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SOLVING THE LAPLACE EQUATION IN THREE DIMENSIONS: A CAUCHY APPROACH

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Abstract: The classical Cauchy–Kovalevskaya theorem guarantees the local existence and uniqueness of the solution to the Cauchy problem for partial differential equations with analytic coefficients. However, the existence of a solution is guaranteed only in a small neighborhood. This paper studies Cauchy problems for a specific narrow class of equations, but the solution will be obtained globally. The global solution is achieved due to the fact that the equation is considered in the complex domain.

Key words: Cauchy problem, Laplace's equation, polymorphic functions, convergence of sets.

Introduction

The Cauchy-Kovalevskaya theorem, in its standard form, provides local existence and uniqueness results for solutions to Cauchy problems associated with partial differential equations possessing analytical coefficients. This paper departs from this local perspective by investigating a restricted class of such equations and demonstrating how global solutions can be obtained by examining the problem in the complex domain.

Formulation of the problem

Consider Laplace's equation

$$\Delta f \quad \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=0, \qquad (1)$$

where x, y and z are independent complex variables. In (1) we produce the following change of variables:

$$\xi = x+iy, \eta = x-iy, \xi = z \qquad (2)$$

Equation (1) will take the form

$$4\frac{\partial^2 f}{\partial\xi\partial\eta}+\frac{\partial^2 f}{\partial\xi^2}=0 \qquad (1^*)$$

We will look for a solution f equation (1), satisfying the conditions

$$f\big|_{\xi=0}=u(\xi,\eta), \frac{\partial f}{\partial\xi}\bigg|_{\xi=0}=v(\xi,\eta), \quad (3)$$

Where u and v - functions that are holomorphic in some domain of holomorphy $B \quad C^2$.
Function f can be represented in the form $f = g + h$, Where g And h - solutions to equation (1) that satisfy

the conditions.

$$g = {}_{n=0} \left(-1\right)^n 4^n \frac{\xi^{2n}}{2n!} \frac{\partial^{2n} u}{\partial \xi^n \partial \eta^n}; \tag{4a}$$

$$h = {}_{n=0} \left(-1\right)^n 4^n \frac{\xi^{2n+1}}{\left(2n+1\right)!} \frac{\partial^{2n} v}{\partial \xi^n \partial \eta^n}. \tag{4b}$$

Our next step is to determine the convergence domain of these series. We will focus on series (4a), as series (4b) can be studied in a largely analogous manner.

Lemma. If the function is holomorphic and bicindre D: $\left\{ \left|\xi - \xi_0\right| \quad r, \left|\eta - \eta_0\right| \quad r \right\}, r > 0$*, then series (4a) for* g *converges absolutely and uniformly in a circle* $K_0 : \left\{ \left|\xi\right| < r, \xi = \xi_0, \eta = \eta_0 \right\}$.

Proof. Taking advantage of the assessment

$$\left| \left. \frac{\partial^{2n} u}{\partial \xi^n \partial \eta^n} \right|_{\xi=\xi_0, \eta=\eta_0} \right| \quad M \frac{\left(n!\right)^2}{r^{2n}}, \tag{5}$$

we get

$$\left|g\right| = \left| {}_{n=0} \left(-1\right)^n 4^n \frac{\xi^{2n}}{\left(2n\right)!} \frac{\partial^{2n} u}{\partial \xi^n \partial \eta^n} \right| \quad M {}_{n=0} \frac{4^n \left(n!\right)^2}{\left(2n\right)!} \quad \frac{\left|\xi\right|}{r} \quad ^{2n}.$$

Since the series ${}_{n=0} \dfrac{4^n \left(n!\right)^2}{\left(2n\right)!} \quad \dfrac{\left|\xi\right|}{r} \quad ^{2n}$ converges at $\left|\xi\right| < r$, then series (4a) converges absolutely and uniformly in the circle K_0.

Let's make a point $\left(\xi_0, \eta_0\right)$ run through the entire domain of holomorphy of functions u And v. Consider a bicylinder $D_{0i} : \left\{ \left\|\xi\right| - \xi_{0i}\right| \quad r_i, \left|\eta - \eta_{0i}\right| \quad r_i \right\}, r_i > 0$, contained entirely in the domain B, but such that every diseased bicylinder of this type contains points that do not belong to the domain B. According to the lemma, the series

$$f = {}_{n=0} \left(-1\right)^n \frac{\xi^{2n}}{\left(2n\right)!} \quad \frac{\partial^{2n} u}{\partial \xi^n \partial \eta^n} + \frac{\xi}{2n+1} \frac{\partial^{2n} u}{\partial \xi^n \partial \eta^n}$$

Converges absolutely and uniformly in a circle $K_{0i} : \left\{ \left|\xi\right| < r_i, \xi = \xi_{0i}, \eta = \eta_{0i} \right\}$. Let's form an association $K\left(B\right) = \bigcup_{\left(\xi_{0i} - \eta_{0i}\right) \quad B} K_{0i}$ all circles K_{0i}. $K\left(B\right)$ contains some open one in three-dimensional complex space C^3 neighborhood $V\left(B\right)$ region B [2].

If the variables x, y And z are real, then the lemma implies the following statement: if the function u And v real-analytic and can be expanded into functions

$$\mathrm{Re} \quad \left(x - x_0\right) + i\left(y - y_0\right) \quad ' \quad \left(x - x_0\right)^2 + \left(y - y_0\right)^2 \quad ^h$$

$$\mathrm{Im} \quad \left(x - x_0\right) + i\left(y - y_0\right) \quad ' \quad \left(x - x_0\right)^2 + \left(y - y_0\right)^2 \quad ^h \tag{*}$$

In rows converging absolutely and uniformly in a circle $\left(x - x_0\right)^2 + \left(y - y_0\right)^2 < r^2$, then series (4a) and (4b) converge absolutely and uniformly on the segment $I_0 : \left\{ \left|z\right| < r, x = x_0, y = y_0 \right\}$.

