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RETROSPECTIVE INVERSE PROBLEMS FOR PARABOLIC EQUATIONS

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Abstract: This article examines inverse retrospective problems, which have important practical applications in various fields where it is necessary to determine the initial state or previous state of dynamic objects based on the available information about the characteristics of the field at the current time. Particular attention is paid to inverse and ill-posed problems for parabolic equations, which are among the most complex and practically significant classes of inverse problems. It is noted that retrospective inverse problems for parabolic equations have various formulations based on the analysis of initial-boundary problems for the equation.

Key words: inverse retrospective problems, parabolic equations, ill-posed problems, hyperbolic equations, retrospective inverse problems, initial-boundary value problems, heat equation, temperature measurement, field, Fredholm integral equation of the first kind, complete orthogonal system, Weierstrass theorem.

Introduction.

Inverse retrospective problems have important practical applications in various fields where it is necessary to determine the initial state or previous state of dynamic objects based on available information about the field characteristics at the current time [1, 2, 3, 4].

Inverse and ill-posed problems for parabolic equations are among the most complex and practically significant classes of inverse problems. They are less stable compared to the corresponding inverse problems for hyperbolic equations.

Retrospective inverse problems for parabolic equations have different formulations based on the analysis of initial boundary value problems for the heat equation for different methods of temperature measurement. A detailed analysis of various types of retrospective inverse problems is presented in monographs [2, 4].

As model examples of such problems, consider the following one-dimensional initial-boundary value problem for the heat equation [1, 2].

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(l,t) = 0, \tag{2}$$

$$u(x,0) = \varphi(x), \tag{3}$$

$$u(x,T) = f(x), x \quad [0,l]. \tag{4}$$

Our main task will be to reconstruct the temporal dynamics of the temperature field. To do this, we need to find the initial temperature distribution using the available information about temperature traces on some sets. In doing so, we will consider the two most common formulations of inverse problems in this area.

1. There is information about the temperature distribution in the rod at a certain point in time.

$T > 0$:
$u(x,T) = f(x), x \in [0,l]$.

2. The temperature field at a certain point is known x_0 inside the medium (segment $[0,l]$): $u(x_0,t) = g(t), t \in [t_1,t_2]$.

From the point of view of setting up an experiment and observing the object of study, the second formulation of the problem is simpler. This is due to the fact that it involves measuring temperature at one point in a certain time period.

At the same time, a comparative analysis of two formulations of problems allows us to identify important features of the reconstruction of the initial condition. These features are important for the correct setup of the experiment and the subsequent reconstruction procedure.

Thus, although the second formulation of the problem is simpler from the point of view of conducting an experiment, comparing the two approaches can provide valuable information that will help improve the process of reconstructing initial conditions and improve the quality of the experiment as a whole.

Let's explore the first setting.

So, let's pose the problem of finding the initial temperature distribution $\varphi(x)$ based on information about this field at some point in time: $t = T > 0$:

$$z(x,T) = f(x), x \in [0,l] \qquad (5)$$

To study the inverse problem, it is necessary to construct the appropriate operator equations that will allow us to connect the given and sought functions. Next, you should construct a solution to the direct problem using the method of separation of variables [2, 5] and find a solution in the form $u(x,t) = X(x)T(t)$. Separating the variables, we get $\frac{X''}{X} = \frac{1}{a}\frac{\dot{T}}{T} = const = -\lambda^2$, , from which we derive the ordinary differential equation $X'' + \lambda^2 X = 0$ and define the general solution in the form $X = C_1 \cos\lambda x + C_2 \sin\lambda x$. .Using boundary conditions, we find $C_2 = 0, C_1 \sin\lambda x = 0$ and therefore we define the eigenvalues and eigenfunctions: $\lambda_n = \frac{\pi n}{l}, X_n = \cos\frac{\pi n x}{l}, T_n = C_n e^{-a\lambda^2_n}$.

Satisfying the initial conditions, we determine the unknown constants C_n :

$$C_n = \frac{2}{l}\int_0^l \varphi(\xi)\cos\lambda_n\xi d\xi, n = 1,2..., C_o = \frac{1}{l}\int_0^l \varphi(\xi)d\xi.$$

Thus, the solution to the direct problem has the form

$$u(x,t) = \int_0^l K(x,\xi,t)\varphi(\xi)d\xi, \qquad (6)$$

Where

$$K(x,\xi,t) = \frac{1}{l} + \frac{2}{l}\sum_{n=1} \cos\lambda_n x \cos\lambda_n\xi e^{-\lambda_n a t} \qquad (7)$$

Relation (7) allows one to calculate the temperature field at any point of the rod at any time if the initial temperature distribution is known. Let's move on to consider inverse problems. When the additional condition (5) is satisfied, the inverse problem is reduced to the Fredholm integral equation of the 1st kind based on formulas (6) - (7).

$$K_1\varphi = \int_0^l K_1(x,\xi)\varphi(\xi)d\xi = f(x), x \in [0,l], \qquad (8)$$

and $K_1(x,\xi)=K(x,\xi,t)$. Note that the kernel $K_1(x,\xi)$ is symmetric, and the operator $K_1\varphi$ – self-adjoint in $L_2[0,l]$. Native kernel functions - $1, \cos\lambda_n x$ – represent a complete orthogonal system in $L_2[0,l]$ and the singular values of the operator K_1 are $\sigma_n^2 = e^{-\lambda_n^2 aT}, n=0,1,2....$, rapidly decreasing at, $n \to$.

