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AN INVENTORY SYSTEM WITH TIME-DEPENDENT HOLDING COST AND DEMAND DYNAMICS UNDER A GENERALIZED TRAPEZOIDAL NEUTROSOPHIC FRAMEWORK

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This study presents an inventory model that incorporates realistic demand factors such as price, time, reliability, and the influence of advertisements. The model allows shortages with full backlogging and considers permissible delays in payment. Deterioration is represented by a two-parameter Weibull distribution, while the holding cost is modeled as a function of both holding time and item reliability. Since inventory cost parameters are often imprecise, this uncertainty is addressed using Generalized Trapezoidal Neutrosophic Numbers (GTrNNs). To handle such imprecision, a novel de-neutrosophication technique is proposed. The payment structure considers two cases: when the credit period is less than or equal to the replenishment cycle time, and when it is greater. Optimal solutions are derived by taking the time at which the inventory level reaches zero as the decision variable. Numerical examples and sensitivity analyses are conducted to examine the impact of variations in key parameters on the optimal inventory policy.

Keywords: Deterioration; Holding cost; Reliability; Neutrosophic set; Delay in Payment; Weibull distribution.

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1. INTRODUCTION

Deterioration refers to the decline in the quality or condition of an item over time. In inventory management, deterioration plays a crucial role in influencing demand. Customers are generally unwilling to purchase deteriorated items, which results in such items remaining in inventory for extended periods and increasing holding costs. Recently, researchers have incorporated various realistic aspects of deterioration into their inventory models, considering a broader range of characteristics to better reflect real-world scenarios. The inventory models proposed by Hung (2011) and Skouri *et al*. (2009) account for time-dependent deterioration modeled using a Weibull distribution. De and Goswami (2006) developed an Economic Order Quantity (EOQ) model with a fuzzy deterioration rate. Sarkar and Sarkar (2013) and Mahata *et al*. (2020) proposed inventory models that consider a time-varying deterioration rate. Wang *et al*. (2011) developed a supply chain model with time-sensitive deterioration rates. Lin and Wang (2018) designed a model for configuring warehouses that manage deteriorating items. Deterioration is an unavoidable phenomenon and must be studied alongside the interrelated aspects of inventory management. Accordingly, this study models item deterioration using a two-parameter Weibull distribution.

In today's market, the sales of an item are significantly influenced by the effectiveness of advertisements. Many consumers make purchasing decisions based on an item's reliability and the influence of advertisements through electronic and print media. When an advertisement has a strong appeal, it leads to an increase in both item sales and demand. An item's value is closely linked to its reliability; if an item is perceived as unreliable, some customers may choose not to buy it, reducing demand. Moreover, if an item is priced too high, many customers may be unable to afford it, which means demand is also affected by the price factor. Researchers have incorporated various types of demand rates, such as fuzzy demand (Vahdani *et al*., 2016; Debnath *et al*., 2018), stock-dependent demand (Hou, 2006; Konstantaras and Skouri, 2011; Shaikh *et al*., 2018b; Pervin *et al*., 2019), ramp-type demand (Manna and Chaudhuri, 2006; Ahmed *et al*., 2013; Shi *et al*., 2019), and quadratic demand rate (Vandana and Sharma, 2016; Shukla *et al*., 2015). Tan and Weng (2012) developed an inventory model with a constant demand rate. Pal and Mahapatra (2017) developed a supply chain production model with stochastic demand. Mishra *et al*. (2019) proposed an inventory model considering a hybrid demand

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rate. Chanda and Kumar (2017) studied a fuzzy EOQ model with advertising and price-sensitive demand rates. Mahata *et al*. (2019) proposed an inventory model with price-dependent demand. Bhunia *et al*. (2017) presented an EOQ model under variable demand. An EOQ model for linear trend demand was developed by Wu and Zhao (2015) and Jaggi *et al*. (2011). Aarya and Sharma (2015) developed an inventory model with power-pattern demand. This study assumes demand to dependent on time, price, the impact of advertisements, and reliability.

Many inventory models assume that holding costs remain constant, but this assumption often does not reflect real-life scenarios. In practice, holding costs can depend on both time and reliability. The reliability of an item refers to its ability to perform its intended function over a specified period without failure. If an item remains in inventory for an extended period, its desirability may decline, leading to reduced customer interest and longer storage times. This results in higher holding costs over time. Conversely, if an item is highly reliable, that is, if it maintains its quality and functionality over time, customers are more likely to purchase it promptly, reducing the time it spends in inventory and, consequently, lowering the holding costs. This dynamic relationship highlights the importance of incorporating time- and reliability-dependent holding costs into inventory models. Muhlemann and Valtis-Spanopoulos (1980) and Pervin *et al*. (2017) developed EOQ models under variable holding costs. Sen and Saha (2018) developed an inventory model with time-dependent holding costs. Cárdenas-Barrón *et al*. (2020) developed a model that considers nonlinear, stock-dependent holding costs. In this scenario, we assume a reliability and time-dependent holding cost, which provides a more effective and practical approach to inventory system management.

In inventory management, payment delays play a vital role in the settlement process. Typically, a supplier offers a credit period to the retailer, during which the retailer does not have to pay any interest on the payment. However, if the retailer settles the payment after the credit period, they must pay interest to the supplier. Several inventory models have been developed that incorporate permissible delays in payment to better reflect real-world credit policies. For instance, Chang *et al*. (2016, 2017) and Liao *et al*. (2016, 2018) analyzed EOQ models with delayed payment and imperfect items, showing their impact on ordering decisions. Kumari and Pakkala (2016) studied uncertain payment times, while Majumder *et al*. (2016) explored two-level trade credit systems. Models by Rezezedeh *et al*. (2016) and Kundu *et al*. (2019) incorporated production reliability and customer default risk under delayed payment terms. Seifert *et al*. (2017) empirically validated the role of payment delay in improving firm profitability. These studies highlight the strategic value of trade credit in inventory systems. This study develops an inventory system that accounts for delayed payments.

Dealing with uncertainties in inventory systems, such as demand, storage, costs, and EOQ, presents significant challenges for researchers. Estimating the value of cost parameters in an inventory management system is difficult because these parameters fluctuate daily. This variability indicates uncertainty in cost parameters, which are modeled as Neutrosophic numbers. Neutrosophic numbers help address these uncertainties by incorporating three membership functions: truth, hesitation, and falsity. These functions enable more accurate estimations of inventory parameters. Jaggi *et al*. (2015) developed an inventory model under uncertain conditions by assuming cost parameters were triangular fuzzy numbers. Shabani *et al*. (2016a) created an inventory model considering both fuzzy deterioration rates and demand rates. Adak and Mahapatra (2021) proposed a supply chain inventory model under fuzzy uncertainty. Shaikh *et al*. (2018a) presented an inventory model for deteriorating items with variable demand in a fuzzy environment. In this paper, we assume the cost parameters are represented by GTrNNs, which capture the truth, hesitation, and falsity ranges of each specific parameter.

2. DEVELOPMENT AND MODELING OF THE INVENTORY SYSTEM

2.1. Research gap and problem definition

Research gap: Several inventory models in the literature assume constant holding costs, which is unrealistic, as holding costs often increase over time for many commodities. Additionally, reliable items tend to experience less deterioration, indicating a dependency of holding costs on reliability that has rarely been considered in earlier studies. Although numerous inventory models have been developed under fuzzy and intuitionistic fuzzy environments, they often fail to address inconsistent or incomplete information effectively. Neutrosophic sets, as generalizations of fuzzy and intuitionistic fuzzy sets, offer a robust framework for handling uncertainties and incomplete information. However, their application in inventory management, particularly with GTrNNs, remains largely unexplored. Compared to fuzzy and intuitionistic fuzzy sets, GTrNNs allow for more flexible and accurate modeling of uncertainty by accommodating varying degrees of truth, indeterminacy, and falsity, especially when data is inconsistent or incomplete.

Moreover, earlier models often overlook the significant impact of advertisements, price, and item reliability on demand rates. These factors are crucial for accurately modeling real-world inventory systems, especially in competitive markets where customer attraction plays a pivotal role. Table 1 summarizes the innovations of this model and provides a comparison with the existing literature. ("NA" indicates that the respective feature is not considered in the referenced inventory systems).

Table 1. Contribution of the proposed model to compare with earlier studies

Articles	Deterioration	Demand	Shortage	Holding cost	Delay payment	Uncertainty nature
Lin *et al*. (2012)	Constant	Ramp type	Fully backlog	Constant	NA	NA
Chuang *et al*. (2013)	Constant	Trapezoidal type	Fully backlog	Constant	NA	NA
Chung *et al*. (2014)	Exponential	Constant	NA	Constant	Applicable	NA
Pal *et al*. (2014)	Time varying (Weibull)	Ramp type	NA	Constant	NA	Triangular fuzzy
Chou and Julian (2015)	Constant	Exponential	Partial backlog	Constant	NA	NA
Sanni and Chukwu (2016)	Time varying (Weibull)	Quadratic	Quasipartial backlog	Constant	NA	NA
Shabani *et al*. (2016b)	Fuzzy number	Fuzzy number	NA	Constant	Applicable	Right-shaped fuzzy
San-José *et al*. (2017)	NA	Power pattern of time	Partial backlog	Constant	NA	NA
Chakraborty *et al*. (2018)	Time varying (Weibull)	Ramp type	Partial backlog	Constant	Applicable	NA
Bardhan *et al*. (2019)	Constant	Stock dependent	NA	Constant	NA	NA
Pervin *et al*. (2020)	Constant	Stock& price dependent	Partial backlog	Constant	NA	NA
Bappa Mondal and Majumder (2021)	Time varying (Weibull)	Time-varying logistic	Partial	Constant	NA	Triangular Neutrosophic
Bhavani *et al*. (2022)	Maximum life-time	Reliability & power pattern of time	NA	Constant	Applicable	Triangular Neutrosophic
Bhavani and Mahapatra (2023)	Maximum life-time	Quality & power pattern of time	NA	Constant	Applicable	Generalized triangular Neutrosophic
Current model	Time varying (Weibull)	Price, time, advertisement impact, and reliability	Fully backlog	Time & reliability dependent	Applicable	Generalized trapezoidal Neutrosophic

Problem Definition: The primary objective of this study is to develop an inventory model that integrates realistic features such as deterioration, time-dependent holding costs, uncertainty, and shortages. The model considers a dynamic demand rate influenced by price, reliability, and advertisements, effectively capturing real-world customer attraction. Specifically, the demand function is given by $D(t,p,r,A) = k(p)(m + bt)r^a A^v$, where $k(p) = \eta\, e^{-p\,\delta}$ represents the price factor, with $0 < \delta < 1$ and $\eta > 1$, while $a, v \in (0,1)$, $m, b > 0$, and $A > 1$. Here, δ is a static price sensitivity parameter satisfying $0 < \delta < 1$, which quantifies how sensitive the demand is to changes in price p. A larger value of δ indicates greater sensitivity, meaning demand decreases more rapidly as price increases. The deterioration rate follows a two-parameter Weibull distribution, and shortages are allowed with complete backlogging, requiring customers to wait until the next replenishment cycle. The holding cost depends on both time and reliability and is given by $H(r,t) = C_h htr^{-c}$, where $0 < h < 1$, with h serving as the scale parameter. Additionally, the model incorporates a delay in payment and addresses uncertainties in cost parameters using a novel de-Neutrosophication method for GTrNNs, ensuring better handling of imprecise and inconsistent data in real-market scenarios. By integrating these features, the proposed model provides a more practical and realistic approach to inventory management.

