0 & a_{45} & -a_{46} & -a_{47} \\ -a_{51} & -a_{52} & a_{53} & -a_{54} & 0 & -a_{56} & a_{57} \\ -a_{61} & -a_{62} & -a_{63} & a_{64} & a_{65} & 0 & a_{67} \\ a_{71} & a_{72} & a_{73} & a_{74} & -a_{75} & -a_{76} & 0 \end{pmatrix}$$



A_9

$$\begin{pmatrix} 0 & -a_{12} & -a_{13} & a_{14} & a_{15} & a_{16} & -a_{17} \\ a_{21} & 0 & -a_{23} & -a_{24} & a_{25} & -a_{26} & a_{27} \\ a_{31} & a_{32} & 0 & -a_{34} & -a_{35} & a_{36} & -a_{37} \\ -a_{41} & a_{42} & a_{43} & 0 & a_{45} & -a_{46} & -a_{47} \\ -a_{51} & -a_{52} & a_{53} & -a_{54} & 0 & a_{56} & a_{57} \\ -a_{61} & a_{62} & -a_{63} & a_{64} & -a_{65} & 0 & -a_{67} \\ a_{71} & -a_{72} & a_{73} & a_{74} & -a_{75} & a_{76} & 0 \end{pmatrix}$$



A_{10}

$$\begin{pmatrix} 0 & -a_{12} & -a_{13} & a_{14} & a_{15} & a_{16} & -a_{17} \\ a_{21} & 0 & -a_{23} & -a_{24} & a_{25} & -a_{26} & a_{27} \\ a_{31} & a_{32} & 0 & -a_{34} & -a_{35} & -a_{36} & a_{37} \\ -a_{41} & a_{42} & a_{43} & 0 & a_{45} & -a_{46} & -a_{47} \\ -a_{51} & -a_{52} & a_{53} & -a_{54} & 0 & a_{56} & a_{57} \\ -a_{61} & a_{62} & a_{63} & a_{64} & -a_{65} & 0 & -a_{67} \\ a_{71} & -a_{72} & -a_{73} & a_{74} & -a_{75} & a_{76} & 0 \end{pmatrix}$$

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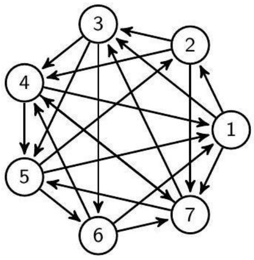
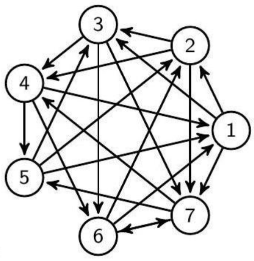


A_{11}

$$\begin{pmatrix} 0 & -a_{12} & -a_{13} & a_{14} & a_{15} & a_{16} & -a_{17} \\ a_{21} & 0 & -a_{23} & -a_{24} & a_{25} & a_{26} & -a_{27} \\ a_{31} & a_{32} & 0 & -a_{34} & -a_{35} & -a_{36} & a_{37} \\ -a_{41} & a_{42} & a_{43} & 0 & a_{45} & -a_{46} & -a_{47} \\ -a_{51} & -a_{52} & a_{53} & -a_{54} & 0 & a_{56} & a_{57} \\ -a_{61} & -a_{62} & a_{63} & a_{64} & -a_{65} & 0 & a_{67} \\ a_{71} & a_{72} & -a_{73} & a_{74} & -a_{75} & -a_{76} & 0 \end{pmatrix}$$



A_{12}

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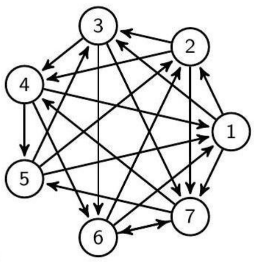
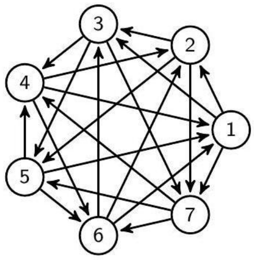


