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ITERATIVE ASSOCIATIVE METHOD OF DYNAMIC CONSEQUENTS (IAMDC) FOR MAMDANI FUZZY SYSTEMS TYPE I AS AN ATTENTION MECHANISM

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The description of a fuzzy inference system (FIS) is qualitative, known as the expert-driven approach. Generally, antecedents are determined combinatorially, while consequents are defined based on expert knowledge, which often involves issues of high interpretability and imprecision. Therefore, this article presents an Iterative Associative Method of Dynamic Consequents (IAMDC) as an Attention Mechanism. The k-means algorithm is used to group premises based on the distance-similarity of their fuzzy values, in numerical and objective form, thereby eliminating the imprecision of experts in defining the consequents' categories, making a significant contribution to the development of FIS. Furthermore, based on this distance, the consequents of each rule are classified. A normalization phase is proposed for the distances obtained in each rule to identify the most significant probability of occurrence, and based on this, estimate the parameters of the consequences in each rule using the corresponding product fuzzy operator, which is another significant contribution of this research. The novel prototype is validated through a simulation model based on the case of an automotive manufacturing company, in which supplier evaluation is developed using four evaluation criteria. Five possible combinations—Prod-Max, Min-Max, Max-Min, Max-Max, and Max-Prod — were used as inference rules for the proposed associative fuzzy inference system (AFIS) and compared with the present evaluation method in the company and with a conventional fuzzy inference system. The results of the proposed system were more accurate and reliable, with lower mean-squared error values.

Keywords: Mamdani Fuzzy Systems; Attention Mechanism; Data-Driven Approach; Expert-Driven Approach; K-Means.

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1. INTRODUCTION

The development of new ideas for improving products or services is based on long-term collaboration between a company and its suppliers, facilitating the exchange of information and allowing for thorough monitoring of the production process. (Nguyen *et al.*, 2020). Moreover, a robust and reliable evaluation framework that integrates qualitative and quantitative criteria for measuring supplier performance needs to be identified. Zeng (2020) identified the relationships that companies should prioritize based on key distinguishing factors. FIS models human opinion on a specific subject or circumstance and addresses uncertain situations. This task is complex because understanding and interpreting qualitative criteria vary subjectively, depending on the experts' experience. In addition, subjective judgments enable the connection between the use of such qualitative criteria (Chatterjee *et al.*, 2019). Finally, a lack of time or knowledge should be considered as a situation or constraint that may arise during the assessment process (Wang *et al.*, 2018). The information collection method may be vague or incomplete due to the way observations are made (Kusi-Sarpong *et al.*, 2018). Another source of variation is the evaluator's perception and ability to define their opinion precisely, even under conditions where the parameters are subjective or do not have clear numerical characteristics (Senkivskyy *et al.*, 2025). Consequently, a set of factors is generated that contribute to ambiguity and variation in the valuation process.

Imprecise and uncertain relationships between variables can be modeled with fuzzy logic theory (Cemek *et al.*, 2025). Similarly, the fuzzy description conveys the importance of relationships between variables in a subjective manner, as it is not possible to represent these relationships in a numerical format (Yang *et al.*, 2022). García-Pardo *et al.* (2024) describe fuzzy logic as a method for determining the predicted numerical values of parameters through linguistic variables in a mathematical language that enables decisions to be made in uncertain environments, whose meaning is initially presented as a set of verbal descriptions. FIS models human opinion on a specific subject or circumstance and addresses uncertain situations by assigning a degree of belonging in ambiguous terms, using an interval of [0, 1] (Feng, 2025). FIS has emerged as a method to effectively provide explainability to data-driven models using linguistic rules that humans can interpret (Salinas *et al.*, 2025). Fuzzy Rule Base Systems comprise two main components: the Knowledge Base (KB) and the fuzzy inference engine. The KB comprises linguistic rules and parameters that define fuzzy sets used in these rules (Gallo *et al.*, 2020). Moreover, Fuzzy Inference Systems (FIS) have been widely proposed in various approaches to

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assist the evaluation of suppliers. This research proposes a novel FIS for evaluating suppliers in the automotive sector, reducing subjectivity in assigning output variable labels by utilizing an intelligent clustering technique.

2. LITERATURE REVIEW

2.1 Mamdani fuzzy inference systems

Supplier selection is a crucial component in supply chain management, as it directly impacts operational efficiency and production continuity. In analyzing a state-of-the-art search for diffuse systems applied to supplier classification, significant works have been found, such as Runtuk *et al.* (2025), who compared two fuzzy techniques to optimize supplier selection for a manufacturing company in Indonesia versus relying on a single supplier. In addition, Qiu *et al.* (2024) concluded that to gain a more nuanced understanding of vendor reliability and implement proactive mitigation strategies, they explored the challenges and opportunities associated with implementing effective supply chain risk management strategies within this industry. The paper highlights that the sports supply industry can significantly benefit from the application of probabilistic language sets to enhance risk assessment and management. Therefore, Sayyadi *et al.* (2018) presented a new approach based on fuzzy inference combined with the simple fuzzy grid method to help decision-making in supplier evaluation for development. This method classifies the supplier's performance by article, following a pattern classification procedure based on decision rules. Moreover, Dobos *et al.* (2019) developed a Mamdani-type Fuzzy Inference System (FIS). The expertise of three senior managers in the supplier department of the company under review was considered to designate the consequences of each fuzzy rule.

Furthermore, Uluskan and Beki (2024) proposed a customized version of a less-preferred methodology in decision-making processes, i.e., the interval type-2 fuzzy ORESTE. The results revealed that IT2F ORESTE assigned the highest ranks to projects with high earning potential, low cost, low number of operations, and high production capacity. In contrast, fuzzy TOPSIS failed to select the best project. Milovanović *et al.* (2024) also developed a model for measuring key performance indicators (KPI) of processes implemented in business organizations. The model was developed in four phases, based on top-order fuzzy sets, more precisely using fermatean fuzzy sets. The first step is to break down the process into sub-processes and define key performance indicators. In the third phase, experts evaluate key performance indicators at the process level. In the fourth phase, KPIs are measured using the Multi-Criteria Optimization and Compromise Solution method.

In each of the cited references, the experts assigned the consequences for each of the fuzzy rules. The results have a high level of interpretation but are not compared with any other method to assess their accuracy. The proposed research is based on analyzing a traditional evaluation method used within the automotive sector organization. By analyzing the input-output relationship, a mechanism of attention is developed that relates the diffuse values in each rule to determine their consequents objectively, yielding high precision when compared to the traditional method.

In a specific interpretation, the behavior is described quantitatively using fuzzy values, and in a general interpretation, natural language behavior explains the FIS behavior in an expert-based approach. It should be noted that fuzzy numbers do not need a linguistic interpretation (Görener *et al.*, 2017). Moreover, Awasthi *et al.* (2010) mention that there are always two contradictory requirements in the modeling process: the model's ability to faithfully represent the natural system (precision) and its ability to express the behavior of the actual system in an understandable way (interpretability).

FIS establishes a knowledge base based on the expert's knowledge to define the relationship between the structure of the antecedent and the consequent through a set of logical rules. The results are not very accurate. Generally, the antecedents of fuzzy rules are established objectively as a combinatorial problem. However, the experts interpret the consequences without knowing the diffuse values in each rule, which do not need any linguistic interpretation but a function to relate them. This research, therefore, presents a hybrid approach to maintain high system interpretation and reliability of results when making comparisons with the current evaluation method by using an IAMDC algorithm as an attention mechanism to identify the relationship between the structure of the antecedents and the structure of the consequents as an attention mechanism with a data-driven approach.

2.2 Takagi-Sugeno-Kang fuzzy inference systems

The basis of precision in any inference system is based on identifying or characterizing the system through a function that relates inputs and outputs. System Identification is essential in problems with a black box approach based on inputs and outputs. Identifying the structure and parameters is necessary because the qualitative model is built without prior knowledge of the intended system. See Table 1.

Table 1. System Identification.

Classification	Structure part	Problem type
Structure identification I	Input candidates	Induction problem
Structure identification I	Input variables	Combinatorial problem
Structure identification II	Input-Output relationship (Number of rules)	Ordinary identification
Structure identification II	Partition of input space	Combinatorial problem
Parameters identification	Input-Output coefficients	Coefficients determination

Based on an expert-driven approach, FIS does not use the black box approach. A pre-analysis of the input-output relationship through system identification is not used. The results of the models are not compared with any previous relationship of the system behavior. System identification is a mechanism that can be used as a complement in diffuse inference systems to improve the accuracy of results, as is the case with fuzzy systems, Takagi-Sugeno-Kang (TSK), which are used to model complex nonlinear systems by combining local rules based on linear models, representing an advanced extension of classical fuzzy logic (Zhang *et al.*, 2024). The overall output of the system is obtained as a weighted combination of the outputs of all activated rules. Each rule is activated based on the degree of belonging of the entries to specific fuzzy sets (Jovanovi'c *et al.*, 2022). Different patterns have their respective homogeneity, and the samples from different groups have explicit or implicit homogeneity. Due to its excellent interpretability and sorting performance, the TSK has garnered significant attention (Gu *et al.*, 2023). In this context, one of the most common approaches to TSK modeling is the use of fuzzy clustering analysis. Several key problems must be addressed to develop a fuzzy-rule-based system using clustering. We address three essential questions: 1) clustering on the appropriate domain(s), 2) determining the number of optimal rules, and 3) determining an initial estimate of system parameters (Sugeno & Yasukawa, 1993). Thus, there are three different approaches in which fuzzy clustering can be used: input space analysis, output space analysis, and combined input-output space analysis (Cordón, 2011). The k-means iterative algorithm is employed as a mechanism of attention in our research to determine the dynamic consequences of the fuzzy rules based on the black box approach when analyzing the input-output relationship using the traditional evaluation method of the organization.

The literature review of research that integrates diffuse TSK-type systems and clustering techniques reveals a notable lack of recent studies. This finding underscores the importance of conducting current research to advance the state of the art in systems of diffuse inference, utilizing clustering techniques. Furthermore, fuzzy clustering algorithms, such as the popular fuzzy c-means algorithm, are frequently used to automatically divide the data space into fuzzy granules (Höppner *et al.*, 2003). This application utilized grouping techniques and identified parameters to partition the system's entry spaces, unlike our research, where the input space is fully defined. Applications of TSK fuzzy models are used for identifying the structure and defining the number of rules relating to grouping in the input-output relationship space, including estimation of the focal points of the rules (antecedent parameters), in which the fuzzy membership functions are obtained, and rules are determined (Rezaee *et al.*, 2010).