The following notations are considered for the proposed single-item inventory system.

Notations:

$D(t,p,r,A)$	Demand rate.
p	Selling price.
T_1	The time when the stock level reaches zero.
M	Credit period (permissible delay for retailer payment to the supplier).
T	Replenishment cycle.

r	Reliability.
C_h	The cost of holding per item per time unit
C_d	The cost of deterioration per item per time unit.
C_s	Shortage cost per time unit.
C_1	Purchase cost per single item.
A	Number of advertisements.
I_p	Interest rate charged for delayed payments.
I_e	Interest rate earned.
ν	Impact of advertisement.

Assumptions:

1. The demand rate of the model is defined as $D(t,p,r,A) = \eta\, e^{-p\,\delta}(m + b\,t)r^a A^\nu$, where $m, b > 0, \delta \in (0,1), \eta > 1, A > 1,$ and $a, \nu \in (0,1)$.
2. The deterioration rate of the item over time is modeled using a two-parameter Weibull distribution.
3. Shortages are permitted and completely backlogged.
4. The holding cost is a function of both time and reliability, given by $H(r,t) = C_h h t r^{-c}$, where $0 < h < 1$, with h serving as the scale parameter.
5. This model incorporates permissible delays in payment.
6. The imprecise cost parameters of the model are represented with the help of GTrNNs.
7. A single item is considered per replenishment cycle.

Lemma 1: *The demand rate $D(t,p,r,A)$ increases with (i) time and reliability for $t > 0$ and $r, a \in (0,1)$, and (ii) the impact of advertisement ν for $A > 1$ and $\nu \in (0,1)$.*

Proof. The demand rate of the inventory model is given by

$$D(t,p,r,A) = \eta\, e^{-p\,\delta}(m + b\,t)r^a A^\nu,$$

where $\nu, a \in (0,1), m > 0, b > 0,$ and $A > 1$.

(i) First, we compute the derivative of demand rate with respect to time

$$\frac{dD(t,p,r,A)}{dt} = \eta e^{-p\delta} b r^a A^\nu > 0 \quad \text{for all}\, t > 0.$$

Therefore, the proposed demand rate is an increasing function of time for $t > 0$.

Next, we compute the derivative of demand rate with respect to reliability

$$\frac{dD(t,p,r,A)}{dr} = a\eta e^{-p\delta}(m + bt) r^{a-1} A^\nu > 0 \quad \text{for all}\, r, a \in (0,1).$$

Thus, the demand rate is an increasing function of reliability for all $r, a \in (0,1)$.

Therefore, the demand rate $D(t,p,r,A)$increases with time and reliability for $t > 0$ and $r, a \in (0,1)$.

(ii) (Now, we compute the derivative of demand rate with respect to the impact of advertisement

$$\frac{dD(t,p,r,A)}{d\nu} = \delta\eta e^{-p\delta}(m + bt) r^a A^\nu \log A > 0 \quad \text{for all}\, A > 1 \text{ and}\, \nu \in (0,1).$$

Hence, the demand rate increases with the impact of advertisement for $A > 1$ and $\nu \in (0,1)$.

Thus, the proof is complete. ■

Lemma 2: *The demand rate $D(t,p,r,A)$ decreases with price at an increasing rate for $\eta > 1$ and $0 < \delta < 1$.*

Proof. The demand rate of the inventory model is given by

$$D(t,p,r,A) = \eta\, e^{-p\,\delta}(m + b\,t)r^a A^\nu,$$

where $\nu, a \in (0,1), m > 0, b > 0$, and $A > 1$.

Now, we compute the first derivative of the demand with respect to price

$$\frac{dD(t,p,r,A)}{dp} = -\delta\eta e^{-p\delta}(m+bt)r^{a}A^{\nu} < 0$$

for all $\eta > 1$ *and* $0 < \delta < 1$. This shows that the demand rate decreases with price for these parameter values.

Next, we compute the second derivative of the demand rate with respect to price:

$$\frac{d^2D(t,p,r,A)}{dp^2} = \delta^2\eta e^{-p\delta}(m+bt)r^{a}A^{\nu} > 0$$

for $\eta > 1$ *and* $0 < \delta < 1$, indicating that the rate of decrease of the demand rate increases with price.
Therefore, the demand rate decreases at an increasing rate with price. This completes the proof. ■

Lemma 3: *The holding cost of an item decreases with reliability r and increases with time t for all* $r, c \in (0,1), 0 < h < 1$, *and* $t > 0$.

Proof. The holding cost per unit of an item in the inventory system is denoted by C_h.
Let the holding cost function be $H(r,t) = C_h htr^{-c}$.
First, we compute the derivative of the holding cost with respect to reliability

$$\frac{dH(r,t)}{dr} = -C_h hctr^{-(c+1)} < 0$$

for all r, t, h, and c, where $r, c \in (0,1), 0 < h < 1$, *and* $t > 0$. This shows that the holding cost decreases with reliability for the given parameter values.
Next, we compute the derivative of the holding cost with respect to time

$$\frac{dH(r,t)}{dt} = C_h hr^{-c} > 0$$

for all r, t, h, and c, where $r, c \in (0,1), 0 < h < 1$, *and* $t > 0$. This shows that the holding cost increases with time.
Thus, the holding cost per item decreases with reliability and increases with time. Therefore, the proof is complete. ■
As shown in Lemmas 1 and 2, the demand rate is influenced by factors such as price, time, reliability, and the impact of advertisement. Additionally, Lemma 3 establishes the relationship between holding cost, time, and reliability, confirming that holding cost decreases with reliability but increases with time. These insights are crucial for understanding the model dynamics, especially when considering the time-dependent nature of demand and the influence of reliability and advertisement on customer behavior.

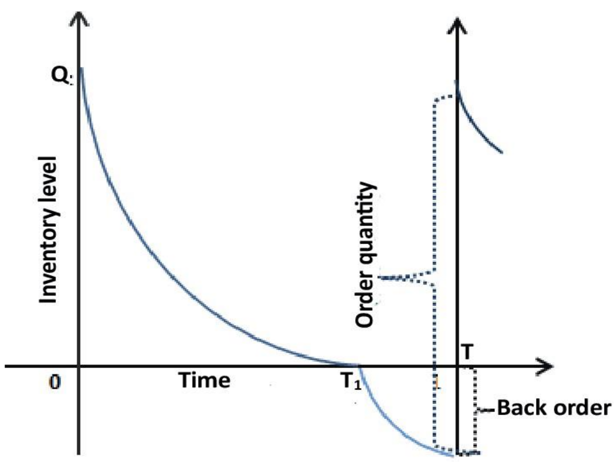


Figure 1. Graphical depiction of the inventory model

2.2. Mathematical formulation of the proposed inventory model

Based on the above discussion, we have established the mathematical formulation of the proposed inventory system. The initial inventory level, denoted by Q, decreases over time because of both demand and deterioration. The total cycle duration T, is divided into two distinct intervals. Within the interval $[0, T_1]$, the inventory level gradually declines and reaches zero at T_1 due to these factors. During the interval $[T_1, T]$, shortages occur, and all unmet demand is fully backlogged, as illustrated in Figure 1. In this interval, demand is influenced by factors such as the impact of advertisement, reliability, time, and price. The backlogged items are fulfilled during the subsequent replenishment cycle.

The time until the deterioration of items follows a two-parameter Weibull distribution, which is considered one of the most suitable models for representing deterioration. The instantaneous deterioration rate of the non-deteriorated inventory at time t, denoted as $\theta(t)$, is given by $\theta(t) = \alpha\beta\, t^{\beta-1}$, where $(0 < \alpha < 1)$, $\beta > 0$, and $t > 0$.

Figure 1 illustrates that at time T_1, the inventory level reaches zero, leading to shortages during the period $[T_1, T]$ due to stock depletion. These shortages result in full backordering, with unmet demand being fulfilled in the next replenishment. The order quantity for the subsequent cycle consists of the initial order quantity Q plus the backordered demand. The inventory system incorporates payment delays, leading to two possible cases:

Case I: Delay period M(the credit period) is less than cycle time T_1: In this case, the retailer does not pay interest during the credit period *M* and earns interest on the delayed payment. After *M*, however, the retailer must pay interest to the supplier.

Case II: Delay period *M* (the credit period) is greater than or equal to cycle time T_1: In this scenario, the retailer is not required to pay interest to the supplier and can continue to earn interest on the outstanding amount. Let $I(t)$ represent the inventory level at time t $(0 \le t \le T)$. Based on the above assumptions, the differential equations governing the inventory model over the interval $[0, T]$ are expressed as follows

$$\frac{dI(t)}{dt} + \theta(t)I(t) = -(m+bt)k(p)r^{a}A^{\nu}, \quad 0 \le t \le T_1 \tag{1}$$

$$\frac{dI(t)}{dt} = -k(p)mr^{a}A^{\nu}, \quad T_1 \le t \le T \tag{2}$$

with the boundary conditions $I(0) = Q$ and $I(T_1) = 0$.

2.2.1. Mathematical Analysis of the Proposed Inventory Model

The solution to the differential equations ((1), (2)) governing the proposed inventory system, with the initial condition $I(0) = Q$ and the terminal condition $I(T_1) = 0$, is

$$I(t) = \begin{cases} (1-\alpha t^{\beta})\left(m(T_1 - t) + \frac{b(T_1^2 - t^2)}{2} + \frac{m\alpha\left(T_1^{\beta+1} - t^{\beta+1}\right)}{\beta+1} + \frac{b\alpha\left(T_1^{\beta+2} - t^{\beta+2}\right)}{\beta+2}\right) R \ if\ 0 \le t \le T_1 \\ R(T_1 - t)m \quad if \ \ T_1 \le t \le T \end{cases}$$

Here, $R = \eta e^{-p\delta} r^{a} A^{\nu}$, and using the boundary condition, the maximum inventory is given by

$$I(0) = Q = \left(mT_1 + \frac{bT_1^2}{2} + \frac{m\alpha T_1^{\beta+1}}{\beta+1} + \frac{b\alpha T_1^{\beta+2}}{\beta+2}\right)R.$$

and the total shortage is $I(T) = R(T_1 - T)m$.