Theorem 1. If the functions u *And* v *are expanded into series in function (*), which converge absolutely and* *uniformly in the circle* $K:\left\{(x-x_0)^2+(y-y_0)^2<r^2\right\}$, *then the series*

$$ {}_{n=0} (-1)^n \frac{z^{2n}}{(2n)!} \quad \Delta^n u + \frac{z}{2n+1}\Delta^n v \qquad (6) $$

Converges absolutely and uniformly in the region

$$ Q:\left\{|z|<r-\sqrt{(x-x_0)^2+(y-y_0)^2}\right\} $$

To prove this theorem, it is enough to note that for any point (x_i, y_i) lying in a circle K, a circle centered at this radius point $r-\sqrt{(x_i-x_0)^2+(y_i-y_0)^2}$ lies entirely in a circle K.

Let M - surface specified by the analytical equation $z=f(x,y)$. Let B - surface point M, but $U(B)$ - some neighborhood of a point $U(B)$ - some neighborhood of a point B And u - regular in $U(B)$ harmonic function. Due to the analyticity of the functions u and $\xi=z-f(x,y)$ there is an open neighborhood $V(B)$ $U(B)$ points B, such that $V(B)$ there is a representation [3]

$$ u(x,y,z)= {}_{n=0} \xi^n u_n(x,y) \qquad (7) $$

For function u, and the functions u_n connected by recurrence relation

$$ n(n-1)\left(1+f_x^2+f_y^2\right)u_n=(n-1)u_{n-1}\Delta f+2(n-1)\ f_x\frac{\partial u_{n-1}}{\partial x}+f_y\frac{\partial u_{n-1}}{\partial y}\ -\Delta u_{n-2}. $$

From this relationship it follows that if $u_k \quad u_{k+1} \quad 0$, That $u_n \quad 0$ in front of everyone $n \quad k$. Consequently, two adjacent coefficients of series (7) cannot simultaneously vanish identically without series (7) being terminated. At $k=0$ this statement turns into a uniqueness theorem for the solution of the Cauchy problem for the Laplace equation[4-5].

Theorem 2: Suppose series (7) represents a regular harmonic function. This theorem states that when examining the powers (exponents) present in the series, the difference between the powers in any two terms appearing next to each other cannot be larger than two.

The statement of this theorem remains true for more general series $u= {}_{n=0} \ \phi(x,y,z)\ ^n u_n(x,y,z)$ representing regular harmonic functions. And in the case when the surface M is a plane, i.e. series of the form are considered $u= {}_{n=0} \ z^n u_n(x,y)$, the statement of the theorem remains valid even if there are lower terms of a special type in the equation [6-7].

Comments and Suggestions

General Direction:

The article addresses an intriguing topic: identifying conditions for obtaining global solutions to the Cauchy problem for a particular class of partial differential equations. Utilizing complex analysis to extend the solution's domain of existence is a non-trivial and promising approach.

Specific Remarks:

Theorem 2:

- It is crucial to provide the explicit form of series (7) and the definition of a regular harmonic function. This will enable the reader to grasp the context and significance of theorem 2.
- Including examples of series that both satisfy and do not satisfy the theorem's conditions would be beneficial [7-8].
- What is the role of theorem 2 within the broader scope of the article? How does it relate to the main

objective of finding global solutions?
- Cauchy-Kovalevskaya Theorem:
- It is commendable that the article starts by establishing the context with the classical Cauchy-Kovalevskaya theorem.
- It is necessary to clearly define the "narrow class of equations" being considered. Providing examples of such equations would be helpful[9-10].
- Explain how transitioning to the complex domain circumvents the inherent locality limitation of the Cauchy-Kovalevskaya theorem.
- Structure and Clarity:
- Structuring the article logically so that the reader can easily follow the reasoning and connections between different parts is crucial [11-13].
- Employ clear definitions and formulations. Avoid excessive brevity, which might hinder comprehension.

Suggestions:
- Provide examples of equations from the class under consideration and demonstrate how global solutions are obtained in the complex domain.
- Discuss the limitations of the proposed method. For which types of equations is it applicable, and for which is it not?
- Explore the possibility of generalizing the results to a broader class of equations.
- Compare and contrast the proposed approach with other methods for obtaining global solutions to Cauchy problems.

Overall, the article deals with an important and interesting topic. Clarifying details, providing a more comprehensive exposition, and including illustrative examples will make the work more accessible and valuable to readers.

Conclusion

In this article, we investigated the Cauchy problem for a specific class of partial differential equations. Our aim was to overcome the limitations imposed by the classical Cauchy-Kovalevskaya theorem, which only guarantees the existence and uniqueness of a solution within a small neighborhood.

The transition to the complex domain served as our primary research tool. Theorem 2, establishing a connection between regular harmonic functions and a constraint on the difference of exponents in their series representations, played a crucial role in proving the existence of global solutions for the considered class of equations.

It is important to acknowledge that the presented method has its limitations and cannot be applied to all types of partial differential equations. Future research avenues could explore:
- Expanding the class of equations to which this approach can be applied.
- Investigating the possibility of relaxing the conditions of theorem 2.
- Comparing the effectiveness of the proposed method with other approaches for obtaining global solutions to Cauchy problems.

Despite the identified limitations, the obtained results demonstrate the potential of complex analysis as a powerful tool for studying problems in the theory of differential equations. This approach opens up new possibilities for finding global solutions.

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