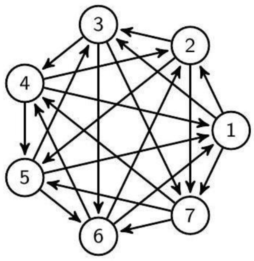
A_{13}

$$\begin{pmatrix} 0 & -a_{12} & -a_{13} & a_{14} & a_{15} & a_{16} & -a_{17} \\ a_{21} & 0 & -a_{23} & -a_{24} & a_{25} & a_{26} & -a_{27} \\ a_{31} & a_{32} & 0 & -a_{34} & a_{35} & -a_{36} & -a_{37} \\ -a_{41} & a_{42} & a_{43} & 0 & -a_{45} & -a_{46} & a_{47} \\ -a_{51} & -a_{52} & -a_{53} & a_{54} & 0 & a_{56} & a_{57} \\ -a_{61} & -a_{62} & a_{63} & a_{64} & -a_{65} & 0 & a_{67} \\ a_{71} & a_{72} & a_{73} & -a_{74} & -a_{75} & -a_{76} & 0 \end{pmatrix}$$



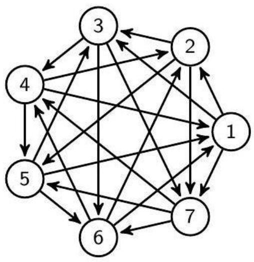
A_{14}

$$\begin{pmatrix} 0 & -a_{12} & -a_{13} & a_{14} & a_{15} & a_{16} & -a_{17} \\ a_{21} & 0 & -a_{23} & a_{24} & -a_{25} & a_{26} & -a_{27} \\ a_{31} & a_{32} & 0 & -a_{34} & -a_{35} & a_{36} & -a_{37} \\ -a_{41} & -a_{42} & a_{43} & 0 & a_{45} & -a_{46} & a_{47} \\ -a_{51} & a_{52} & a_{53} & -a_{54} & 0 & -a_{56} & a_{57} \\ -a_{61} & -a_{62} & -a_{63} & a_{64} & a_{65} & 0 & a_{67} \\ a_{71} & a_{72} & a_{73} & -a_{74} & -a_{75} & -a_{76} & 0 \end{pmatrix}$$



A_{15}

$$\begin{pmatrix} 0 & -a_{12} & -a_{13} & a_{14} & a_{15} & a_{16} & -a_{17} \\ a_{21} & 0 & -a_{23} & a_{24} & -a_{25} & a_{26} & -a_{27} \\ a_{31} & a_{32} & 0 & -a_{34} & a_{35} & -a_{36} & -a_{37} \\ -a_{41} & -a_{42} & a_{43} & 0 & -a_{45} & a_{46} & a_{47} \\ -a_{51} & a_{52} & -a_{53} & a_{54} & 0 & -a_{56} & a_{57} \\ -a_{61} & -a_{62} & a_{63} & -a_{64} & a_{65} & 0 & a_{67} \\ a_{71} & a_{72} & a_{73} & -a_{74} & -a_{75} & -a_{76} & 0 \end{pmatrix}$$



A_{16}

$$\begin{pmatrix} 0 & -a_{12} & -a_{13} & a_{14} & a_{15} & a_{16} & -a_{17} \\ a_{21} & 0 & -a_{23} & a_{24} & -a_{25} & a_{26} & -a_{27} \\ a_{31} & a_{32} & 0 & -a_{34} & a_{35} & -a_{36} & -a_{37} \\ -a_{41} & -a_{42} & a_{43} & 0 & -a_{45} & a_{46} & a_{47} \\ -a_{51} & a_{52} & -a_{53} & a_{54} & 0 & -a_{56} & a_{57} \\ -a_{61} & -a_{62} & a_{63} & -a_{64} & a_{65} & 0 & a_{67} \\ a_{71} & a_{72} & a_{73} & -a_{74} & -a_{75} & -a_{76} & 0 \end{pmatrix}$$

4 Conclusion

This classification provides valuable insight into the structural diversity of regular tournaments of a small fixed order. Such classifications are helpful in theoretical investigations and applications in social choice theory, ranking systems, and dynamical systems. Their connection to the Lotka-Volterra map highlights the practical significance of such discrete models in the analysis of biological and economic systems. Each vertex in the tournament can represent a strategy or species, and directed edges represent competitive dominance. Regular tournaments,

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particularly homogeneous ones, are useful in analyzing equilibria and cyclic behaviors in these systems. The structural properties of these tournaments influence the stability and periodicity of the Lotka-Volterra dynamics. Hence, classifying non-isomorphic regular tournaments helps understand the diversity of interaction models possible under symmetric competitive rules.

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