The cost incurred for holding items during the period $[0, T_1]$ is

$$HC = C_h \int_0^{T_1} htr^{-c}\, I(t)dt$$
$$= \frac{C_h R h T_1^3}{r^c}\left(\frac{m}{6} + \frac{bT_1}{8} + \frac{m\alpha\beta T_1^{\beta}}{2(\beta+2)(\beta+3)} + \frac{b\alpha\beta T_1^{\beta+1}}{2(\beta+2)(\beta+4)} - \frac{m\alpha^2 T_1^{2\beta}}{(2\beta+3)(\beta+2)} - \frac{b\alpha^2 T_1^{2\beta+1}}{2(\beta+2)^2}\right).$$

The cost incurred due to the non-availability of stock during the period $[T_1, T]$ is

$$SC = -C_s \int_{T_1}^{T} I(t)\,dt = \frac{C_S Rm}{2}[(T - T_1)^2].$$

The cost of purchasing items is

$$PC = C_1 Q + C_1 \int_0^{T-T_1} mk(p)A^{\nu}\, r^{a} dt = C_1 R\left(mT + \frac{m\alpha T_1^{\beta+1}}{\beta+1} + \frac{bT_1^2}{2} + \frac{b\alpha T_1^{\beta+2}}{\beta+2}\right).$$

The cost of decayed goods is

$$DC = C_d\left(Q - \int_0^{T_1} D(t,p,r,A)dt\right) = C_d R\left(\frac{m\alpha T_1^{\beta+1}}{\beta+1} + \frac{b\alpha T_1^{\beta+2}}{\beta+2}\right).$$

The cost of ordering goods remains fixed for each cycle.

Ordering cost (OC) is $= A_0$.

Case I: Delay period *M* (the credit period) is less than cycle time T_1: In this case, the retailer is allowed a credit period *M*, which is shorter than the cycle time T_1. During this period, the retailer does not have to pay interest on the outstanding amount and may even earn interest on the money that would otherwise be paid to the supplier. Once the credit period *M* expires, the retailer becomes liable for interest charges on the outstanding amount, as the payment is delayed beyond the allowed credit period. Thus, the retailer can enjoy interest-free payment within the period *M*, but interest will accrue once the delay exceeds *M*.

$$IE_1 = C_1 I_e \int_0^{T_1} tD(t,p,r,A)dt = C_1 I_e R\left(\frac{mT_1^2}{2} + \frac{bT_1^3}{3}\right).$$

In this scenario, the retailer is required to pay interest to the supplier once the credit period ends.

$$IP_1 = C_1\left(\int_M^T I(t)dt\right) I_p = C_1\left(\int_M^{T_1} I(t)dt + \int_{T_1}^{T} I(t)dt\right) I_p.$$

$$IP_1 = C_1 I_P R\Bigg[\left(mT_1 + \frac{bT_1^2}{2} + \frac{m\alpha T_1^{\beta+1}}{\beta+1} + \frac{b\alpha T_1^{\beta+2}}{\beta+2}\right)\left((T_1 - M) - \frac{\alpha\left(T_1^{\beta+1} - M^{\beta+1}\right)}{\beta+1}\right) + \left(\frac{m\alpha^2\left(T_1^{2\beta+2} - M^{2\beta+2}\right)}{2(\beta+1)^2}\right) - \frac{m(T-T_1)^2}{2} + \left(\frac{b\alpha^2\left(T_1^{2\beta+3} - M^{2\beta+3}\right)}{(\beta+2)(2\beta+3)}\right) + \left(\frac{b\alpha\beta\left(T_1^{\beta+3} - M^{\beta+3}\right)}{2(\beta+2)(\beta+3)}\right) - m\left(\frac{T_1^2 - M^2}{2}\right) - b\left(\frac{T_1^3 - M^3}{6}\right) + \left(\frac{m\alpha\beta\left(T_1^{\beta+2} - M^{\beta+2}\right)}{(\beta+1)(\beta+2)}\right)\Bigg].$$

The cost of advertising items is $AC = gA$, where g is the cost per advertisement.
The total inventory cost of the system per unit of time is given by

$$\begin{aligned} TC_1(T_1) &= \frac{1}{T}[OC + DC + SC + HC + PC + IP_1 + gA - IE_1] \\ &= \frac{1}{T}\Bigg[\frac{C_h Rh T_1^3}{r^c}\left(\frac{m}{6} + \frac{bT_1}{8} + \frac{m\alpha\beta T_1^{\beta}}{2(\beta+3)(\beta+2)} + \frac{b\alpha\beta T_1^{\beta+1}}{2(\beta+2)(\beta+4)} - \frac{m\alpha^2 T_1^{2\beta}}{(2\beta+3)(\beta+2)} - \frac{b\alpha^2 T_1^{2\beta+1}}{2(\beta+2)^2}\right) + \frac{C_S Rm(T-T_1)^2}{2} + \\ &\quad C_1 R\left(mT + m\alpha\frac{T_1^{\beta+1}}{\beta+1} + \frac{bT_1^2}{2} + \frac{b\alpha T_1^{\beta+2}}{\beta+2}\right) + C_d R\left(\frac{m\alpha T_1^{\beta+1}}{\beta+1} + \frac{b\alpha T_1^{\beta+2}}{\beta+2}\right) - RC_1 I_e\left(\frac{mT_1^2}{2} + \frac{bT_1^3}{3}\right) + A_0 + \\ &\quad gA + C_1 I_P R\Bigg[\left(mT_1 + \frac{bT_1^2}{2} + \frac{m\alpha T_1^{\beta+1}}{\beta+1} + \frac{b\alpha T_1^{\beta+2}}{\beta+2}\right)\left(T_1 - M - \frac{\alpha\left(T_1^{\beta+1} - M^{\beta+1}\right)}{\beta+1}\right) - \frac{m}{2}(T_1^2 - M^2) - \\ &\quad \frac{b}{6}(T_1^3 - M^3) - \frac{m(T-T_1)^2}{2} + \frac{m\alpha^2\left(T_1^{2\beta+2} - M^{2\beta+2}\right)}{2(\beta+1)^2} + \frac{b\alpha\beta\left(T_1^{\beta+3} - M^{\beta+3}\right)}{2(\beta+2)(\beta+3)} + \frac{m\alpha\beta\left(T_1^{\beta+2} - M^{\beta+2}\right)}{(\beta+1)(\beta+2)} + \\ &\quad \frac{b\alpha^2\left(T_1^{2\beta+3} - M^{2\beta+3}\right)}{(\beta+2)(2\beta+3)}\Bigg]\Bigg]. \end{aligned} \tag{3}$$

To evaluate the optimal total average inventory cost when the permissible delay period is less than the cycle time T_1, the above Equation (3) needs to be optimized.

Case II: Delay period *M* (the credit period) is greater than or equal to cycle time T_1: In this scenario, the credit period M is not shorter than the cycle time T_1, meaning that the retailer is allowed to delay payment without incurring any interest during the credit period. During this period, the retailer can use the money without paying interest and may even earn interest on the amount that would otherwise be paid to the supplier. However, once the credit period M expires, the retailer becomes liable for interest charges on the outstanding payment. Thus, M represents the credit period, within which no interest is charged, and any payment delayed beyond this period incurs interest. Interest earned (IE_2) is given by

$$IE_2 = C_1 I_e\left[\int_0^{T_1} tD(t,p,r,t,A)dt + (M - T_1)\int_0^{T_1} D(t,p,r,A)dt\right]$$
$$= RC_1 I_e\left[\frac{mT_1^2}{2} + \frac{bT_1^3}{3} + (M - T_1)\left(mT_1 + \frac{bT_1^2}{2}\right)\right].$$

In this case, the retailer sells Q units during the period $[0, T_1]$ and pays C_1Q to the supplier. Since the supplier is paid in full at time M, the retailer does not incur any interest charges, resulting in zero interest cost.

$$IP_2 = 0.$$

The total inventory cost of the system per unit of time is expressed as

$$TC_2(T_1) = \frac{1}{T}[OC + DC + SC + HC + PC + IP_2 + gA - IE_2]$$
$$= \frac{1}{T}\Bigg[\frac{C_hRhT_1^3}{r^c}\left(\frac{m}{6} + \frac{bT_1}{8} + \frac{m\alpha\beta T_1^{\beta}}{2(\beta+3)(\beta+2)} + \frac{b\alpha\beta T_1^{\beta+1}}{2(\beta+2)(\beta+4)} - \frac{m\alpha^2 T_1^{2\beta}}{(2\beta+3)(\beta+2)} - \frac{b\alpha^2 T_1^{2\beta+1}}{2(\beta+2)^2}\right) + \frac{C_S Rm(T-T_1)^2}{2} +$$
$$C_1R\left(mT + m\alpha\frac{T_1^{\beta+1}}{\beta+1} + \frac{bT_1^2}{2} + \frac{b\alpha T_1^{\beta+2}}{\beta+2}\right) + C_dR\left(\frac{m\alpha T_1^{\beta+1}}{\beta+1} + \frac{b\alpha T_1^{\beta+2}}{\beta+2}\right) - RC_1I_e\Bigg(\frac{mT_1^2}{2} + \frac{bT_1^3}{3} + (M -$$
$$T_1)\left(mT_1 + \frac{bT_1^2}{2}\right)\Bigg) + gA + A_0\Bigg]. \tag{4}$$

To evaluate the optimal total average inventory cost when the permissible delay in payment exceeds cycle time T_1, Equation (4) needs to be optimized.

2.2.2. Inventory Model under Generalized Trapezoidal Neutosophic Environment

In real-life inventory systems, many parameters are imprecise, meaning they may be inexact, invalid, or inaccurate. This imprecision creates challenges in developing reliable mathematical models and making accurate decisions. The crisp model discussed earlier assumes precise values for cost, demand, deterioration, and other parameters, which may not hold in practice. To address these limitations, Neutrosophic numbers provide a robust framework for modeling uncertainty. They effectively capture and represent the truth, indeterminacy, and falsity associated with imprecise parameters, thereby offering a more comprehensive and realistic formulation. By incorporating GTrNNs, the proposed inventory model accounts for uncertainty in decision variables and parameters. This extension enhances the decision-making process by enabling a more practical and realistic representation of uncertain or ambiguous data, bridging the gap between theoretical models and real-world scenarios.