Additionally, a neuroflex system is developed based on an input-output data set. First, the data set is grouped based on input and output similarity tests. Finally, a diffuse IF-THEN rule is extracted from each group to form a knowledge base (Lee *et al.*, 2003). In addition, Mendes *et al.* (2018) presented an approach based on the use of a set of fuzzy rules, which extracts knowledge to learn and generate new fuzzy rules in a fuzzy controller capable of mimicking the control action of an expert human operator. Similarly, Hernandez-Julio *et al.* (2019) employed clusters and pivot tables to construct the knowledge base, and the pivot table facilitates the extraction of the fuzzy rule base from the input/output data naturally. Data and its relationships represent the knowledge-based management subsystem in a decision support system. Finally, Wadhawan *et al.* (2013) presented the application of memetic algorithms (MAs) for identifying complete fuzzy models, which include the design of membership functions for input and output variables, and generating fuzzy rules from the numeric data set. The above research studies used the clustering approach to identify the structure and quantitatively determine the number of fuzzy rules. Our research defines the number of rules in combinatorial form using the total number of possible combinations. The objective is to identify the relationship between the predecessor's structure and the consequent's structure to determine its parameters using the k-means algorithm.

TSK fuzzy model applications can be observed when the cluster estimation method is combined with a linear least squares estimation procedure to identify fuzzy models from numerical data, yielding an initial estimate of the parameters of fuzzy systems. With fixed antecedent parameters, the TSK model is transformed into a linear model (Chiu, 1994). In addition, constructing interpretable Takagi-Sugeno (TS) models using clustering is addressed. The groups obtained by the Gath-Geva (GG) algorithm enable the identification of previous fuzzy sets and their corresponding consequential parameters in the TS model (Abonyi *et al.*, 2002). Furthermore, the simultaneous definition of fuzzy subspaces and the determination of parameters in the consecutive parts of the TSK rules are based on the Adaptive Fuzzy Regression Clustering algorithm (Chuang, 2003). Each of the three research studies cited employed the clustering approach to identify the structure and determine the parameters of the consequents, which is also relevant to our research. However, a different approach is employed, preserving the classification defined by the expert and which is used to determine the number of clusters required through a mechanism of attention that relates the diffuse values of the knowledge base. A conceptualization was made after analyzing the importance of system identification when using the black box approach for TSK fuzzy modeling, adapting it to Mamdani systems while preserving its linguistic interpretation in determining the consequents. It is essential to note that in the expert-driven approach, the determination of the consequents introduces a limitation to the model's precision, given that the fuzzy values of the rules do not have and do not need a linguistic interpretation. Therefore, it is concluded that expert-based models have high interpretation and low accuracy.

An attention mechanism is a machine learning technique that directs deep learning models to prioritize the most relevant parts of input data. The attention mechanisms have two central processes: A process of reading raw data streams and converting them into vector embeddings, in which each element of the stream is represented by its own vector of characteristics. The process of quantifying similarities and relationships between vectors is called alignment. By a softmax probability distribution, all values are normalized in a range between 0 and 1, where an attention weight of 1 means that this element should receive 100% attention.

Transformer architectures use an alignment scoring function based on multiplication, calculating the similarity between hidden state vectors using their point product. If the vectors P and K are aligned, that is, they have similar meanings, multiplying them yields a significant value, and the softmax function assigns a considerable value as an attention weight. If they are not well aligned, your spot product will be small or negative, and the subsequent softmax function will result in a low attention weight.

The proposed attention mechanism uses the fuzzy values of each variable in the antecedents of the knowledge base as input vectors into the raw data reading process. Euclidean distance measurement is used as a mechanism to quantify similarity between input vectors by obtaining the weights of alignment. Finally, the relationship between each fuzzy rule is weighted using a normalized stage to evaluate the associative iterative method for dynamic consequents.

In summary, the major contributions of this paper are:

1. Identification of the structure by determining an input-output relationship using the k-means algorithm as an attention mechanism in the proposed fuzzy system.
2. Classification of the consequent parameters dynamically maintains the linguistic classification proposed by the expert's interpretation.
3. A normalization phase is proposed to identify the most significant probability of occurrence in the knowledge base and return to a fuzzy domain in a range of 0 to 1, using the SoftMax function.
4. Development of a hybrid proposal of an expert-driven approach and a data-driven approach to preserve high interpretability and accuracy for a Mamdani fuzzy system.

The rest of this paper is organized as follows. Section 3 describes the proposed method. Section 4 presents and discusses the results of IAMDC, and Section 5 concludes the proposed method.

4. METHODOLOGY

This paper presents a methodology for developing the proposed Associative Fuzzy Inference System (AFIS) using a novel IAMDC algorithm as an attention mechanism. AFIS is structured to compare the results obtained concerning the Current Evaluation System (CES) used in the case study. Finally, once the evaluation system has been structured, a simulation model is carried out to test the operation of the proposed system by evaluating its performance in multiple cases. A flow diagram of the methodology used is shown in Figure 1. The expert-driven approach and the data-driven approach are used in the confirmation of the AFIS proposed approach: An iterative associative method is proposed to determine the consequents of each fuzzy rule based on the construction of a geometric rule base.

The proposed approach begins with the fuzzification of linguistic variables. A fuzzy rule base is constructed using an expert-driven approach. The k-means algorithm is applied until convergence to group the antecedents, determine the optimal consequents, and construct the geometric rule base as an attention mechanism. A normalization phase is proposed for the distances obtained in each rule to identify the most significant probability of occurrence, and based on it, to estimate the corresponding product. The Cartesian product $\mathcal{F}(\mathbf{X} \times \mathbf{Y})$ uses mathematical product operators. A maximum argument is used for aggregating the normalized rule base. Finally, a weighted evaluation is performed, resulting in a highly interpretable and precise system.

3.1 Conforming a Current Evaluation System (CES) of an automotive sector company.

This research is developed considering the case of an automotive company dedicated to manufacturing auto parts, where supplier evaluation is constantly required to select the best components and suppliers for the new models. It is crucial to evaluate aspects related to quality to avoid defects that could develop during the process and to minimize scrap rates. Similarly, it is essential to evaluate delivery times to meet production plans and customer orders. Likewise, production should avoid missing materials or confusion in inventory management due to excess material or a disagreement in the request. Finally, the supplier must use the innovation criterion to submit proposals for improvements in its products and processes to enhance its performance and reduce costs. Additionally, production stops resulting from missing materials or confusion in inventory management, such as excess material or a disagreement in the request, should be avoided.

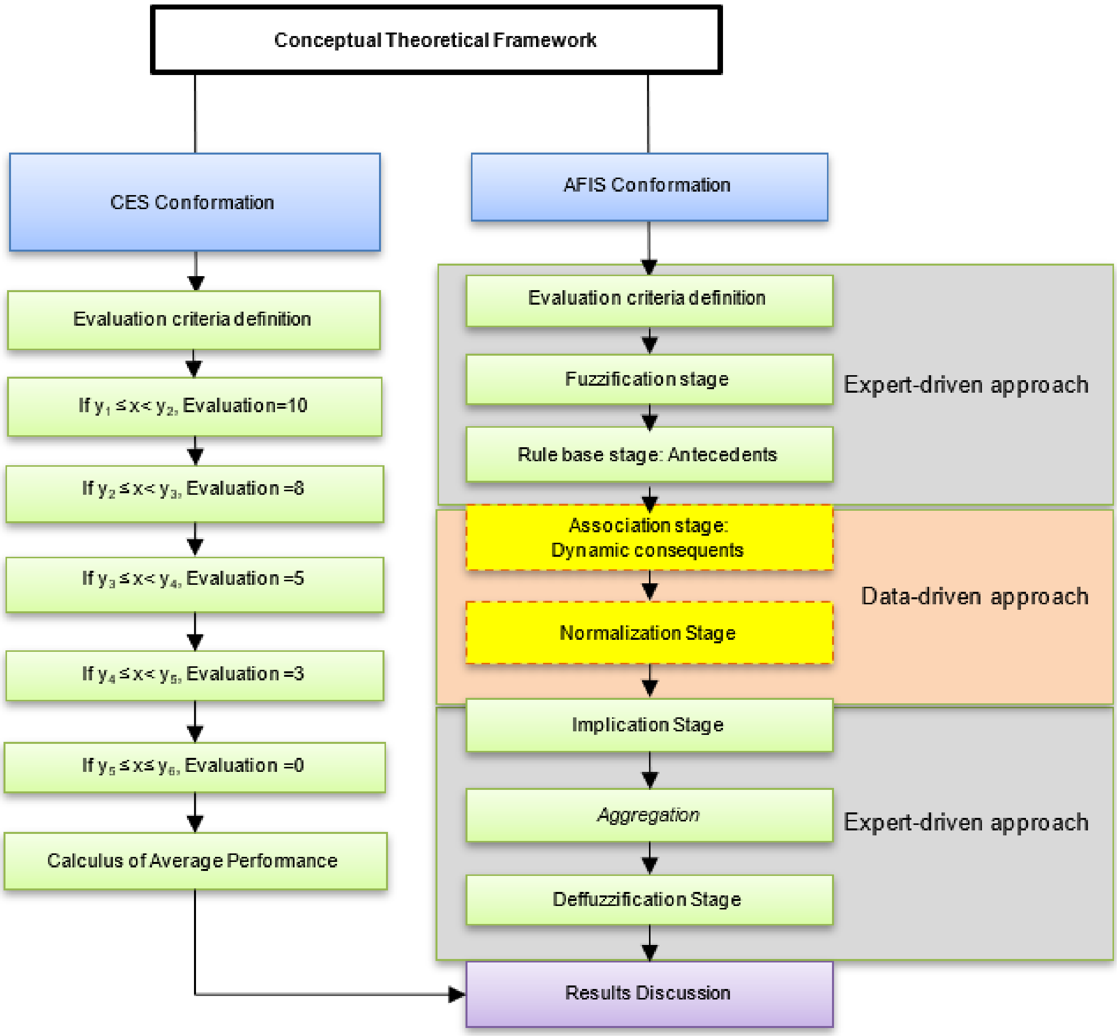


Figure 1. Methodology.

A straightforward quantitative evaluation system is employed, constructed using simple logical rules and based on a performance scale established for a given evaluation criterion, and using logical operators: less than, equal to, and greater than. Therefore, the evaluator assigns an initial rating for each criterion based on the supplier's performance. Lastly, a Final Score (FS) is assigned based on the average of the criteria. See Equation (1).

$$\mathrm{FS} = \frac{\sum_{i=1}^{n} x_i}{n} \tag{1}$$

CES is observed in Table 2. The four evaluation criteria are: Delivery time ($\mathbf{x}_1$), Quality ($\mathbf{x}_2$), Parts Discrepancy ($\mathbf{x}_3$) and Innovation ($\mathbf{x}_4$).

The CES method used is simple, consistent, and logical. However, the system is limited to 5 rules for each criterion. This approach to evaluating suppliers offers an opportunity to enhance the evaluation by considering each criterion as a linguistic label and increasing the number of fuzzy rules. Jointly evaluate the four criteria based on the extraction of knowledge from the data and obtain more precise results while preserving a perfect interpretation of the model.

Table 2. Current Evaluation System.