By virtue of assessment $\left|K_1(x,\xi)\right| \quad \frac{1}{l}+\frac{2}{l} \quad {}_{n=1} \quad e^{-\lambda_n^2 aT}$ and the convergence of the series on the right side of this inequality in accordance with the Weierstrass theorem [2] kernel, $K_1(x,\xi)$ is a continuous function, therefore the operator $K_1(x,\xi)$ is completely continuous from $L_2[0,l]$ V $L_2[0,l]$,That's why K_1^{-1} is unlimited and the problem of solving the operator equation (8) requires regularization. Note that, using the orthogonality of the system $\{1,\cos\lambda_n x\}$ it is not difficult to write down the formal solution (8) using a linear operator of the form

$$\varphi(\xi)=\int_0^l K_1^*(x,\xi) f(\xi) d\xi, \qquad (9)$$

since the Fourier coefficients φ_k and f_k functions are respectively related by dependence

$$\varphi(x)\sigma_n^2 = f_k, k=0,1... \qquad (10)$$

Analyzing representation (9)–(10) for the function $\varphi(x)$, it can be noted that for the convergence of the series in this representation it is necessary to fulfill rather stringent conditions on the nature of the decrease in the Fourier coefficients of the function f_k . This condition is not met for many functions from $L_2[0,l]$, even infinitely differentiable. Therefore, the formal solution (10) cannot be used for practical calculations when the function $f(\xi)$ given in a finite set of points. This solution structure reflects the fact that the operator $K_1(x,\xi)$, the inverse of the operator describing the problem, is unbounded.

Apparently, one of the most effective ways to construct a regularized solution (8) is the method of truncated singular expansions, according to which this solution has the form

$$\varphi_\delta(x)=R_N f_\delta = \frac{1}{l}\int_0^l f_\delta(\xi)d\xi + \frac{2}{l} \quad {}_{n=1} \quad f_\delta \cos\lambda_n\xi \cos\lambda_n x e^{-\lambda_n^2 aT}, \quad (11)$$

and the number N (the regularization parameter in this case) is chosen to be consistent with the error δ of specifying the function $f\,(\left\|f-f_\delta\right\| \quad \delta)$. Let us find from what conditions it is determined. Let's estimate the error in the norm

$$L_2[0,l]:\left\|R_N f_\delta - \varphi\right\|_{L_2[0,l]} \quad \left\|R_N\right\| \; \left\|f-f_\delta\right\|_{L_2[0,l]} + \left\|R_N f - \varphi\right\|_{L_2[0,l]},$$

and $\left\|R_N\right\| = (1+ \sum_{n=1}^{N} e^{-\lambda_n^2 aT})^{\frac{1}{2}} = M(N)$ monotonically increases with increasing N a $\left\|R_N f - \varphi\right\|_{L_2[0,l]} = C(N)$ represents the norm of the remainder of a convergent series and, therefore, tends to 0 at $N \to$.

Thus, the error of the regularized solution is estimated through the quantity $m(N,\delta)=M(N) \quad \delta + C(N)$ and for a given δ there is always N*δ that delivers the minimum $m(N,\delta)$. If it is necessary to construct a solution to the inverse problem with an error not exceeding ε, provided that the error in specifying the input data is δ, then the order of magnitude of N is as follows: $N = c^{-1}\ln(\frac{\varepsilon}{\delta}), c = \pi l^{-1}(aT)^{0.5}$. So, for certain relationships between the parameters ε and δ, it may turn out that the required integer N does not exist. This means that it is impossible to achieve the required accuracy of the solution with a given error in the input data. An important role in the study of the problem is played by the parameter characterizing the rate of decrease of singular numbers. As the observation time T increases, the value of this parameter also increases, and at some point the change in the current temperature becomes significantly less than the error in its measurement. The temperature levels out, tending to an equilibrium state, and it becomes impossible to determine its initial distribution.

Notes and comments on the topic: „Retrospectiveinverse problems for parabolic equations."

In principle, to solve the posed retrospective problem, it is enough to make a change of variables, $\tau = T - t$ where T is the known observation time of the research object, and solve the problem with reversed time. The corresponding initial-boundary value problem with respect to the function $v(\tau) = u(T - \tau)$ looks like

$$\frac{\partial^2 v}{\partial x^2} = -\frac{1}{a}\frac{\partial v}{\partial t}, \quad (12)$$

$$\frac{\partial v}{\partial x}(0,\tau) = \frac{\partial v}{\partial x}(l,\tau) = 0, \quad (13)$$

$$v(x,0) = f(x). \quad (14)$$

To determine temperature $v(x,T) = \varphi(x)$ rod at moment T, it is necessary to solve the initial-boundary value problem described by equations (12)-(14). This can be done in two ways:

1. Solve the Fredholm equation of the 1st kind with a smooth kernel, given by formula (7).
2. Directly solve problem (12)-(14) using the method of separation of variables, presenting the solution in the form of a series. The terms of this series, according to (11), contain exponentially growing factors, therefore, for the convergence of the series, very strict restrictions on the function are required $f(x)$ in the form of a very rapid decrease in the coefficients of its Fourier series. Even infinite differentiability of a function $f(x)$ does not guarantee the convergence of this series.

Conclusion.

Inverse retrospective problems have important practical applications in various fields where it is necessary to determine the initial or previous state of dynamic objects based on available information about the field characteristics at the current time. Inverse and ill-posed problems for parabolic equations, such as the heat equation, are among the most complex and practically significant classes of inverse problems. These problems are less stable compared to inverse problems for hyperbolic equations. There are many different formulations of retrospective inverse problems for parabolic equations, based on the analysis of initial boundary value problems for different methods of temperature measurement. A detailed analysis of these types of inverse problems is presented in specialized literature.

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