Definition 1 ***Neutrosophic set:*** *A single-valued Neutrosophic set ($\tilde{P}$) associated with a single independent variable(x) is defined as $\tilde{P} = \{\langle x; [\pi_{\tilde{P}}(x), \theta_{\tilde{P}}(x), \eta_{\tilde{P}}(x)]\rangle : x \in X\}$, where $\pi_{\tilde{P}}(x), \theta_{\tilde{P}}(x), \eta_{\tilde{P}}(x)$ represent the membership functions for truth, falsity, and indeterminacy, respectively. Specifically, $\pi_{\tilde{P}}: R \to [0,1]$ denotes the truth membership function, $\theta_{\tilde{P}}: R \to [0,1]$ represents the falsity membership function, and $\eta_{\tilde{P}}: R \to [0,1]$ corresponds to the indeterminacy membership function.*

Definition 2 **(α, β, γ) *cut:*** *The (α, β, γ) cut of a Neutrosophic set is defined as $\tilde{P}_{(\alpha,\beta,\gamma)} = \{x \in X | \pi_{\tilde{P}}(x) \geq \alpha, \theta_{\tilde{P}}(x) \leq \beta, \eta_{\tilde{P}}(x) \leq \gamma\}$.*

Definition 3 ***Generalized trapezoidal Neutrosophic number (GTrNN):*** *A GTrNN ($\tilde{F}$) is defined as $\tilde{F} = \langle(m_{11}, m_{12}, m_{13}, m_{14};\ \mu_\delta),\ (n_{11}, n_{12}, n_{13}, n_{14};\ \nu_\delta),\ (p_{11}, p_{12}, p_{13}, p_{14};\ \zeta_\delta)\rangle$, where $\mu_\delta,\ \nu_\delta,\ \zeta_\delta \in [0,1]$. Here, $\pi_{\tilde{F}}: R \to [0, \mu_\delta]$ is the membership function for truth, $\theta_{\tilde{F}}: R \to [\nu_\delta, 1]$ is the membership function for falsity, and $\eta_{\tilde{F}}: R \to [\zeta_\delta, 1]$ is* *the membership function for indeterminacy. The following mathematical expressions define these membership functions, and their graphs are depicted in Figure 2.*

$$\pi_{\tilde F}(x)=\begin{cases}\frac{(x-m_{11})}{(m_{12}-m_{11})}\mu_\delta, for\ m_{11}\le x\le m_{12}\\ \mu_\delta, \quad for\ m_{12}\le x\le m_{13}\\ \frac{(m_{14}-x)}{(m_{14}-m_{13})}\mu_\delta, for\ m_{13}\le x\le m_{14}\\ 0,\ otherwise\end{cases},\ \theta_{\tilde F}(x)=\begin{cases}\frac{(n_{12}-x)+\nu_\delta(x-n_{11})}{(n_{12}-n_{11})},\ for\ n_{11}\le x\le n_{12}\\ \nu_\delta, \quad for\ n_{12}\le x\le n_{13}\\ \frac{(x-n_{13})+\nu_\delta(n_{14}-x)}{(n_{14}-n_{13})}, for\ n_{13}\le x\le n_{14}\\ 1, \quad otherwise\end{cases}$$

$$and\ \eta_{\tilde F}(x)=\begin{cases}\frac{(p_{12}-x)+\zeta_\delta(x-p_{11})}{(p_{12}-p_{11})},\ for\ p_{11}\le x\le p_{12}\\ \zeta_\delta, \quad for\ p_{12}\le x\le p_{13}\\ \frac{(x-p_{13})+\zeta_\delta(p_{14}-x)}{(p_{14}-p_{13})}, for\ p_{13}\le x\le p_{14}\\ 1, \quad otherwise\end{cases}$$

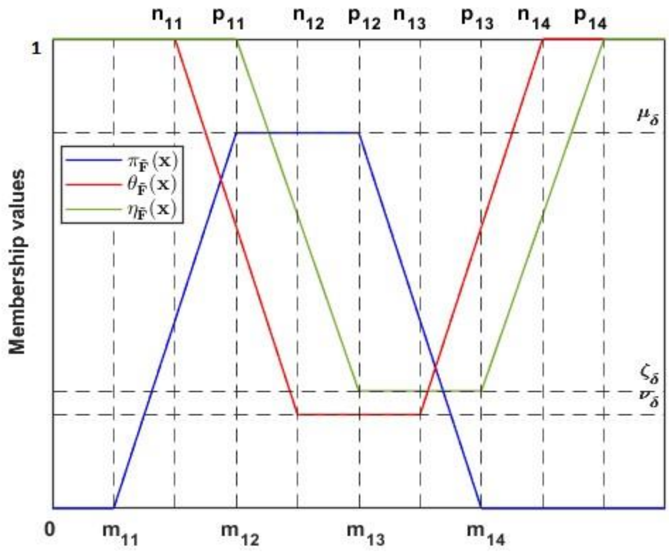


Figure 2. Visual depiction of GTrNN

2.2.3. De-Neutrosophication of GTrNNs

Numerous methods for de-Neutrosophication of Neutrosophic numbers have been proposed in the literature. In this article, a new de-Neutrosophication method for GTrNNs is introduced by applying Roubens (Roubens, 1990) ranking function, which is defined as follows:

$$R_{r\delta}(\mu_{\tilde C})=(1/2)\int_0^1\left(inf\widetilde{C_\alpha}+sup\widetilde{C_\alpha}\right)d\alpha \tag{5}$$

for a fuzzy number $\tilde C$ having a membership function $\mu_{\tilde C}$, where $\widetilde{C_\alpha}$ denotes the alpha-cut of the fuzzy number $\tilde C$.

Definition 4: *The de-Neutrosophication of Neutrosophic number $\widetilde N$, which has the truth membership function $\pi_{\widetilde N}$, the falsity membership function $\theta_{\widetilde N}$, and the indeterminacy membership function $\eta_{\widetilde N}$ is defined as follows*

$$R(\widetilde N)=\frac{1}{3}[R_{r\delta}(\pi_{\widetilde N})+R_{r\delta}(\theta_{\widetilde N})+R_{r\delta}(\eta_{\widetilde N})], \tag{6}$$

where $R_{r\delta}$ represents Roubens ranking function, as defined in Equation (5).
Using Equation (6), *if $\tilde F=\langle(m_{11},m_{12},m_{13},m_{14};\ \mu_\delta),\ (n_{11},n_{12},n_{13},n_{14};\ \nu_\delta),\ (p_{11},p_{12},p_{13},p_{14};\ \zeta_\delta)\rangle$ is a GTrNN, then*

$$R(\tilde F)=\frac{1}{3}[R_{r\delta}(\pi_{\tilde F})+R_{r\delta}(\theta_{\tilde F})+R_{r\delta}(\eta_{\tilde F})] \tag{7}$$

$$= \frac{1}{3}\left(\frac{1}{4\mu_\delta}\big(m_{12} + m_{13} + (2\mu_\delta - 1)(m_{11} + m_{14})\big) + \frac{1}{4(1-\nu_\delta)}\big(n_{12} + n_{13} + (1 - 2\nu_\delta)(n_{11} + n_{14})\big) + \frac{1}{4(1-\zeta_\delta)}(p_{12} + p_{13} + (1 - 2\zeta_\delta)(p_{11} + p_{14}))\right).$$

This paper models holding, deterioration, purchase, shortage, and ordering costs as GTrNNs to address the uncertainty in real-world cost parameters. By introducing GTrNNs, we aim to more accurately represent actual market conditions and evaluate their impact on the proposed inventory system. In this study, the holding cost (C_h), shortage cost (C_s), deterioration cost (C_d), purchase cost (C_1), and ordering cost (A_0) are considered as GTrNNs. The cost parameters and the total cost of the inventory system under a generalized Neutrosophic environment are denoted as $\widetilde{C_h}, \widetilde{C_s}, \widetilde{C_d}, \widetilde{C_1}, \widetilde{A_0}, \widetilde{TC}_1(T_1)$, and $\widetilde{TC}_2(T_1)$. The representation of the generalized trapezoidal Neutrosophic cost parameters is given as follows:

$$\widetilde{C_h} = <(h_{1\tau}, h_{2\tau}, h_{3\tau}, h_{4\tau}; \mu_\delta), (h_{1\sigma}, h_{2\sigma}, h_{3\sigma}, h_{4\sigma}; \nu_\delta), (h_{1\rho}, h_{2\rho}, h_{3\rho}, h_{4\rho}; \zeta_\delta)>,$$
$$\widetilde{C_s} = <(s_{1\tau}, s_{2\tau}, s_{3\tau}, s_{4\tau}; \mu_\delta), (s_{1\sigma}, s_{2\sigma}, s_{3\sigma}, s_{4\sigma}; \nu_\delta), (s_{1\rho}, s_{2\rho}, s_{3\rho}, s_{4\rho}; \zeta_\delta)>,$$
$$\widetilde{C_d} = <(d_{1\tau}, d_{2\tau}, d_{3\tau}, d_{4\tau}; \mu_\delta), (d_{1\sigma}, d_{2\sigma}, d_{3\sigma}, d_{4\sigma}; \nu_\delta), (d_{1\rho}, d_{2\rho}, d_{3\rho}, d_{4\rho}; \zeta_\delta)>,$$
$$\widetilde{C_1} = <(a_{1\tau}, a_{2\tau}, a_{3\tau}, a_{4\tau}; \mu_\delta), (a_{1\sigma}, a_{2\sigma}, a_{3\sigma}, a_{4\sigma}; \nu_\delta), (a_{1\rho}, a_{2\rho}, a_{3\rho}, a_{4\rho}; \zeta_\delta)>,$$
$$\text{and } \widetilde{A_0} = <(b_{1\tau}, b_{2\tau}, b_{3\tau}, b_{4\tau}; \mu_\delta), (b_{1\sigma}, b_{2\sigma}, b_{3\sigma}, b_{4\sigma}; \nu_\delta), (b_{1\rho}, b_{2\rho}, b_{3\rho}, b_{4\rho}; \zeta_\delta)>.$$

The Neutrosophic parameters $\widetilde{C_h}, \widetilde{C_s}, \widetilde{C_d}, \widetilde{C_1}$ and $\widetilde{A_0}$ can be applied to the inventory model, making the total inventory cost (TC) inherently Neutrosophic. Consequently, this leads to the identification of the Neutrosophic total expected cost $(\widetilde{TC})$ per unit time.