$\mathbf{x}_1$ Day's Range	Score	$\mathbf{x}_2$ Defects Range	Score	$\mathbf{x}_3$ Parts Range	Score	$\mathbf{x}_4$ Innovations Range	Score
$0 \leq X < 5$	10	$0 \leq X < 10$	10	$-1 \leq X \leq 1$	10	$13 \leq X < 17$	10
$5 \leq X < 10$	8	$10 \leq X < 20$	8	$-3 \leq X \leq -2$ OR $2 \leq X \leq 3$	8	$10 \leq X \leq 12$ OR $18 \leq X \leq 20$	8

x_1 Day's Range	Score	x_2 Defects Range	Score	x_3 Parts Range	Score	x_4 Innovations Range	Score
$10 \le X < 15$	5	$20 \le X < 30$	5	$-6 \le X \le -4$ OR $4 \le X \le 6$	5	$6 \le X \le 9$ OR $21 \le X \le 23$	5
$15 \le X < 20$	3	$30 \le X < 40$	3	$-8 \le X \le -7$ OR $7 \le X \le 8$	3	$3 \le X \le 5$ OR $24 \le X \le 27$	3
$20 \le X < 25$	0	$40 \le X \le 50$	0	$-10 \le X \le -9$ OR $9 \le X \le 10$	0	$0 \le X \le 2$ OR $28 \le X \le 30$	0

3.2 AFIS confirmation.

The AFIS proposed in the present research was constructed using MATLAB R2021a and an Excel spreadsheet environment. System validation is done through a simulation with multiple random values for each of the four criteria. It compares the results of the CES-AFIS to analyze the prediction level of the developed systems. The purpose of the development is to obtain a reliable and accurate evaluation of the supplier's global performance, considering its performance in each of the defined evaluation criteria, while integrating the use of the k-means algorithm to objectively structure the consequent parts of each of the fuzzy rules as a data-driven approach.

3.2.1 Evaluation criteria definition.

The same evaluation criteria from CES were used to confirm AFIS. Each of them is represented as a vector whose elements will be the fuzzy values of each fuzzy rule. Each of the criteria represents a linguistic variable of the system.

x_1 : After receiving the purchase order, the number of business days required to deliver the material.
x_2 : Number of defects present in the requested batch of merchandise.
x_3 : Precision in the number of parts delivered concerning the purchase order, an exact quantity, is required to avoid inventory problems.
x_4 : Innovations are carried out on a product in which periodic modifications are required to improve its performance or optimize its manufacture.

Three linguistic labels are defined for each linguistic variable based on an expert-driven approach.
x_{1i} : Slow (S), Normal(N), Fast (Fa)
x_{2i} : Few (F), Regular (R), Many (M)
x_{3i} : Less (Le), Just (J), Extra (E)
x_{4i} : Low (L), Optimum (O), Too many (T)

3.2.2 Fuzzification Stage.

3.2.2.1 Membership functions definition.

A fuzzy set in the domain universe U is characterized by a $\mu_A(x)$ membership function that takes values in the interval [0.1]. It can be represented as a set of ordered pairs of element x and its membership value. See Equation (2). The form of the characteristic function used depends on the criterion applied to resolve each problem. It will vary according to the user's culture, geography, time, or perspective.

$$A = \{(x, \mu_A(x)) | x \in U\} \tag{2}$$

Analysis of Criterion $\mathbf{x}_1$. The goal is to receive the raw material as soon as possible. For a shorter delivery time, the supplier will have a better evaluation. Otherwise, the longer the delivery time, the lower the qualifications. The membership function with similar behavior is the Inverse Sigmoidal membership function, as lower values in time allocate a higher degree of membership, and higher values in time assign lower degrees of membership. See Figure 3. The inflection points, parameters, and the lower limit make the difference in the interpretation of each linguistic label. See Table 3.

Analysis of Criterion $\mathbf{x}_2$. There should be no defect in the supplier's delivery of the raw material. If the company receives fewer defects, suppliers will receive a better rating. Otherwise, higher defects will be assigned a lower rating. The reverse sigmoidal membership function exhibits the same behavior, as lower values on the horizontal scale result in a better degree of belonging; conversely, higher values in the domain of the variable of interest correspond to a lower degree of membership. The mathematical representation is evident in Equation (3). The behavior of the Inverse Sigmoidal membership function is shown in Figure 2.

$$\mu_{a,b,c}(x) = \begin{cases} 1, & if\ x = a \\ 1 - \left(2\left(\frac{c-x}{c-a}\right)^2\right), & if\ a \le x \le b \\ 2\left(\frac{c-x}{c-a}\right)^2, & if\ b \le x \le c \\ 0, & if\ \ c \le x \end{cases} \tag{3}$$

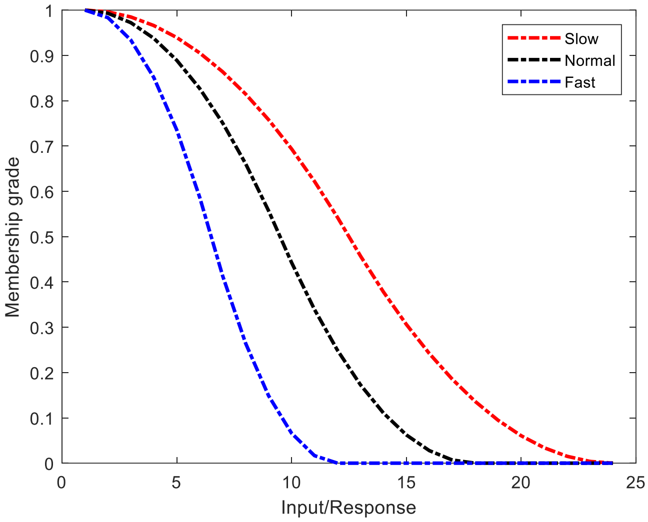


Figure 2. Inverse Sigmoidal membership behavior.

Analysis of Criterion $\mathbf{x}_3$. The goal is that the supplier has no discrepancies in the order's delivery or that they are minimal, that is, that they do not deliver additional parts or that something is missing from the order. In this case, an allowable amount is allocated in the order discrepancy, which will have the best evaluation. The more significant the discrepancy in additional or missing parts, the less assessment will be granted. The membership function that has similar behavior is the Triangular type. A modal or allowable value is assigned, which will have the maximum degree of membership; any higher or lower value will have a lower degree of membership. The mathematical representation is evident in Equation (4). The behavior of the Triangular membership function is shown in Figure 3.

$$\mu_{a,b,c}(x) = \begin{cases} 0, & if\ x \le a \\ \dfrac{x-a}{b-a}, & if\ a \le x \le b \\ \dfrac{c-x}{c-b}, & if\ b \le x \le c \\ 0, & if\ c \le x \end{cases} \tag{4}$$

Analysis of Criterion $\mathbf{x}_4$. The goal of the supplier is to stay within a selected range of innovations made to its process, thereby improving the product's functionality. Being within the expected range will have a better rating. Larger or smaller quantities will have a lower qualification, as it is undesirable to make frequent changes to the processes and to maintain their execution constantly. The Triangular type is the membership function with maximum behavior for a desired. Values within the maximum time range have the highest degree of membership; values lower or higher than this range have lower membership grade values. The mathematical representation can be seen in Equation (4). The behavior of the Triangular membership function is shown in Figure 4.

3.2.2.2 Fuzzy rule base.

The structure of a fuzzy rule base consists of a universe of antecedents (X) and a universe of consequents (Y). The collections of fuzzy sets of X and Y are denoted as $\mathscr{F}$(X) and $\mathscr{F}$(Y), respectively. Two standard approaches exist to model a given fuzzy rule base by an appropriate fuzzy relation on the Cartesian product of both universes $\mathrm{R} \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$. The first approach constructs the unclear relation defined by Equation (5) (Dvořák *et al.*, 2021).

$$\widehat{\mathrm{R}}(\mathbf{X}, \mathbf{Y}) = \wedge_{i=1}^{n}\left(A_i(x) \to B_i(y)\right) \tag{5}$$

In Equation (5), → is its adjoint fuzzy implication, and ∧ is the conjunction of n implications (Chiu, 1994).

The leading operators complying with the conditions to be t-norms are the minimum operator and the algebraic product. These fuzzy relations are formed as the conjunction of implications (known as the implicative model). The second approach to modeling a given fuzzy rule base, initiated by a successful experimental application by Mamdani and Assilian, consists of constructing the fuzzy relation $\widehat{\mathrm{R}} \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$ defined by Equation (6) (Baczyński *et al.*, 2008).

$$\widehat{\mathrm{R}}(\mathbf{X}, \mathbf{Y}) = \bigvee_{i=1}^{n}\big(A_i(x) \otimes B_i(y)\big) \tag{6}$$

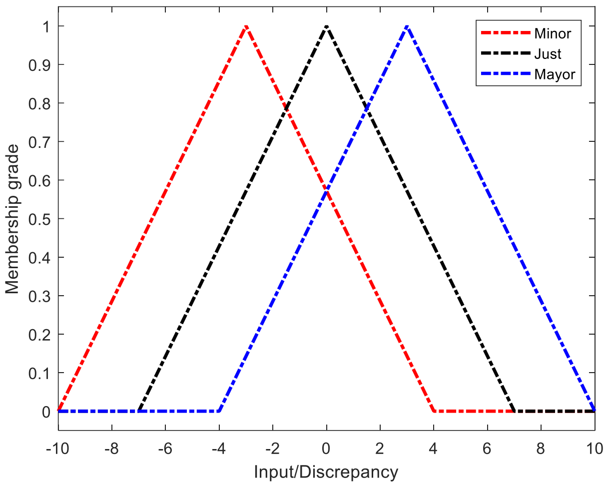


Figure 3. Triangular membership behavior.

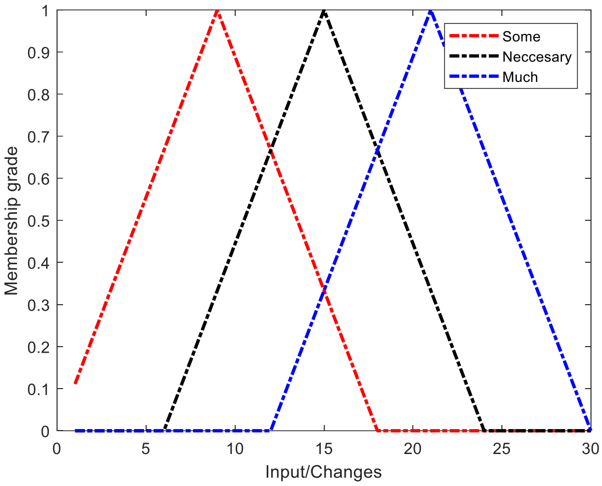


Figure 4. Triangular membership behavior.

The membership functions assigned for each criterion and the parameters for each linguistic label are shown in Table 3.

Table 3. Linguistic variables and linguistic labels.