Case I: Delay period *M* (the credit period) is less than cycle time (T_1):

$$\widetilde{TC}_1(T_1) = \frac{1}{T}\Bigg[\frac{\widetilde{C_h}RhT_1^3}{r^c}\left(\frac{m}{6} + \frac{bT_1}{8} + \frac{m\alpha\beta T_1^{\beta}}{2(\beta+3)(\beta+2)} + \frac{b\alpha\beta T_1^{\beta+1}}{2(\beta+2)(\beta+4)} - \frac{m\alpha^2 T_1^{2\beta}}{(2\beta+3)(\beta+2)} - \frac{b\alpha^2 T_1^{2\beta+1}}{2(\beta+2)^2}\right) + \frac{\widetilde{C_s}Rm(T-T_1)^2}{2} + \widetilde{C_1}R\left(mT + m\alpha\frac{T_1^{\beta+1}}{\beta+1} + \frac{bT_1^2}{2} + \frac{b\alpha T_1^{\beta+2}}{\beta+2}\right) + \widetilde{C_d}R\left(\frac{m\alpha T_1^{\beta+1}}{\beta+1} + \frac{b\alpha T_1^{\beta+2}}{\beta+2}\right) - R\widetilde{C_1}I_e\left(\frac{mT_1^2}{2} + \frac{bT_1^3}{3}\right) + \widetilde{A_0} + gA + \widetilde{C_1}I_PR\Bigg[\left(mT_1 + \frac{bT_1^2}{2} + \frac{m\alpha T_1^{\beta+1}}{\beta+1} + \frac{b\alpha T_1^{\beta+2}}{\beta+2}\right)\left(T_1 - M - \frac{\alpha\left(T_1^{\beta+1} - M^{\beta+1}\right)}{\beta+1}\right) - \frac{m}{2}(T_1^2 - M^2) - \frac{b}{6}(T_1^3 - M^3) - \frac{m(T-T_1)^2}{2} + \frac{m\alpha^2(T1^{2\beta+2} - M^{2\beta+2})}{2(\beta+1)^2} + \frac{b\alpha\beta\left(T_1^{\beta+3} - M^{\beta+3}\right)}{2(\beta+2)(\beta+3)} + \frac{m\alpha\beta\left(T_1^{\beta+2} - M^{\beta+2}\right)}{(\beta+1)(\beta+2)} + \frac{b\alpha^2\left(T_1^{2\beta+3} - M^{2\beta+3}\right)}{(\beta+2)(2\beta+3)}\Bigg]\Bigg]. \quad (8)$$

Case II: Delay period *M* (the credit period) is greater than or equal to cycle time T_1:

$$\widetilde{TC}_2(T_1) = \frac{1}{T}\Bigg[\frac{\widetilde{C_h}RhT_1^3}{r^c}\left(\frac{m}{6} + \frac{bT_1}{8} + \frac{m\alpha\beta T_1^{\beta}}{2(\beta+3)(\beta+2)} + \frac{b\alpha\beta T_1^{\beta+1}}{2(\beta+2)(\beta+4)} - \frac{m\alpha^2 T_1^{2\beta}}{(2\beta+3)(\beta+2)} - \frac{b\alpha^2 T_1^{2\beta+1}}{2(\beta+2)^2}\right) + \frac{\widetilde{C_s}Rm(T-T_1)^2}{2} + \widetilde{C_1}R\left(mT + m\alpha\frac{T_1^{\beta+1}}{\beta+1} + \frac{bT_1^2}{2} + \frac{b\alpha T_1^{\beta+2}}{\beta+2}\right) + gA + \widetilde{C_d}R\left(\frac{m\alpha T_1^{\beta+1}}{\beta+1} + \frac{b\alpha T_1^{\beta+2}}{\beta+2}\right) - R\widetilde{C_1}I_e\left(\frac{mT_1^2}{2} + \frac{bT_1^3}{3} + (M - T_1)\left(mT_1 + \frac{bT_1^2}{2}\right)\right) + \widetilde{A_0}\Bigg]. \quad (9)$$

Here, $\widetilde{TC}_1(T_1)$ and $\widetilde{TC}_2(T_1)$ represent the total cost of the inventory systems per unit time under a Neutrosophic environment.

2.2.4. Procedure for optimal solution under Neutrosophic and crisp circumstances

The procedure outlined in this section provides a step-by-step algorithm 1 for determining the optimal cost of an inventory system under Neutrosophic and crisp environments.

Algorithm 1: Procedure for finding optimal cost of the inventory system

Start;
Step 1: Initialize the inventory system by specifying all necessary parameters and input variables;
Step 2: Determine if there is uncertainty in the cost parameters;
Step 3: if *there is uncertainty* then
- **Step 3a:** Choose total cost Equations (8) and (9);

else
- **Step 3b:** Choose total cost Equations (3) and (4);

Step 4: Determine if the delay period is greater than T_1;
Step 5: if *the delay period is longer than* T_1 then
- **Step 5a:** For a crisp environment, select total cost Equation (3), and for a Neutrosophic environment, select total cost Equation (8). Solve both equations using MATLAB R2024a;

else
- **Step 5b:** For a crisp environment, select total cost Equation (4), and for a Neutrosophic environment, select total cost Equation (9). Solve both Equations (4), and (9) using MATLAB R2024a;

Step 6: For Equation (8), the optimal cycle duration for the total cost $\widetilde{TC_1}$ is t_1^*, and the optimal cost is $\widetilde{TC_1^*}(t_1^*)$. Similarly, for Equation (9), the optimal cycle duration for the total cost $\widetilde{TC_2}$ is t_2^*, and the corresponding optimal cost is $\widetilde{TC_2^*}(t_2^*)$.
Step 7: For Equation (3), the optimal cycle duration for the total cost TC_1 is t_1^*, and the optimal cost is $TC_1^*(t_1^*)$. Similarly, for Equation (4), the optimal cycle duration for the total cost TC_2 is t_2^*, and the corresponding optimal cost is $TC_2^*(t_2^*)$.
Step 8: End

3. NUMERICAL EXAMPLE WITH PRACTICAL IMPLICATION

The inventory model is demonstrated for a newly launched product whose demand depends on price, advertising efforts, reliability, and time. This situation commonly occurs in consumer electronics, fast-moving consumer goods, or durable goods, where marketing and product reliability strongly influence demand patterns. This context aligns well with the assumptions and demand structure used in the model.

This numerical example illustrates the inventory model developed using the generalized trapezoidal Neutrosophic methodology described in the preceding sections. Consider a company that has recently launched an item, where the demand rate is influenced by time, the impact of advertisements, price, and reliability. The demand rate is given $D(t,p,r,A) = R(m+bt)$, where $R = \eta e^{-p\delta} r^a A^v$ with the following values: $v = 0.7$, $\delta = 0.6$, $A = 50$ (number of advertisements), $\eta = 150$ (scaling factor), $a = 0.8$, $m = 250$, $r = 0.9$, $p = \$15$ per unit, $b = 0.5$, and the advertisement cost for advertisement $g = \$10$. The interest earned by the retailer over a one-year cycle time is 12% ($I_e = 0.12$), while the interest paid is 25% ($I_P = 0.25$). Additional parameters are $c = 0.8$, $T = 1$ year, $M = 0.167$ years (approximately 2 months), $\alpha = 0.1$, $\beta = 1$ and $h = 0.84$. The generalized trapezoidal Neutrosophic inventory cost parameters are

$$\widetilde{C_h} =< (0.5,1.5,2.5,3.5;0.75),(0.5,1.5,2.5,3.5;0.2),(1,2,3,3.5;0.38) >,$$
$$\widetilde{C_d} =< (13,15,17,19;0.8),(6,8,10,12;0.25),(11,13,15,17;0.4) >,$$
$$\widetilde{C_1} =< (12,14,16,18;0.8),(6,8,10,12;0.25),(10,13,14,17;0.3) >,$$
$$\widetilde{C_s} =< (12,14,18,20;0.7),(5,7,10,11;0.45),(12,14,16,18;0.3) >,$$
$$\text{and } \widetilde{A_0} =< (400,450,550,600;0.7),(350,475,525,650;0.2),(300,400,600,700;0.3) >.$$

Using Equation (7), the generalized trapezoidal Neutrosophic cost parameters are converted into de-Neutrosophic (crisp) values. The de-Neutrosophic values of the cost parameters are $\widetilde{A_0}$ = \$500 / order, $\widetilde{C_1}$ = \$12.5 / item, $\widetilde{C_h}$ = \$2.1/ unit / unit time, $\widetilde{C_d}$ = \$13 / unit / unit time, $\widetilde{C_s}$ = \$13.15 / unit time.

Table 2: Optimal average inventory cost under Neutrosophic environment

$T_1^* = t_1^*\ (years)$	$\widetilde{TC_1^*}(t_1^*)$ (\$)	$T_1^* = t_2^*\ (years)$	$\widetilde{TC_2^*}(t_2^*)$ (\$)
0.704028	1906.25	0.745896	1921.36

Table 3: Average optimal inventory cost for different payment delay periods

Permissible delay period (M)	$t_1^*\ (years)$	$\widetilde{TC_1^*}(t_1^*)$ (\$)	$t_2^*\ (years)$	$\widetilde{TC_2^*}(t_2^*)$ (\$)
2 months $(M = 0.167)$	0.704028	1906.25	0.745896	1921.36
4 months $(M = 0.333)$	0.738407	1889.79	0.759208	1909.02

Permissible delay period (M)	t_1^* $(years)$	$\widetilde{TC}_1^*(t_1^*)$ (\$)	t_2^* $(years)$	$\widetilde{TC}_2^*(t_2^*)$ (\$)
6 months ($M = 0.5$)	0.772456	1877.97	0.772581	1896.39
8 months ($M = 0.667$)	0.805961	1870.85	0.785935	1883.54
10 months ($M = 0.833$)	0.838724	1868.38	0.79919	1870.55
12 months ($M = 1$)	0.871138	1870.47	0.812507	1857.37

Table 2 presents the optimal inventory cost under a Neutrosophic environment for the given numerical example, while Table 3 shows the optimal total cost for different values of the delay period (*M*). At a 2-month delay period, the delay period M $(= 0.167)$ is less than both t_1^* and t_2^*, so Case I holds, and the optimal cost is $\widetilde{TC}_1^*(t_1^*)$. Similarly, for 4, 6, and 8 months, where M remains less than both t_1^* and t_2^*. Case-I continues to apply, with the optimal cost being $\widetilde{TC}_1^*(t_1^*)$. At a 10-month delay period, M is less than t_1^* but greater than t_2^*, so both Case-I and Case-II are valid. In this case, the optimal cost for Case I is $\widetilde{TC}_1^*(t_1^*)$, and for Case II, it is $\widetilde{TC}_2^*(t_2^*)$. Finally, at a 1-year delay period, where M exceeds both t_1^* and t_2^*, only Case II holds, and the optimal cost is $\widetilde{TC}_2^*(t_2^*)$. The retailer should account for these variations when making inventory and payment decisions to ensure cost optimization.

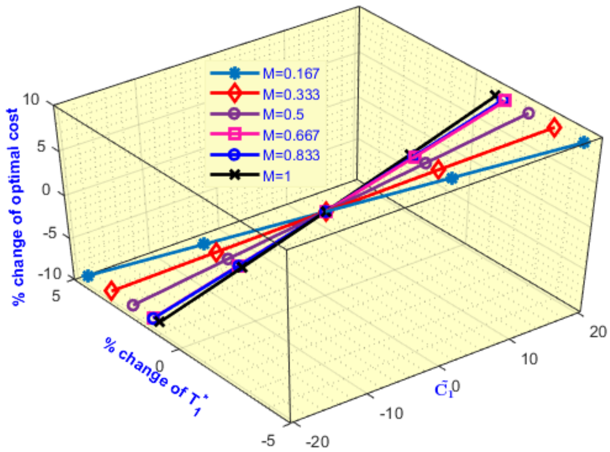
4. SENSITIVITY ANALYSIS AND MANAGERIAL IMPLICATIONS

By varying one parameter from -20% to 20% while holding the other parameters constant, we performed a sensitivity analysis on the parameters used in the fundamental structure of the proposed inventory system under GTrNN circumstances. The changes in optimal total costs with respect to all the key parameter variations are presented in Tables 4 and 5.

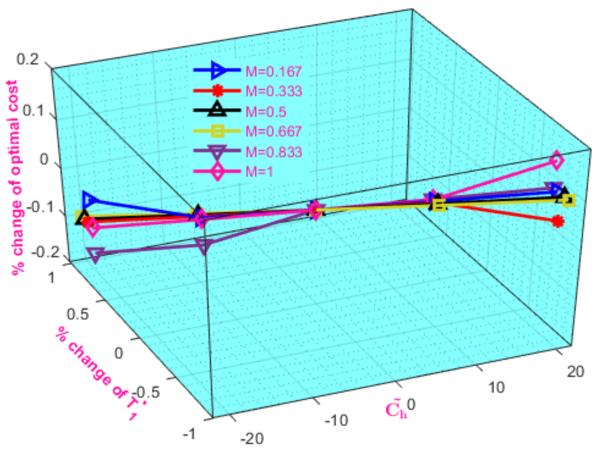
Table 4. Analysis of sensitivity to various inventory-related cost parameters

Parameter	% change	t_1^* $(years)$	$\widetilde{TC}_1^*(t_1^*)$ (\$)	t_2^* $(years)$	$\widetilde{TC}_2^*(t_2^*)$ (\$)	Conclusion	% change in cost
$\widetilde{C_1}$	-20	0.735444	1737.80	0.765926	1748.93	$t_1^* > M$	-8.84
	-10	0.719760	1822.15	0.755766	1835.23	$t_1^* > M$	-4.41
	10	0.688245	1990.08	0.736305	2007.37	$t_1^* > M$	4.40
	20	0.672409	2073.66	0.726982	2093.27	$t_1^* > M$	8.78
$\widetilde{C_h}$	-20	0.710331	1904.73	0.751856	1919.55	$t_1^* > M$	-0.08
	-10	0.707152	1905.49	0.748853	1920.46	$t_1^* > M$	-0.04
	10	0.700958	1906.99	0.742986	1922.24	$t_1^* > M$	0.04
	20	0.697940	1907.73	0.740119	1923.12	$t_1^* > M$	0.08
$\widetilde{C_s}$	-20	0.644532	1897.15	0.704252	1914.86	$t_1^* > M$	-0.48
	-10	0.677067	1902.11	0.726677	1918.35	$t_1^* > M$	-0.22
	10	0.726758	1909.74	0.762560	1923.96	$t_1^* > M$	0.18
	20	0.746194	1912.74	0.777151	1926.4	$t_1^* > M$	0.34
$\widetilde{A_0}$	-20	0.704028	1806.25	0.745876	1821.58	$t_1^* > M$	-5.25
	-10	0.704028	1856.25	0.745876	1871.36	$t_1^* > M$	-2.62
	10	0.704028	1956.25	0.745876	1971.36	$t_1^* > M$	2.62
	20	0.704028	2006.25	0.745876	2021.36	$t_1^* > M$	5.25
$\widetilde{C_d}$	-20	0.715828	1901.93	0.756412	517.544	$t_1^* > M$	-0.23
	-10	0.709881	1904.13	0.751119	517.550	$t_1^* > M$	-0.11
	10	0.698266	1908.35	0.6342	0.740743	$t_1^* > M$	0.11
	20	0.692592	1910.44	0.6334	0.735658	$t_1^* > M$	0.22

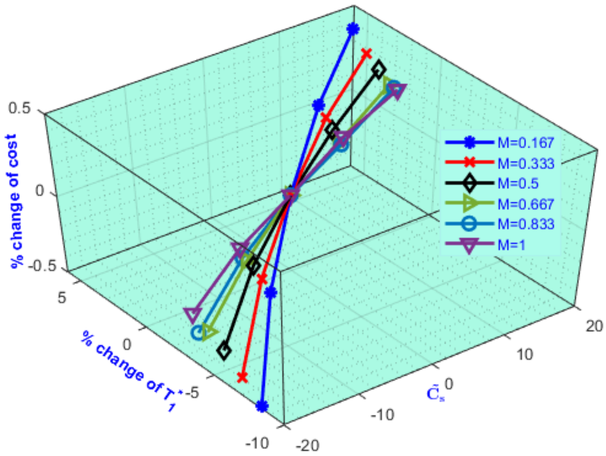
As shown in Table 4, the inventory system is moderately sensitive to the ordering cost ($\widetilde{A_0}$) and the purchasing cost ($\widetilde{C_1}$). If these costs increase, the total cost of the inventory system rises proportionally for various values of *M*. The increase in purchasing cost directly contributes to the rise in overall inventory cost, as illustrated in Figure 3(a). This highlights the importance of controlling ordering and purchasing costs to maintain cost efficiency. Managers should focus on negotiating favorable terms with suppliers, optimizing order quantities, and leveraging economies of scale to mitigate the impact of these costs.



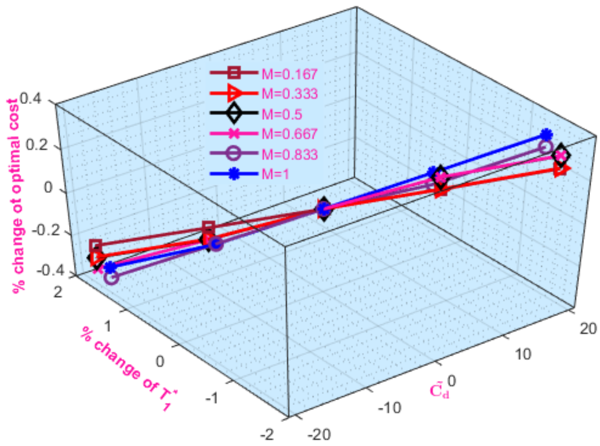
(a) Parameter $\widetilde{C_1}$



(b) Parameter $\widetilde{C_h}$



(c) Parameter $\widetilde{C_s}$



(d) Parameter $\widetilde{C_d}$

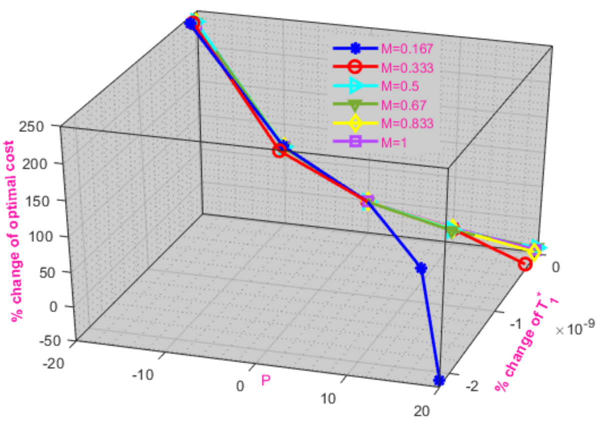
Figure 3. Graphical representation of the impact of varying cost parameters on optimal inventory cost across different credit periods

On the other hand, the system exhibits only slight sensitivity to other inventory cost parameters. Changes in these parameters, whether increases or decreases, do not significantly influence the overall cost of the inventory system, as illustrated in Figures 3(b), 3(c), and 3(d) for different values of *M*. This observation suggests that managerial resources should be focused on addressing highly sensitive parameters while keeping a watchful eye on less impactful ones to anticipate any unexpected variations. By prioritizing high-sensitivity factors, managers can allocate efforts more effectively to enhance cost efficiency and ensure the stability of the inventory system.

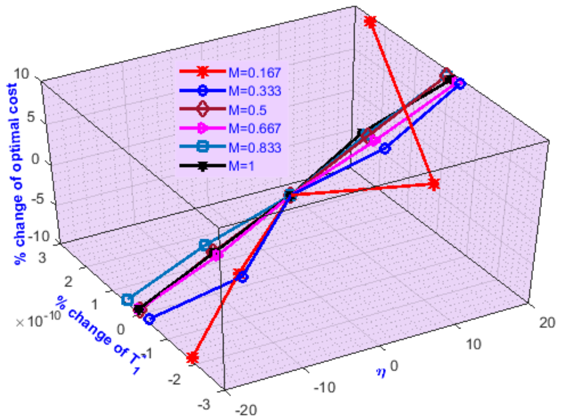
Table 5. Analysis of sensitivity to various inventory-related parameters

Parameter	% change	t_1^* $(years)$	$\widetilde{TC}_1^*(t_1^*)$ (\$)	t_2^* $(years)$	$\widetilde{TC}_2^*(t_2^*)$ (\$)	Conclusion	% change in cost
p	-20	0.704028	6482.47	0.745896	6573.87	$t_1^* > M$	240.06
	-10	0.704028	3229.01	0.745896	3266.17	$t_1^* > M$	69.39
	10	0.704028	1368.45	0.745896	1375.03	$t_1^* > M$	-28.21
	20	0.704028	1149.8	0.745896	1152.48	$t_1^* > M$	-39.68
r	-10	0.701325	1833.59	0.743334	1847.60	$t_1^* > M$	-3.72
	-5	0.702735	1870.11	0.744672	1884.67	$t_1^* > M$	-1.90
	5	0.705217	1942.01	0.747023	1957.67	$t_1^* > M$	1.88
	10	0.706316	1977.45	0.748062	1993.64	$t_1^* > M$	3.76
I_e	-20	0.690864	1911.05	0.755325	1918.27	$t_1^* > M$	0.25
	-10	0.697387	1908.67	0.750574	1919.83	$t_1^* > M$	0.13
	10	0.710790	1903.78	0.741291	1922.86	$t_1^* > M$	-0.13
	20	0.717676	1901.26	0.736757	1924.83	$t_1^* > M$	-0.26
ν	-20	0.704028	1524.07	0.745896	1532.81	$t_1^* > M$	-20.05
	-10	0.704028	1689.16	0.745896	1700.65	$t_1^* > M$	-11.39