Evaluation Criteria		
x_1 (Days) Inverse Sigmoidal membership function: upper limit (a), inflection point (b), lower limit (c)		
Slow	Normal	Fast
1, 12, 24	1, 9, 18	1, 6, 12
x_2 (Defects) Inverse Sigmoidal membership function: upper limit (a), inflection point (b), lower limit (c)		
Few	Regular	Many
1, 15, 30	1, 21, 40	1, 25, 50
x_3 (Parts) Triangular membership function: lower limit (a), modal value (b), upper limit (c)		
Less	Just	Extra
-10, -3, 4	-7, 0, 7	-4, 3, 10
x_4 (Modifications) Triangular membership function: lower limit (a), modal value (b), upper limit (c)		
Low	Optimum	Too many
0, 9, 18	6, 15, 24	12, 21, 30

In Equation (6), $\otimes$ is a left-continuous t-norm, and $\bigvee$ is the disjunction of n conjunctions (Klement *et al.*, 2000).

In Mamdani-Assilian inference, each rule is interpreted as a Cartesian product of fuzzy sets, then takes the degree to which a rule's condition is fulfilled and computes the output of a rule by truncating the output fuzzy set with this degree. Finally, the output of the rule base is calculated by aggregating the fuzzy output sets of the rules using a t-conorm (often the maximum). The last step is usually a defuzzification utilizing the center of gravity COG method (Bodenhofer *et al.*, 2007).

A t-norm is an appropriate interpretation of conjunction, not of implication; moreover, the maximum operation disjunctively aggregating all rules has nothing in common with the logical operation AND. Then, the maximum in (6) expresses the accumulation of data. This fact, together with the known fact that the maximum operation and other t-conorms are appropriate interpretations of disjunction (the logical connective OR). The leading operators qualifying as t-conorms are the maximum operator and the algebraic sum. These fuzzy relations are formed as the disjunction of conjunctions (so-called Mamdani-Assilian model) (Dvořák *et al.*, 2021).

Obviously, $\widehat{\mathrm{R}}(\mathbf{X},\mathbf{Y})$ the degree Y is the output of the rule base for input X. Moreover, we see that in R, there is nothing else but the disjunction of n fuzzy relations, which are Cartesian products of input and output fuzzy sets. The most common choice is to use the minimum (Gödel) t-norm for the Cartesian products (corresponding to truncating B_i with the degree $A_i(x)$) and the maximum for aggregation. Also popular is the combination of maximum for aggregation and product t-norm for building Cartesian products (Bodenhofer *et al.*, 2007). It is more than evident that the Mamdani-Assilian inference does not use any implication concept.

The number of rules (R) necessary for the system's operation was defined based on the number of input variables and linguistic labels, using a combinatorial approach. Having four criteria (c) and three linguistic labels per variable (p), 81 fuzzy rules were required to constitute the knowledge base, considering the possible combinations. See Equation (7). The sentences of the type If-and-then are established to cover all the possible scenarios a supplier could play concerning each criterion.

$$\begin{aligned} \mathrm{R} &= \mathrm{p}^{\mathrm{c}} \\ \mathrm{R} &= 3^4 = 81 \end{aligned} \tag{7}$$

3.2.3 Association stage: IAMDC as an attention mechanism.

3.2.3.1 Number of centroids designation.

At this stage, the paper's main scientific contribution is described: the determination of the dynamic consequent parts using the k-means algorithm as an attention mechanism. The fuzzy values of the antecedents are related and transformed to geometric distances, passing from a fuzzy domain to a positive real domain. The proposed grouping real sets of $\mathcal{F}(\mathbf{X})$ and $\mathcal{F}(\mathbf{Y})$ are denoted as $\mathbb{R}(\mathbf{T})$ and $\mathbb{R}(\mathbf{Z})$ respectively.

To start the operation of the IAMDC algorithm and determine the consequents of each fuzzy rule based on their performance, it is necessary to define the number of groups k that the k-means algorithm will use to perform the corresponding groupings. The number of groupings is determined by the number of linguistic labels used to define the system's output, in this case, the vendor performance categorization using three performance labels: Low, Medium, and High. This information proposes three centroids: Low centroid $\mathbf{m}_1$, Medium centroid $\mathbf{m}_2$ and High centroid $\mathbf{m}_3$. Each antecedent of the fuzzy rule shall be classified as its consequent as Low group S_1, Medium group S_2 and High group S_3. The number of input variables that configure each centroid; in this case, four values configure each centroid: $m_{i1}, m_{i2}, m_{i2}, m_{i4}$. This conceptualization of preserving the labels of the output variables by the expert allows retaining the expert-driven approach while using it to determine the number of centroids that group structures with similar values at their distances, thereby integrating the data-driven approach.

3.2.3.2 Definition of initial position for each centroid.

Typically, the operation of the k-means algorithm is initialized with the definition of the number of groups to be formed and the assignment of a random initial position for the centroid of each group, based on which the grouping begins. In contrast, the centroid's position is optimized through a series of iterations.

3.2.3.3 k-means algorithm application.

It is very important to highlight that the algorithm can randomly start the initial position of the centroids with any of the fuzzy values of any antecedent structure, if desired. Expected values ($\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3, \mathrm{E}_4$) are assigned to each criterion in each of the centroid groups. See Table 4. In addition, the respective fuzzification is performed for each criterion in its respective linguistic labels. Then the maximum value for each of them is found. These values correspond to the initial positions of each centroid $\mathbf{m}_1^1, \mathbf{m}_2^1, \mathbf{m}_3^1$. Additionally, the k-means algorithm is applied to determine the output groups corresponding to the consequent of the fuzzy rules. In this way, a geometric transformation is performed, and three distances are determined in a positive real domain for each fuzzy rule as $\mathbb{R}(\mathbf{T})$. Considering the output labels proposed by the expert, in S_1 will be the distance for a Low group. Successively, in S_2 will be the distance for a Medium group. Finally, in S_3 will be the distance for a High group.

Therefore, the first step is presented to obtain the initial position of $\mathbf{m}_1^1$. We begin by assigning a high score to each of the input variables with the purpose of obtaining low fuzzy values, then the fuzzy values of the three linguistic labels of each variable are reviewed. Finally, the maximum fuzzy value is chosen to be considered as the initial coordinate of the $\mathbf{m}_1^1$. See Table 5. In the same way, for $\mathbf{m}_2^1$, intermediate scores are entered in each of the input variables and is chosen maximum fuzzy value produced in the linguistic labels of each variable. See Table 6. Successively for $\mathbf{m}_3^1$, low scores will be assigned to each of the variables to select the maximum fuzzy value as coordinate. See Table 7.

Table 4. Expected values for the centroids classification.

	S_1				S_2				S_3			
Criteria	E_1	E_2	E_3	E_4	E_1	E_2	E_3	E_4	E_1	E_2	E_3	E_4
Performance	23	49	9	29	12	25	5	26	2	2	0	15

Table 5 shows the allocation of high expected values, such as 23, 49, 9, and 29, which means low performance in the evaluation system. Its fuzzy values are determined for each of the three linguistic labels, and the maximum value for each criterion is identified, which will be used as the initial value of the centroid 1 for low performance. In this way, Table 5 shows the assignment of high values, which means a low performance for each of the criteria in $\mathbf{m}_1$. The assignment of intermediate values, which means an average performance for each of the criteria in $\mathbf{m}_2$. Finally, the assignment of low values, which means a high performance for each of the criteria in $\mathbf{m}_3$.

Table 5. High expected values to identify the initial position of $\mathbf{m}_1$.

KNOWLEDGE BASE											
E_1:		23	E_2:		49	E_3:		9	E_4:		29
E_1	Label	Fuzzy Value	E_2	Label	Fuzzy Value	E_3	Label	Fuzzy Value	E_4	Label	Fuzzy Value
23	S	**0.003**	49	F	0.000	9	Le	0.000	29	L	0.000
23	N	0.000	49	R	0.000	9	J	0.000	29	O	0.000
23	Fa	0.000	49	M	**0.001**	9	E	**0.143**	29	T	**0.167**

Table 6 shows the allocation of medium expected values, such as 12, 25, 5, and 26, which mean average performance in the evaluation system. Its fuzzy values are determined for each of the three linguistic labels, and the maximum value for each criterion is identified, which will be used as the initial values of the centroid 2 for medium performance.

Table 6. Medium expected values to identify the initial position of $\mathbf{m}_2$.

KNOWLEDGE BASE											
E_1:		12	E_2:		25	E_3:		5	E_4:		26
E_1	Label	Fuzzy Value	E_2	Label	Fuzzy Value	E_3	Label	Fuzzy Value	E_4	Label	Fuzzy Value
12	S	**0.500**	25	F	0.059	5	Le	0.000	26	L	0.000
12	N	0.222	25	R	0.296	5	J	0.286	26	O	0.000
12	Fa	0.000	25	M	**0.520**	5	E	**0.714**	26	T	**0.667**

Table 7 shows the allocation of low expected values, such as 2, 2, 0, and 15, which means high performance in the evaluation system. Its fuzzy values are determined for each of the three linguistic labels, and the maximum value for each criterion is identified, which will be used as the initial values of the centroid 3 for high performance.

Table 7. Low expected values to identify the initial position of $\mathbf{m}_2$.

KNOWLEDGE BASE											
E_1:		2	E_2:		2	E_3:		0	E_4:		15
E_1	Label	Fuzzy Value	E_2	Label	Fuzzy Value	E_3	Label	Fuzzy Value	E_4	Label	Fuzzy Value
2	Slow	**0.986**	2	Few	0.998	0	Less	0.571	15	Low	0.500
2	Normal	0.975	2	Regular	0.999	0	Just	**1.000**	15	Optimum	**1.000**
2	Fast	0.944	2	Many	**0.999**	0	Extra	0.571	15	Too Many	0.500

In summary, the initial positions for each centroid are established as

$$\mathbf{m}_1 = (0.003, 0.001, 0.143, 0.167), \mathbf{m}_2 = (0.5, 0.52, 0.714, 0.667), \mathbf{m}_3 = (0.986, 0.999, 1.0, 1.0).$$

Given a set of observations $(\mathbf{x}_1, \mathbf{x}_2, \ldots . \mathbf{x}_n)$, where each observation is a real vector of d dimensions, k-means constructs a partition of the observations into k sets (k≤n) in order to minimize the sum of the squares within each group (WCSS): S = $\{S_1, S_2, \ldots . S_n, \}$. See Equation (8), where $\boldsymbol{\mu}_i$ is the average of points in S_i.

$$\underset{\mathbf{S}}{\arg\min} \sum_{i=1}^{k} \sum_{\mathbf{x}_j \epsilon S_i} \left\| \mathbf{x}_j - \boldsymbol{\mu}_i \right\|^2 \tag{8}$$

k-means algorithm
Given an initial set of k centroids $\mathbf{m}_1^1, \mathbf{m}_2^1, \ldots, \mathbf{m}_k^1$, the algorithm continues alternating between two steps.
Assignment step: Assign each observation to the group with the nearest means. See Equation (9).

$$S_i^{(t)} = \left\{x_p: \left\|x_p - m_i^{(t)}\right\|^2 \leq \left\|x_p - m_j^{(t)}\right\|^2 \; \forall j, 1 \leq j \leq k\right\} \tag{9}$$

Update step: Recalculate means (centroids) for observations assigned to each cluster. See Equation (10).

$$\mathbf{m}_i^{(t+1)} = \frac{1}{\left|S_i^{(t)}\right|} \sum_{\mathbf{x}_j \in S_i^{(t)}} \mathbf{x}_j \tag{10}$$

The algorithm has converged when the assignments no longer change (MacKay, 2003).

As an attention mechanism, the k-means algorithm performs a geometric transformation of the fuzzy domain for each antecedent structure at three distances in a positive real domain. This stage is one of the most significant in the proposed approach to developing a Geometric rule base. See Figure 6.

Once the initial position of each centroid has been defined as $\mathbf{m}_1^1, \mathbf{m}_2^1, \mathbf{m}_3^1$. It is possible to begin to calculate the distances between each of the fuzzy values $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ with each of the initial position of the centroids $m_{1,1}^1, m_{1,2}^1, m_{1,3}^1, m_{1,4}^1, m_{2,1}^1, m_{2,2}^1, m_{2,3}^1, m_{2,4}^1, m_{3,1}^1, m_{3,2}^1, m_{3,3}^1, m_{3,4}^1$ to assign it to the nearest centroid and determine the grouping of the consequents according to their performance. After determining the distances, the comparison is made, and the smallest value is assigned to the nearest centroid. In this case, the structure of the antecedent is added to the centroid based on the comparison of the real values for each classification group. This process is iterative until the algorithm converges, ensuring the minimum sum of distances, the grouping of antecedent rules, and knowledge extraction from data, thereby strengthening accuracy and maintaining the evaluation system's interpretation as a data-driven approach.

Figure 5 shows the construction of the Geometric rule base proposed in this research. The Fuzzy rule base is related and transformed into a Geometric rule base based on applying the k-means algorithm as an attention mechanism. The fuzzy set $\mathscr{F}$(X) is transformed into a real $\mathbb{R}(\mathbf{T})$ set. Each geometric rule is represented by three distances in a positive real domain: T_1, T_2, T_3. Each is the distance from each of the fuzzy rules to the centroids passing from the fuzzy domain to the real domain. The three distances found in a real domain $\mathbb{R}(\mathbf{T})$ in each box in Figure 6. Each box now represents a geometric rule structure.

In summary, the fuzzy values of each rule are transformed at n distances in the real domain depending on the n linguistic labels of the output variable. In this case, n represents 3 linguistic labels. The 4 fuzzy values of each rule $\mathscr{F}$(X) are transformed to 3 distances in the real domain represented by the real set $\mathbb{R}(\mathbf{T})$.

3.2.3.4 Dynamic consequents determination by performance based on the results of the k-means algorithm.

Once the algorithm has been applied and the relevant iterations have been carried out until convergence, a consequent allocation is made to each antecedent of the fuzzy rules based on their grouping. The consequences of each evaluation are dynamic. The fuzzy values on each assessment are different; therefore, their distances differ, allowing dynamic classification of the consequent. By changing the criteria assignments, their values vary across linguistic labels and cause them to be grouped differently, introducing new knowledge to establish their respective consequent. The dynamism of the consequents in each evaluation makes the system more precise by focusing on their corresponding diffuse values rather than the consequent deterministic values based on the experts' heuristic knowledge. Figure 6 shows a 9×9 matrix. Each cell represents the classification of the consequent of each of the 81 fuzzy rules in a dynamic way using the proposed method. Additionally, Figure 6 shows the final groupings for each knowledge base structure resulting from a convergence of the k-means algorithm over 5 iterations. The following allocations were made to the four evaluation criteria: TD= 7, Q=15, PD=0, I=10.

Alternatively, the results of the classification of consequents proposed by the experts based on their experience and interpretation are shown in Figure 7.

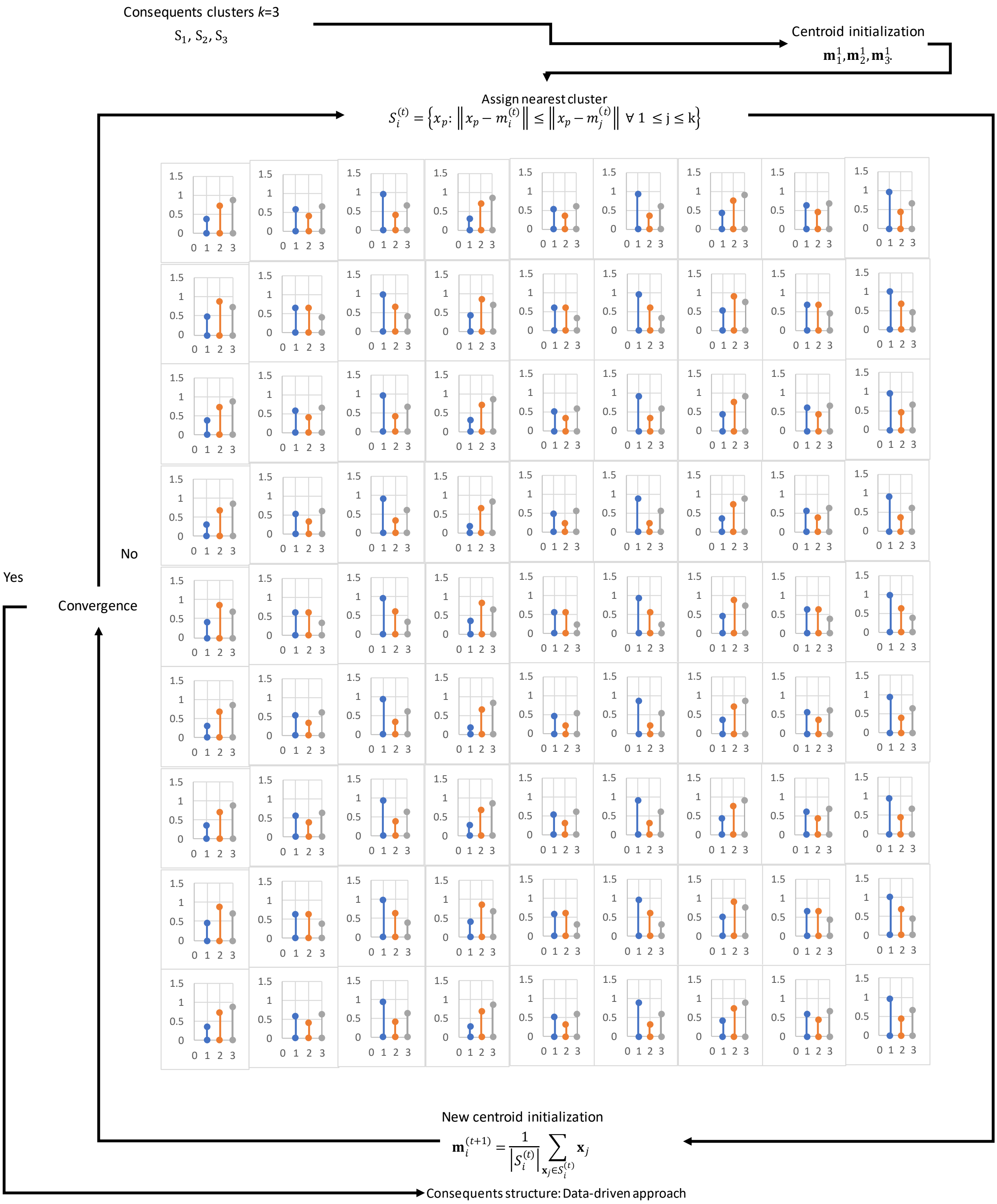
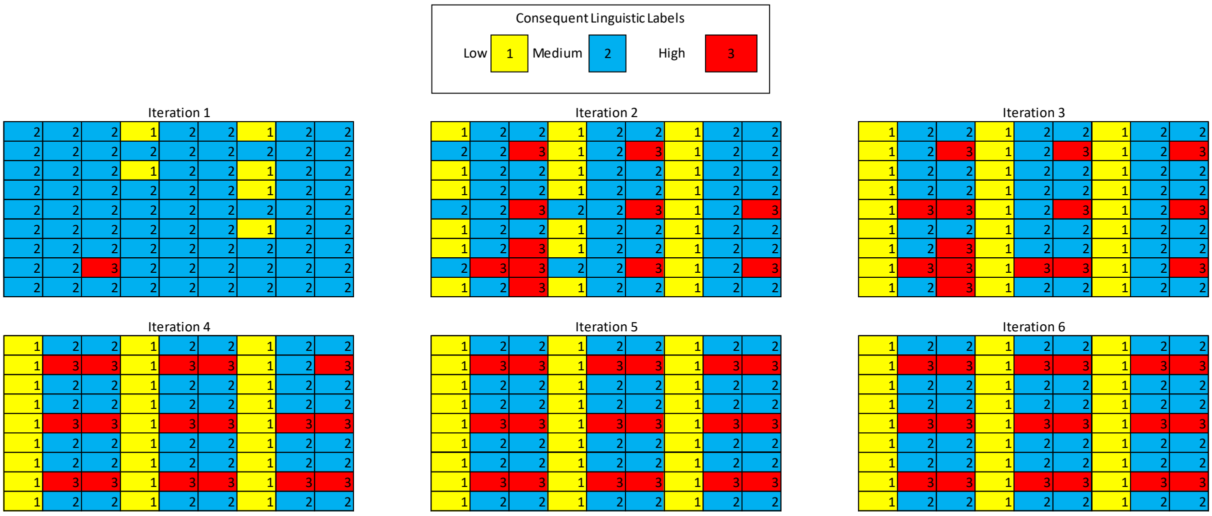


Figure 5. Geometric rule base.