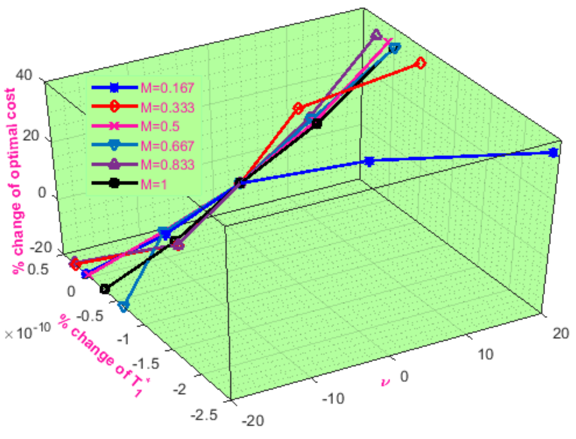
Parameter	% change	t_1^* $(years)$	$\widetilde{TC}_1^*(t_1^*)$ (\$)	t_2^* $(years)$	$\widetilde{TC}_2^*(t_2^*)$ (\$)	Conclusion	% change in cost
	10	0.704028	2191.72	0.745896	2211.59	$t_1^* > M$	14.98
	20	0.704028	2567.12	0.745896	2593.25	$t_1^* > M$	34.67
α	-20	0.728161	1897.53	0.76691	1911.72	$t_1^* > M$	-0.46
	-10	0.715889	1901.96	0.756261	1916.68	$t_1^* > M$	-0.23
	10	0.692556	1910.39	0.735806	1925.97	$t_1^* > M$	0.22
	20	0.681454	1914.40	0.725979	1930.47	$t_1^* > M$	0.43
m	-20	0.703684	1725.09	0.745593	1737.19	$t_1^* > M$	-9.5
	-10	0.703875	1815.67	0.745761	1829.17	$t_1^* > M$	-4.75
	10	0.704153	1996.83	0.746007	2013.44	$t_1^* > M$	4.75
	20	0.704258	2087.40	0.746099	2105.52	$t_1^* > M$	9.5
δ	-20	0.704028	6482.47	0.745896	6573.87	$t_1^* > M$	240
	-10	0.704028	3229.01	0.745896	3266.17	$t_1^* > M$	69.39
	10	0.704028	1368.45	0.745896	1374.60	$t_1^* > M$	-28.24
	20	0.704028	1149.80	0.745896	1152.30	$t_1^* > M$	-39.68
η	-20	0.704028	1725.00	0.745896	1737.08	$t_1^* > M$	-9.51
	-10	0.704028	1815.62	0.745896	1829.22	$t_1^* > M$	-4.75
	10	0.704028	1996.87	0.745896	2013.49	$t_1^* > M$	4.75
	20	0.704028	2087.50	0.745896	2015.63	$t_1^* > M$	9.51



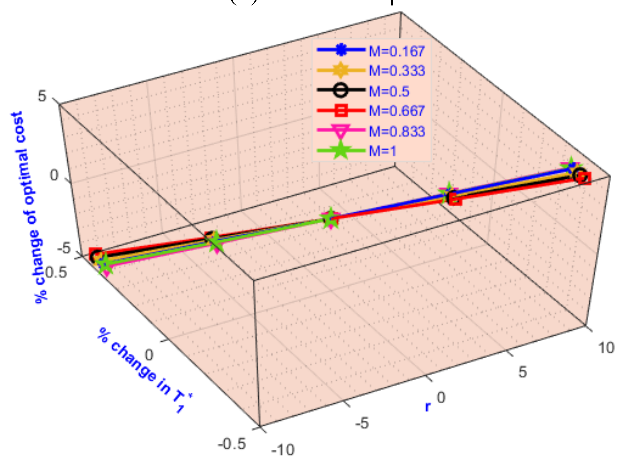
(a) Parameter p



(b) Parameter η



(c) Parameter ν



(d) Parameter r

Figure 4. Graphical representation of the impact of varying inventory-related parameters on optimal inventory cost under different credit periods

As shown in Table 5, the overall cost of the inventory system is significantly influenced by the selling price (p). An increase in the selling price leads to a substantial reduction in total inventory cost for various values of the credit period (M), as illustrated in Figure 4(a). However, this reduction in cost comes at the expense of decreased customer demand, necessitating a careful evaluation of pricing strategies. Managers must strike a balance between achieving cost efficiency and maintaining market competitiveness by analyzing customer price sensitivity. The price-sensitive parameter (δ) further impacts the inventory model, as variations in δ result in notable fluctuations in the overall cost. This underscores the need for managers to consider demand elasticity and adapt pricing strategies to optimize profitability while ensuring steady demand. The significant 240% change in total cost is primarily due to the nonlinear exponential impact of the selling price on demand, which directly influences the ordering quantity and inventory levels, thereby amplifying cost variations.

In contrast, the parameters m, η, v, and r exhibit moderate sensitivity to the optimal cost, as depicted in Figures 4(b), 4(c), and 4(d). These parameters reflect the demand's time dependency, price factor, reliability, and advertising impact, respectively, and should be carefully monitored to maintain cost stability. On the other hand, parameters such as α and I_e show only slight sensitivity, causing minor variations in total cost with changes. While these parameters may not require immediate adjustments, they should still be monitored periodically to prevent unforeseen inefficiencies. Managers should prioritize optimizing high-sensitivity parameters like the selling price and δ, while allocating sufficient resources to monitor moderate-sensitivity parameters like m, η, v, and r. By aligning pricing, advertising, reliability, and credit policies with demand dynamics, managers can effectively minimize costs while maintaining system stability and a competitive market position.

5. CONCLUSION

This study developed an inventory model that effectively incorporates time-dependent demand and holding costs, leveraging the two-parameter Weibull distribution to account for deterioration. The model is particularly suitable for newly introduced products, where advertising and reliability influence demand patterns. By integrating time and reliability-dependent holding costs, it provides a flexible framework to address diverse item characteristics. Additionally, price-dependent demand offers a strategic advantage for managing market competitiveness. The inclusion of delayed payment options adds practical utility by facilitating better financial collaboration between retailers and suppliers.

To address uncertainties in cost parameters, the model employs GTrNNs, enabling robust decision-making under imprecise conditions. This approach aids decision-makers in estimating reasonable costs in uncertain scenarios. The integration of time-varying demand makes the model particularly suitable for seasonal goods, while price dependence facilitates effective pricing strategies.

The current model is formulated as an unconstrained optimization problem for a single item with specific demand and cost structures, excluding real-world factors such as inventory capacity, service levels, and storage constraints. It does not address multi-item or multi-echelon supply chains. Future research can enhance the model's applicability by incorporating practical constraints, exploring uncertain cost parameters, and adapting it to complex scenarios, such as stochastic demand, varying market conditions, evolving customer preferences, and multi-echelon systems, thereby increasing its industrial relevance.

REFERENCES

Aarya, D. and Sharma, V. (2015). An Optimal Policy for Weibull Deteriorating Items with Power Demand Pattern and Permissible Delay on Payments. *Global Journal of Pure and Applied Mathematics*, 11(5):3275–3285.

Adak, S. and Mahapatra, G. S. (2021). Two-Echelon Imperfect Production Supply Chain with Probabilistic Deterioration Rework and Reliability Under Fuzziness. *Journal of Management Analytics*, 0(0):1–25.

Ahmed, M. A., Al-Khamis, T. A., and Benkherouf, L. (2013). Inventory Models with Ramp Type Demand Rate, Partial Backlogging and General Deterioration Rate. *Applied Mathematics and Computation*, 219(9):4288–4307.

Bappa Mondal, Arindam Garai, A. M. and Majumder, S. K. (2021). Inventory Policies for Seasonal Items with Logistic-Growth Demand Rate Under Fully Permissible Delay in Payment: A Neutrosophic Optimization Approach. *Soft Computing*, 25:3725–3750.

Bardhan, S., Pal, H., and Giri, B. C. (2019). Optimal Replenishment Policy and Preservation Technology Investment for A Non-Instantaneous Deteriorating Item with Stock-Dependent Demand. *Operational Research*, 19(2):347–368.

Bhavani, G. D., Georgise, F. B., Mahapatra, G. S., and Maneckshaw, B. (2022). Neutrosophic Cost Pattern of Inventory System with Novel Demand Incorporating Deterioration and Discount on Defective Items Using Particle Swarm Algorithm. *Computational Intelligence and Neuroscience*, 2022.

Bhavani, G. D. and Mahapatra, G. S. (2023). Inventory System with Generalized Triangular Neutrosophic Cost Pattern Incorporating Maximum Life-Time-Based Deterioration and Novel Demand Through Pso. *Soft Computing*, 27(5):2385–2402.

Bhunia, A. K., Shaikh, A. A., and Cárdenas-Barrón, L. E. (2017). A Partially Integrated Productioninventory Model with Interval Valued Inventory Costs, Variable Demand, and Flexible Reliability. *Applied Soft Computing*, 55:491–502.

Chakraborty, D., Jana, D. K., and Roy, T. K. (2018). Two-Warehouse Partial Backlogging Inventory Model with Ramp Type Demand Rate, Three-Parameter Weibull Distribution Deterioration Under Inflation and Permissible Delay in Payments. *Computers & Industrial Engineering*, 123:157-179.

Chanda, U. and Kumar, A. (2017). Optimisation of Fuzzy EOQ Model for Advertising and Price Sensitive Demand Model Under Dynamic Ceiling on Potential Adoption. *International Journal of Systems Science: Operations & Logistics*, 4(2):145–165.

Chang, C. T., Cheng, M. C., and Soong, P. Y. (2016). Impacts of Inspection Errors and Trade Credits on The Economic Order Quantity Model for Items with Imperfect Quality. *International Journal of Systems Science: Operations and Logistics*, 3(1):34–48.

Chang, C. T., Soong, P. Y., and Cheng, M. C. (2017). The Influences of Defective Items and Trade Credits on Replenishment Decision. *International Journal of Information and Management Sciences*, 28(2):113–132.

Chou, W. S. and Julian, P. (2015). Inventory System of Deteriorating Items with Exponential Declining Demand and Partial Backlogging. *Journal of Interdisciplinary Mathematics*, 18:89–96.

Chuang, K. W., Lin, C. N., and Lan, C. H. (2013). Order Policy Analysis for Deteriorating Inventory Model with Trapezoidal Type Demand Rate. *Journal of Networks*, 8(8):1838–1844.

Chung, K. J., Cárdenas-Barrón, L. E., and Ting, P. S. (2014). An Inventory Model with Noninstantaneous Receipt and Exponentially Deteriorating Items for An Integrated Three Layer Supply Chain System Under Two Levels of Trade Credit. *International Journal of Production Economics*, 155:310–317.

Cárdenas-Barrón, L. E., Shaikh, A. A., Tiwari, S., and Treviño-Garza, G. (2020). An EOQ Inventory Model with Nonlinear Stock Dependent Holding Cost, Nonlinear Stock Dependent Demand, and Trade Credit. *Computers & Industrial Engineering*, 139:105557.

De, S. K. and Goswami, A. (2006). An EOQ Model with Fuzzy Inflation Rate and Fuzzy Deterioration Rate When A Delay in Payment Is Permissible. *International Journal of Systems Science*, 37(5):323–335.