Consequent Linguistic Labels

Low 1 Medium 2 High 3

Iteration 1

2	2	2	1	2	2	1	2	2
2	2	2	2	2	2	2	2	2
2	2	2	1	2	2	1	2	2
2	2	2	2	2	2	1	2	2
2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	1	2	2
2	2	2	2	2	2	2	2	2
2	2	3	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2

Iteration 2

1	2	2	1	2	2	1	2	2
2	2	3	1	2	3	1	2	2
1	2	2	1	2	2	1	2	2
1	2	2	1	2	2	1	2	2
2	2	3	2	2	3	1	2	3
1	2	2	1	2	2	1	2	2
1	2	3	1	2	2	1	2	2
2	3	3	2	2	3	1	2	3
1	2	3	1	2	2	1	2	2

Iteration 3

1	2	2	1	2	2	1	2	2
1	2	3	1	2	3	1	2	3
1	2	2	1	2	2	1	2	2
1	2	2	1	2	2	1	2	2
1	3	3	1	2	3	1	2	3
1	2	2	1	2	2	1	2	2
1	2	3	1	2	2	1	2	2
1	3	3	1	3	3	1	2	3
1	2	3	1	2	2	1	2	2

Iteration 4

1	2	2	1	2	2	1	2	2
1	3	3	1	3	3	1	2	3
1	2	2	1	2	2	1	2	2
1	2	2	1	2	2	1	2	2
1	3	3	1	3	3	1	3	3
1	2	2	1	2	2	1	2	2
1	2	2	1	2	2	1	2	2
1	3	3	1	3	3	1	3	3
1	2	2	1	2	2	1	2	2

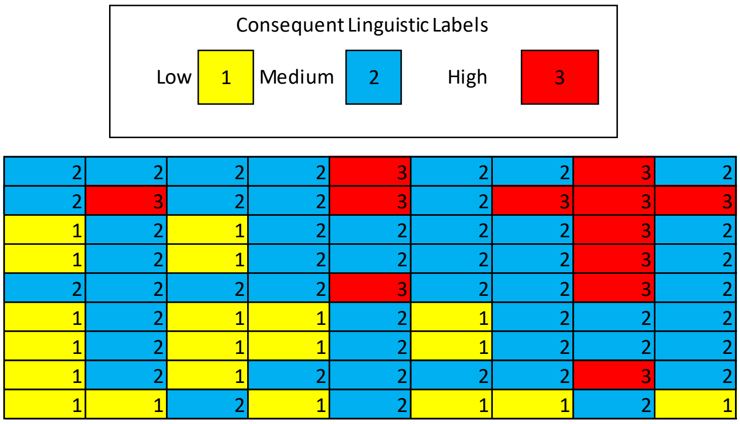
Iteration 5

1	2	2	1	2	2	1	2	2
1	3	3	1	3	3	1	3	3
1	2	2	1	2	2	1	2	2
1	2	2	1	2	2	1	2	2
1	3	3	1	3	3	1	3	3
1	2	2	1	2	2	1	2	2
1	2	2	1	2	2	1	2	2
1	3	3	1	3	3	1	3	3
1	2	2	1	2	2	1	2	2

Iteration 6

1	2	2	1	2	2	1	2	2
1	3	3	1	3	3	1	3	3
1	2	2	1	2	2	1	2	2
1	2	2	1	2	2	1	2	2
1	3	3	1	3	3	1	3	3
1	2	2	1	2	2	1	2	2
1	2	2	1	2	2	1	2	2
1	3	3	1	3	3	1	3	3
1	2	2	1	2	2	1	2	2

Figure 6. Consequents based on a data-driven approach



Consequent Linguistic Labels

Low 1 Medium 2 High 3

2	2	2	2	3	2	2	3	2
2	3	2	2	3	2	3	3	3
1	2	1	2	2	2	2	3	2
1	2	1	2	2	2	2	3	2
2	2	2	2	3	2	2	3	2
1	2	1	1	2	1	2	2	2
1	2	1	1	2	1	2	2	2
1	2	1	2	2	2	2	3	2
1	1	2	1	2	1	1	2	1

Figure 7. Consequents based on an expert-driven approach

3.2.4 Compositional rule of inference

Each rule is interpreted as a real set $\mathbb{R}(\mathbf{T})$ in the geometric relationship proposed as an attention mechanism. See Equation (11).

$$T_p^{(t)} = \left\{ \left\| x_p - m_j^{(t)} \right\|^2 \forall j, 1 \leq j \leq k \right\} \tag{11}$$

In this step, it is proposed to return to the fuzzy domain $\mathcal{F}(\mathbf{X})$ and use the SoftMax function on each fuzzy rule to determine the probabilities. Finally, use them as a representation of fuzzy values of each geometric rule and employ a Bayesian classifier, such as the determination of the parameters of the consequences.

It is used to "compress" a K-dimensional real values vector, z, in a K-dimensional vector $\mathbf{X}$ with components in the range [0, 1]. The function is given in Equations (12) and (13).

$$\boldsymbol{X} : \mathbb{R}^K \longrightarrow [0,1]^K \tag{12}$$

$$\boldsymbol{X} = \frac{\mathrm{e}^{\mathrm{z_j}}}{\sum_{\mathrm{k=1}}^{\mathrm{K}} \mathrm{e}^{\mathrm{z_k}}} \tag{13}$$

At this stage, a Bayesian classifier was established. Since all probabilistic values from diffuse rules are merged or combined in the attention mechanism, each rule is analyzed as a single variable or criterion under study. It computes its algebraic product for each rule and analyzes its behavior. Thus, the joint model can be expressed in Equation (14).

$$Y_i = \prod_{i=1}^{3}(X_i) \tag{14}$$

Figure 8 shows the normalization of the 81 rules in the knowledge base. There is variation in the probability distribution functions and in the product of their respective values in each fuzzy rule.

It is assumed that each X_i is independent of any other X_j for $j \neq i$ when conditioned to a classification group, then takes a product to compute the output of each rule by aggregating the rules using an argument maximum. The maximum value corresponds to the rule that yields the optimal values for each criterion. The compositional rule of inference by AFIS is shown in Equation (15).

$$\mathcal{F}(\mathbf{X}\,\mathrm{x}\,\mathbf{Y}) = arg\max\,(Y_i) \tag{15}$$

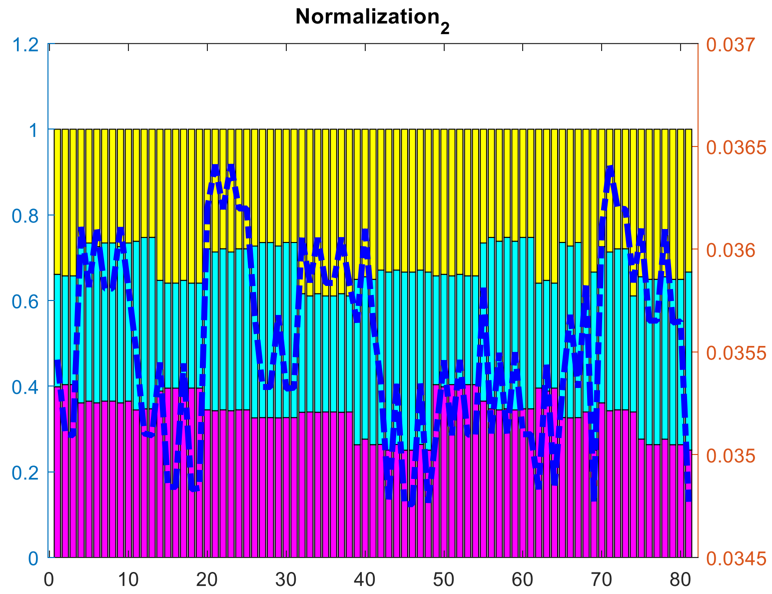


Figure 8. Cartesian product $\mathcal{F}(\mathbf{X}\,\mathrm{x}\,\mathbf{Y})$ with fuzzy product operator.

3.2.5 Weighted evaluation and Defuzzification Stage by AFIS

Finally, the Center-of-Gravity (COG) determines the Weighted evaluation (We) output for the AFIS and the defuzzification stage using the FIS. See Equation (16). The determination of the COG defuzzified value is simplified if we consider the finite universe of discourse U and, thus, a discrete membership function $A_i(x)$.

$$COG(A) = \frac{\sum We_i\, A_i(x)}{\sum A_i(x)} \tag{16}$$

4. RESULTS

First, the evaluations were conducted using the company's CES. Subsequently, the results of the proposed AFIS with CES were compared. In addition, a comparison with a traditional fuzzy inference system (FIS) and the current evaluation system is presented. In the development of the AFIS y FIS, the product operator was used for the Cartesian product $\mathcal{F}(\mathbf{X}\,\mathrm{x}\,\mathbf{Y})$, and the maximum operator was used for aggregation as a compositional rule of inference. The mean square error (MSE) indicator was used to estimate the reliability of the fuzzy models presented. The proposed associative fuzzy inference system (AFIS) has a lower MSE value, validating better reliability and accuracy in the results.

The vertical axis represents the evaluations of the systems under study. The circular axis represents each of the simulations with random assignment to the four system input criteria. In summary, 150 different allocations and classifications were assigned. See Figure 9. Finally, the results show the reliability in evaluating suppliers' performance between the current traditional evaluation system and the fuzzy inference systems built. See Figure 10. The proposed system retains very good interpretability and greater precision in its results by extracting knowledge from data. In each evaluation, the consequents change dynamically.

Figure 9. Simulation runs Prod-Max comparisons.

150 simulation runs

Runs	TD	Q	PD	I	Current	FIS Prod-Max	**AFIS Prod-Max**	Runs	TD	Q	PD	I	Current	FIS Prod-Max	**AFIS Prod-Max**
1	7	15	0	10	8.50	5.813	9.499	**76**	13	43	3	25	4.00	1.301	4.879
2	4	50	4	26	4.50	0.000	6.357	**77**	17	4	10	6	4.50	0.000	4.752
3	8	22	-4	9	5.75	3.165	6.441	**78**	11	7	-10	0	3.75	0.000	2.739
4	11	39	-9	13	4.50	2.571	5.150	**79**	0	38	-5	19	6.50	4.423	4.966
5	16	46	-1	20	5.25	3.000	5.328	**80**	12	46	-9	26	2.00	0.000	2.000
6	2	13	0	10	9.00	6.652	9.500	**81**	24	16	-6	5	4.00	0.000	4.723
7	15	3	3	9	6.50	4.519	5.333	**82**	20	23	4	21	3.75	2.103	5.006
8	20	26	-2	5	4.00	3.409	4.854	**83**	20	22	4	10	4.50	2.881	5.174
9	12	29	-9	20	4.50	1.714	5.351	**84**	0	32	-8	7	5.25	3.750	4.500
10	17	20	-9	6	3.25	1.903	4.825	**85**	11	13	-1	19	7.75	5.164	5.420
11	4	33	-5	29	4.50	2.072	4.125	**86**	12	35	6	1	3.25	1.500	4.699
12	22	37	7	13	4.00	0.582	2.719	**87**	8	40	-3	23	5.25	2.450	4.716
13	16	38	2	29	3.50	0.636	5.236	**88**	7	13	-6	11	7.25	4.134	5.444
14	11	28	-2	3	5.25	3.446	4.893	**89**	19	10	-2	20	6.75	4.737	5.139
15	7	23	10	27	4.00	0.000	4.906	**90**	21	40	-4	11	3.25	2.500	5.031
16	24	26	6	28	2.50	0.000	4.706	**91**	23	26	3	8	4.50	2.797	4.563
17	16	10	10	11	4.75	0.000	5.068	**92**	0	32	-10	16	5.75	0.000	5.969
18	20	23	-9	20	3.25	2.235	5.395	**93**	15	31	-4	1	2.75	2.500	4.024
19	18	20	8	19	4.75	2.887	4.981	**94**	18	21	3	0	4.00	0.000	4.141
20	20	25	-7	6	3.25	0.769	4.823	**95**	22	40	-8	15	3.25	0.000	5.036
21	23	3	1	8	6.25	4.995	8.245	**96**	12	7	-2	21	7.00	4.552	6.643
22	3	23	-10	0	3.75	0.000	3.228	**97**	16	11	-10	27	3.50	0.000	4.856
23	4	1	3	27	7.75	4.892	8.907	**98**	22	14	-2	9	5.25	4.309	5.431
24	15	20	-4	3	4.00	2.664	4.436	**99**	1	43	-6	13	6.25	4.434	4.941
25	18	28	-5	26	4.00	1.186	4.917	**100**	22	22	8	1	2.00	1.454	2.000
26	18	1	-3	3	6.00	3.750	5.392	**101**	8	45	5	21	4.50	2.412	5.053
27	1	33	9	8	4.50	3.730	3.927	**102**	23	8	-10	22	3.75	0.000	2.892
28	24	15	-2	11	6.00	0.000	4.910	**103**	9	40	2	30	4.00	0.000	3.413
29	16	33	-4	20	4.75	1.935	5.223	**104**	16	34	5	20	4.75	1.622	5.284
30	9	26	-6	26	5.25	1.137	4.854	**105**	9	6	-5	10	7.75	4.433	6.034
31	7	22	0	29	5.75	3.766	5.117	**106**	0	9	8	11	7.75	5.448	9.451
32	1	43	6	10	5.75	4.210	5.062	**107**	5	41	8	29	2.75	0.000	4.960
33	3	36	6	1	4.50	3.656	4.473	**108**	0	29	-1	18	8.25	5.833	8.245
34	16	34	-9	13	4.00	1.362	5.145	**109**	7	43	-2	5	4.75	3.409	4.801
35	17	16	1	27	6.00	3.482	4.849	**110**	14	15	-5	12	6.50	4.297	4.823
36	20	3	-6	29	3.75	1.498	4.389	**111**	11	32	5	14	5.75	2.876	5.012
37	12	18	8	23	5.25	1.252	4.641	**112**	15	8	1	3	6.50	4.624	6.631
38	17	14	-8	20	5.50	2.823	5.287	**113**	15	35	1	18	6.00	3.856	4.795
39	7	19	5	13	7.75	4.251	5.960	**114**	20	19	3	8	5.25	3.059	4.530
40	13	31	7	15	5.25	2.015	5.035	**115**	11	35	10	6	3.25	0.000	4.887
41	20	49	-3	12	4.00	2.727	4.871	**116**	4	8	-5	7	7.50	4.599	9.308
42	4	25	8	30	4.50	0.000	4.310	**117**	11	6	3	0	5.75	0.000	5.605
43	24	17	0	5	5.25	0.000	4.698	**118**	3	30	5	10	6.50	4.283	5.191
44	7	17	10	13	6.50	0.000	5.037	**119**	14	20	-8	1	3.25	1.903	5.228
45	18	42	-8	16	4.00	0.000	4.222	**120**	1	1	0	10	9.50	6.802	9.503
46	18	18	4	4	4.75	2.500	4.312	**121**	7	49	-2	20	6.00	3.920	5.129
47	13	5	10	3	4.50	0.000	4.813	**122**	8	13	5	16	7.75	4.458	7.441
48	10	6	2	23	7.00	3.837	8.363	**123**	22	15	8	14	5.25	3.526	5.586
49	23	19	1	12	6.50	5.136	4.774	**124**	17	37	4	16	5.25	2.508	4.834
50	5	41	5	15	5.75	4.433	5.307	**125**	4	43	-3	27	5.25	2.647	5.388
51	23	24	4	30	2.50	0.000	4.067	**126**	17	3	6	2	4.50	3.767	4.532
52	12	4	-6	12	7.00	4.430	4.710	**127**	24	39	-7	1	1.50	0.000	2.000
53	20	21	-5	6	3.75	2.143	4.864	**128**	18	20	-3	27	4.75	2.253	4.475
54	22	23	6	15	5.00	3.123	4.693	**129**	19	48	7	4	2.25	0.000	2.000
55	13	30	7	17	5.25	2.122	5.082	**130**	7	41	4	7	4.50	2.670	4.722
56	11	42	3	8	4.50	2.748	4.583	**131**	17	40	-6	0	2.00	0.000	2.000
57	20	6	5	9	5.00	4.239	5.534	**132**	20	9	3	15	7.00	4.991	8.245
58	6	49	-1	4	5.25	4.091	4.785	**133**	2	3	-7	2	5.75	3.738	8.245
59	11	49	-4	1	2.50	2.500	4.205	**134**	16	25	6	21	4.50	1.752	4.226
60	1	42	0	6	6.25	4.773	8.245	**135**	2	41	2	2	4.50	3.669	5.145
61	11	18	10	30	3.25	0.000	2.000	**136**	19	22	8	25	3.50	0.000	4.834
62	8	12	-7	2	4.75	3.315	5.046	**137**	8	36	8	28	3.50	0.266	5.191
63	14	18	-4	30	4.50	0.000	4.123	**138**	6	47	-8	7	4.00	2.727	4.583
64	11	28	-7	9	4.50	1.553	4.287	**139**	22	19	5	6	4.50	2.143	4.852
65	1	29	-3	27	6.50	2.739	5.335	**140**	16	15	-7	2	3.50	2.921	4.865
66	24	24	3	13	5.75	0.000	5.068	**141**	13	6	-1	30	6.25	0.000	8.245
67	19	41	10	25	1.50	0.000	2.000	**142**	8	12	-7	13	7.25	4.292	5.082
68	6	4	-5	22	7.00	4.200	9.241	**143**	9	26	8	5	4.75	2.040	4.843
69	19	14	-1	30	5.25	0.000	5.399	**144**	24	39	6	26	2.75	0.000	4.858
70	9	16	-5	9	6.50	3.471	6.564	**145**	18	15	5	17	6.50	4.297	5.073
71	16	13	-1	25	6.00	3.596	4.717	**146**	19	35	3	19	5.50	2.419	5.010
72	14	28	3	1	4.50	2.745	4.186	**147**	11	22	4	23	5.00	2.063	5.290
73	1	22	-6	5	5.75	4.215	4.710	**148**	11	31	-9	23	3.25	0.600	4.717
74	14	35	-7	10	4.75	1.196	5.267	**149**	13	49	-5	12	4.50	2.143	4.890
75	24	38	-5	23	3.25	0.000	4.712	**150**	22	42	8	12	2.75	0.000	2.000

Figure 10 shows the comparison of the evaluation results of CES-AFIS and CES-FIS with the Prod-Max fuzzy inference rule. The proposed AFIS system has better accuracy and reliability with a mean quadratic error indicator of 0.6.

In addition, different combinations of inference rules were tested using various fuzzy operators to validate the efficiency of the proposed model using the same assignments of the four criteria in the 150 combinations of Table 4. In each combination, the proposed AFIS has better accuracy and reliability than the conventional FIS system.

Figure 11 shows the comparison of the evaluation results of CES-AFIS and CES-FIS with the Min-Max fuzzy inference rule. The proposed AFIS system has better accuracy and reliability with a mean quadratic error indicator of 0.62. Figure 12 shows the comparison of the evaluation results of CES-AFIS and CES-FIS with the Max-Min fuzzy inference rule. The proposed AFIS system has better accuracy and reliability with a mean quadratic error indicator of 0.67. Figure 13 shows the comparison of the evaluation results of CES-AFIS and CES-FIS with the Max-Max fuzzy inference rule. The proposed AFIS system has better accuracy and reliability with a mean quadratic error indicator of 0.62. Figure 14 shows the comparison of the evaluation results of CES-AFIS and CES-FIS with the Max-Prod fuzzy inference rule. The proposed AFIS system has better accuracy and reliability with a mean quadratic error indicator of 0.86.

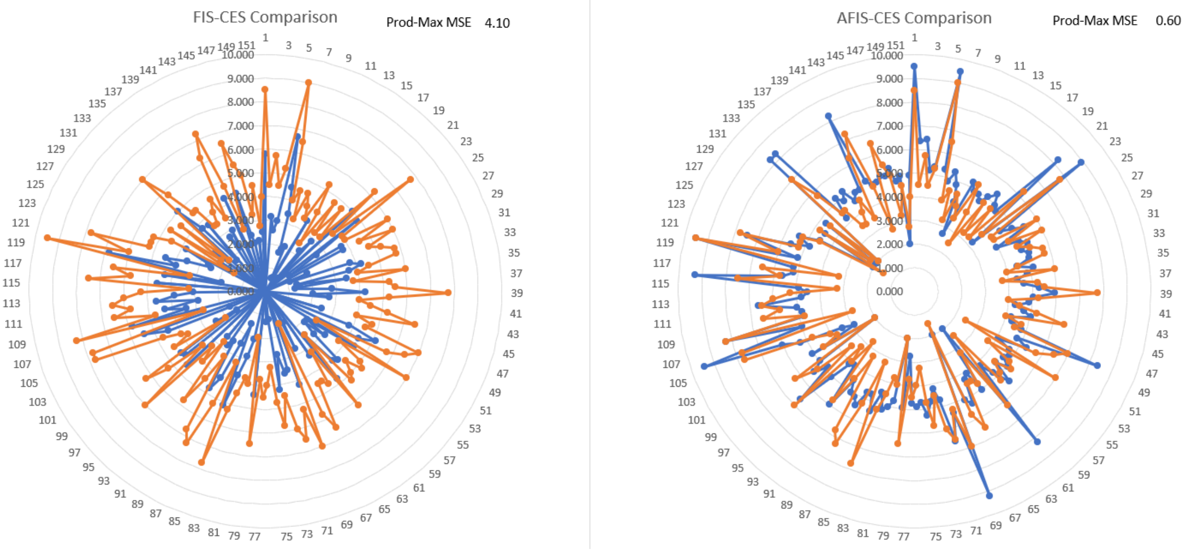


Figure 10. Prod-Max results comparison.

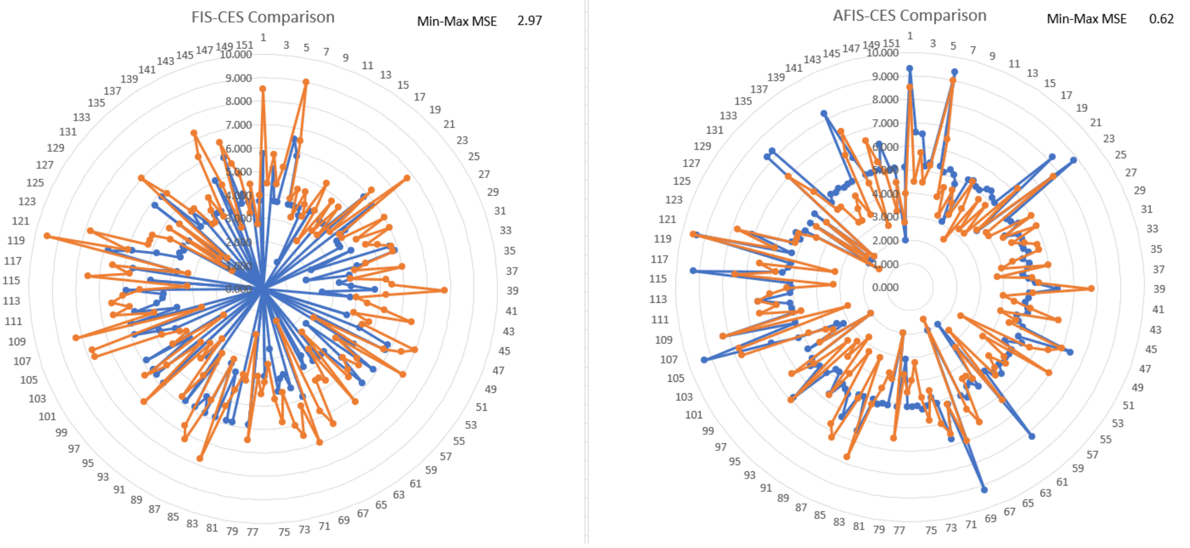


Figure 11. Min-Max results comparison.