Debnath, B. K., Majumder, P., Bera, U. K., and Maiti, M. (2018). Inventory Model with Demand As a Type-2 Fuzzy Number: A Fuzzy Differential Equation Approach. *Iranian Journal of Fuzzy Systems*, 15(1):1–24.

Hou, K. L. (2006). An Inventory Model for Deteriorating Items with Stock-Dependent Consumption Rate and Shortages Under Inflation and Time Discounting. *European Journal of Operational Research*,

168(2):463 – 474.

Hung, K. C. (2011). An Inventory Model with Generalized Type Demand, Deterioration and Backorder Rates. *European Journal of Operational Research*, 208(3):239 – 242.

Jaggi, C. K., Khanna, A., and Verma, P. (2011). Two-Warehouse Partial Backlogging Inventory Model for Deteriorating Items with Linear Trend in Demand Under Inflationary Conditions. *International Journal of Systems Science*, 42(7):1185–1196.

Jaggi, P. D. C., Pareek, S., Goel, S., and Singhal, N. (2015). An Inventory Model for Deteriorating Items with Ramp Type Demand Under Fuzzy Environment. *Int. J. Logistics Systems and Management*, 22:436–463.

Konstantaras, I. and Skouri, K. (2011). A Note on A Production-Inventory Model Under Stock-Dependent Demand, Weibull Distribution Deterioration, and Shortage. *International Transactions in Operational Research*, 18(4):527–531.

Kumari, M. and Pakkala, T. P. M. (2016). Inventory Policy for Deteriorating Items Under Trade Credit When Time of Payment Is Uncertain. *OPSEARCH*, 53(1):178–196.

Kundu, A., Guchhait, P., Panigrahi, G., and Maiti, M. (2019). An EOQ Model for Deteriorating Item with Promotional Effort and Credit-Linked Demand. *European Journal of Industrial Engineering*, 13(3):368–399.

Liao, J. J., Huang, K. N., Chung, K. J., Lin, S. D., Ting, P. S., and Srivastava, H. M. (2018). Retailer's Optimal Ordering Policy in The EOQ Model with Imperfect-Quality Items Under Limited Storage Capacity and Permissible Delay. *Mathematical Methods in The Applied Sciences*, 41(17):7624–7640.

Liao, J. J., Huang, K. N., Chung, K. J., Ting, P. S., Lin, S. D., and Srivastava, H. M. (2016). Some Mathematical Analytic Arguments for Determining Valid Optimal Lot Size for Deteriorating Items with Limited Storage Capacity Under Permissible Delay in Payments. *Applied Mathematics and Information Sciences*, 10(3):915–925.

Lin, J., Chao, H. C. J., and Julian, P. (2012). Improved Solution Process for Inventory Model with Ramp-Type Demand Under Stock-Dependent Consumption Rate. *Journal of The Chinese Institute of Industrial Engineers*, 29(4):219–225.

Lin, Y. S. and Wang, K. J. (2018). A Two-Stage Stochastic Optimization Model for Warehouse Configuration and Inventory Policy of Deteriorating Items. *Computers and Industrial Engineering*, 120:83–93.

Mahata, P., Mahata, G. C., and De, S. K. (2020). An Economic Order Quantity Model Under Two-Level Partial Trade Credit for Time-Varying Deteriorating Items. *International Journal of Systems Science: Operations and Logistics*, 7(1):1–17.

Mahata, P., Mahata, G. C., and Mukherjee, A. (2019). An Ordering Policy for Deteriorating Items with Price-Dependent Iso-Elastic Demand Under Permissible Delay in Payments and Price Inflation. *Mathematical and Computer Modelling of Dynamical Systems*, 25(6):575–601.

Majumder, P., Bera, U. K., and Maiti, M. (2016). An EPQ Model for Two-Warehouse in Unremitting Release Pattern with Two-Level Trade Credit Period Concerning Both Supplier and Retailer. *Applied Mathematics and Computation*, 274:430–458.

Manna, S. K. and Chaudhuri, K. S. (2006). An EOQ Model with Ramp Type Demand Rate, Time Dependent Deterioration Rate, Unit Production Cost, and Shortages. *European Journal of Operational Research*, 171(2):557 – 566.

Mishra, U., Wu, J.-Z., and Tseng, M.-L. (2019). Effects of A Hybrid-Price-Stock Dependent Demand on The Optimal Solutions of A Deteriorating Inventory System and Trade Credit Policy on Re-Manufactured Product. *Journal of Cleaner Production*, 241:118282.

Muhlemann, A. P. and Valtis-Spanopoulos, N. P. (1980). A Variable Holding Cost Rate EOQ Model. *European Journal of Operational Research*, 4(2):132 – 135.

Pal, S. and Mahapatra, G. S. (2017). A Manufacturing-Oriented Supply Chain Model for Imperfect Quality with Inspection Errors, Stochastic Demand Under Rework and Shortages. *Computers & Industrial Engineering*, 106:299 – 314.

Pal, S., Mahapatra, G. S., and Samanta, G. P. (2014). An EPQ Model of Ramp Type Demand with Weibull Deterioration Under Inflation and Finite Horizon in Crisp and Fuzzy Environment. *International Journal of Production Economics*, 156:159 – 166.

Pervin, M., Roy, S. K., and Weber, G. W. (2017). A Two-Echelon Inventory Model with Stock Dependent Demand and Variable Holding Cost for Deteriorating Items. *Numerical Algebra, Control & Optimization*, 7(1):21.

Pervin, M., Roy, S. K., and Weber, G. W. (2019). Multi-Item Deteriorating Two-Echelon Inventory Model with Price- and Stock-Dependent Demand: A Trade-Credit Policy. *Journal of Industrial &*

Management Optimization, 15(3):1345–1373.

Pervin, M., Roy, S. K., and Weber, G. W. (2020). Deteriorating Inventory with Preservation Technology Under Price- and Stock-Sensitive Demand. *Journal of Industrial & Management Optimization*, 16(4):1585–1612.

Rezezedeh, B., Mirzazadeh, A., and Mohammadi, M. (2016). An EPQ Model with Reliability in An Imperfect Production System At The Presence of Two Sale Policies of Prepayment and Delay in Payment. *International Journal of Operational Research*, 27(4):530–551.

Roubens, M. (1990). Inequality Constraints Between Fuzzy Numbers and Their Use in Mathematical Programming, Stochastic Versus Fuzzy Approaches to Multi-Objective Mathematical Programming Under Uncertainty, Eds. R. Slowinski and J. Teghem. *Dordrecht: Kluwer Academic Publishers*, Page 321–330.

San-José, L. A., Sicilia, J., La Rosa, M. G.-D., and Febles-Acosta, J. (2017). Optimal Inventory Policy Under Power Demand Pattern and Partial Backlogging. *Applied Mathematical Modelling*, 46:618–630.

Sanni, S. S. and Chukwu, W. I. E. (2016). An Inventory Model with Three-Parameter Weibull Deterioration, Quadratic Demand Rate and Shortages. *American Journal of Mathematical and Management Sciences*, 35(2):159–170.

Sarkar, B. and Sarkar, S. (2013). Variable Deterioration and Demand—An Inventory Model. *Economic Modelling*, 31:548 556.

Seifert, D., Seifert, R. W., and Isaksson, O. H. D. (2017). A Test of Inventory Models with Permissible Delay in Payment. *International Journal of Production Research*, 55(4):1117–1128.

Sen, N. and Saha, S. (2018). An Inventory Model for Deteriorating Items with Time-Dependent Holding Cost and Shortages Under Permissible Delay in Payment. *International Journal of Procurement Management*, 11(4):518–531.

Shabani, S., Mirzazadeh, A., and Sharifi, E. (2016a). A Two-Warehouse Inventory Model with Fuzzy Deterioration Rate and Fuzzy Demand Rate Under Conditionally Permissible Delay in Payment. *Journal of Industrial and Production Engineering*, 33(2):134–142.

Shabani, S., Mirzazadeh, A., and Sharifi, E. (2016b). A Two-Warehouse Inventory Model with Fuzzy Deterioration Rate and Fuzzy Demand Rate Under Conditionally Permissible Delay in Payment. *Journal of Industrial and Production Engineering*, 33(2):134–142.

Shaikh, A. A., Bhunia, A. K., Cárdenas-Barrón, L. E., Sahoo, L., and , S. (2018a). A Fuzzy Inventory Model for A Deteriorating Item with Variable Demand, Permissible Delay in Payments, and Partial Backlogging with Shortage Follows Inventory (SFI) Policy. *International Journal of Fuzzy Systems*, 20(5):1606–1623.

Shaikh, A. A., Cárdenas-Barrón, L. E., and , S. (2018b). Some Observations On: Improving Production Policy for A Deteriorating Item Under Permissible Delay in Payments with Stock-Dependent Demand Rate. *International Journal of Applied & Computational Mathematics*, 4(1).

Shi, Y., Zhang, Z., Zhou, F., and Shi, Y. (2019). Optimal Ordering Policies for A Single Deteriorating Item with Ramp-Type Demand Rate Under Permissible Delay in Payments. *Journal of The Operational Research Society*, 70(10):1848–1868.

Shukla, H. S., Tripathi, R. P., Yadav, S. K., and Shukla, V. (2015). Inventory Model for Deteriorating Items with Quadratic Time-Dependent Demand Rate and Composed Shortages. *Journal of Applied Probability and Statistics*, 10(2):135–147.

Skouri, K., Konstantaras, I., Papachristos, S., and Ganas, I. (2009). Inventory Models with Ramp Type Demand Rate, Partial Backlogging and Weibull Deterioration Rate. *European Journal of Operational Research*, 192(1):79–92.

Tan, Y. and Weng, M. X. (2012). A Discrete-In-Time Inventory Model with Deterioration and Backlog. *International Journal of Services Operations and Informatics*, 7(1):37–51.

Vahdani, B., Sadigh Behzadi, S., and Allahviranloo, T. (2016). Optimal Multi-Product Dynamic Pricing and Inventory Policies for Coordinating Production and Marketing Under Fuzzy Environment: Fuzzy Expansion Methods. *Scientia Iranica*, 23(5):2297–2317.

Vandana and Sharma, B. K. (2016). An Inventory Model for Non-Instantaneous Deteriorating Items with Quadratic Demand Rate and Shortages Under Trade Credit Policy. *Journal of Applied Analysis and Computation*, 6(3):720–737.

Wang, K. J., Lin, Y. S., and Yu, J. C. P. (2011). Optimizing Inventory Policy for Products with Time-Sensitive Deteriorating Rates in A Multi-Echelon Supply Chain. *International Journal of Production Economics*, 130(1):66–76.

Wu, C. F. and Zhao, Q. H. (2015). An Inventory Model for Deteriorating Items with Inventory-Dependent and Linear Trend Demand Under Trade Credit. *Scientia Iranica*, 22(6):2558–2570.