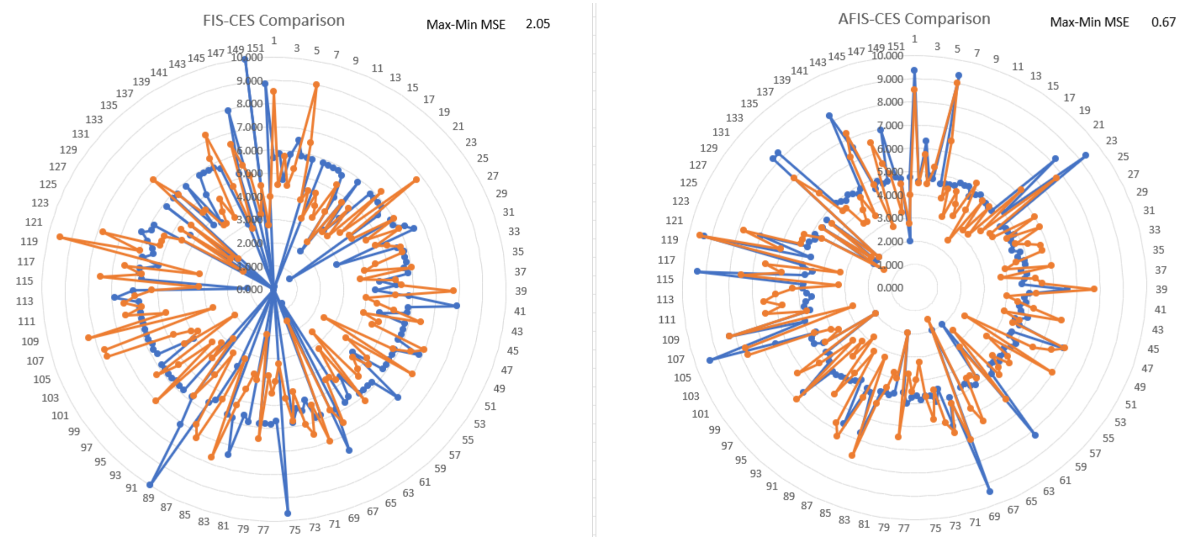


Figure 12. Max-Min results comparison.

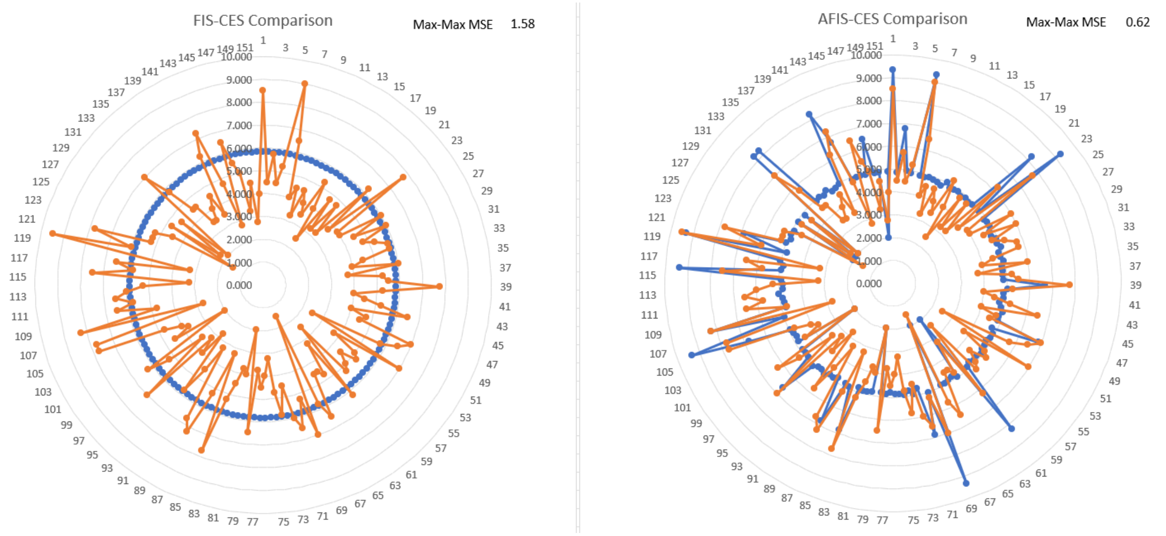


Figure 13. Max-Max results comparison.

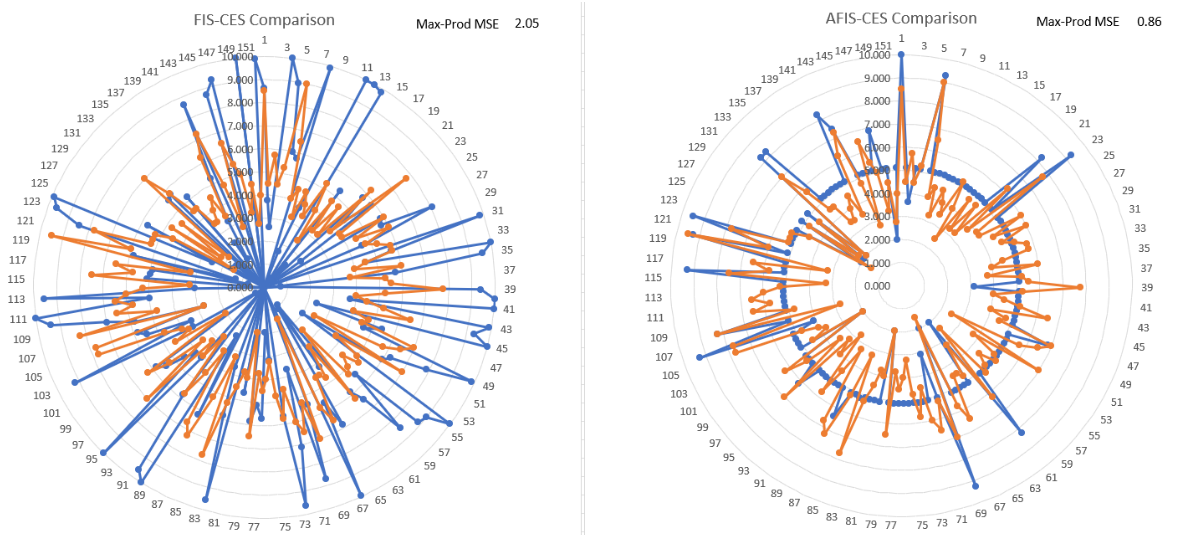


Figure 14. Max-Prod results comparison.

5. DISCUSSION

Fuzzy inference systems are based on an expert's use of knowledge to create a knowledge base and establish relationships between antecedents and consequents based on his or her experience. The proposed IAMDC eliminates the expert's involvement in determining the classification of the consequent by extracting knowledge from the data. The results are more accurate and reliable when developing a prediction system. However, it also reduces the expert's involvement in determining the linguistic labels for each linguistic variable and specifying the ranges for each label. The proposed approach improves the precision and reliability of prediction systems but limits the involvement of human resources, which is the core of expert system development.

6. CONCLUSIONS

Evaluating suppliers is a significant activity with implications for an organization's performance and competitiveness. In contrast, developing an evaluation system based on qualitative and quantitative criteria is complex. The experts participate in the development of the evaluations, and each has their own interpretation, making the subjectivity and imprecision of their partitions evident. Generally, the behavior of a fuzzy system is described using natural language. Otherwise, the behavior of a system is described with fuzzy quantities.

An expert-driven approach is used to design and develop systems with high interpretability but low accuracy, since fuzzy quantities require no interpretation. In black-box approaches where there is no prior knowledge of the process, it is essential to identify the system based on its structure and parameters. They are also known as data-based approaches.

Prior conceptualization allowed the development of the proposed algorithm. The expert approach in building the knowledge base was maintained. Each fuzzy rule has an antecedent structure and a consequent structure. In addition, the data-driven approach was integrated to determine the relationship between the antecedent and consequent structures as an attention mechanism. The proposed attention mechanism consists of two central processes, in which vectors with diffuse values for each linguistic variable are used as input to the data reading process. As a second central process, the Euclidean distance is used to identify relationships between the system's input vectors and to determine alignment weights during the quantification of variable relationships. Finally, the parameters of the consequences were dynamically identified. Each time the evaluation system is used, it establishes a new relationship between the structures, thereby making the system more robust and reliable.

A new stage of association where a geometric rule base is set to the proposed fuzzy system, in which the distances between the centroids and each of the fuzzy values for each rule are optimally determined, grouping the rules based on their similarity and thus choosing the corresponding consequent ones as an attention mechanism. Since the fuzzy values differ each time an evaluation is made, the grouping and determination of consequents change dynamically, thereby improving the precision of the results by measuring the similarity of the diffuse values. The geometric distances correspond to a real domain $\mathbb{R}(\mathbf{T})$. A normalization step is used to return to a diffuse domain in a range from 0 to 1, using the SoftMax function as a diffuse domain recovery mechanism $\mathcal{F}(\mathbf{X})$. In this way, using the mathematical product operator as a Cartesian product $\mathcal{F}(\mathbf{X} \text{ x } \mathbf{Y})$ is proposed. Then, the maximum argument is used in aggregating the output geometric sets. Finally, the centroid method is used as an alternative to perform a weighted evaluation, and the results are compared with those of the current evaluation system. The results demonstrate the accuracy of the proposed AFIS system.

A comparison was performed with a conventional fuzzy inference system, using five different combinations of fuzzy inference rules: Prod-Max, Min-Max, Max-Min, Max-Max, and Max-Prod. In each case, the results of the proposed AFIS system were more accurate and reliable, as indicated by the mean square error, validating the efficiency of the associative fuzzy inference system.

We are highlighting the construction of the Iterative Associative Method of Dynamic Consequents IAMDC for Mamdani Fuzzy Systems Type I as a Data-Driven Approach. In the future, we will focus our research on developing probabilistic and dynamic fuzzy systems through the approach of fuzzy fractional differential Equations.

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