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Asymptotic Stability of Linear Neutral Differential Equations with only Memory-type Neutral Terms

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Abstract

In this paper, we investigate the asymptotic stability of linear neutral differential equations featuring only a memory-type neutral term, without additional internal or boundary damping. By combining the properties of a generalized positive definite kernel (abbreviated to GPDK) with the classical multiplier method, an auxiliary system is constructed to estimate the kinetic energy, potential energy, and convolution terms in the energy $E(t)$, thus proving that the energy function is integrable and decays to zero at least by the rate of $\frac{1}{t+1}$. The ideas presented in this paper can be applied to other neutral differential problems and improve the related decay estimates.

Keywords: neutral differential equations, generalized positive definite kernel, memory-type neutral term, stability estimate

1. Introduction

In this paper, we are concerned with the asymptotic stability of linear neutral differential equations with a memory-type neutral term (see the following equation (1.1)):

$$\begin{cases} \left(u_t+\int_0^t \alpha(t-s)u_t(s)ds\right)_t-\Delta u=0, & in \quad \Omega\times(0,+\infty), \\ u=0, & on \quad \Gamma\times(0,+\infty), \\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), & x\in\times\Omega. \end{cases} \tag{1.1}$$

Let Ω be an open bounded set of $R^n(n\geq 1)$ with a smooth boundary Γ. The kernel function $\alpha(t)$ is a positive non-increasing function, u_0 and u_1 are given initial values. Throughout the literature, C stands for some positive constants and may be different line to line.

We review existing literature on neutral differential equations. In [1], Kun-Peng Jin et al. investigated the uniform polynomial stability of second-order integro-differential equations in Hilbert spaces with positive definite kernels：

$$\begin{cases} u_{tt}(t)+Au(t)-\int_0^t g(t-s)Au(s)ds=f(u(t)), \\ u(0)=u_0, u_t(0)=u_1. \end{cases}$$

they introduced the concept of Generalized Positive Definite Kernel (GPDK), and utilizing GPDK and its properties, they established an efficient criterion regarding the polynomial stability of evolution equations that incorporate this general, more complex, and valuable memory.

In [2], Kun-Peng Jin et al. discussed the neutral viscoelastic system with a boundary memory damping:

$$\begin{cases} \left(u_t+\int_0^t \alpha(t-s)u_t(s)ds\right)_t-\Delta u=0, & (x,t)\in\Omega\times(0,+\infty), \\ u=0, & (x,t)\in\Gamma_1\times(0,+\infty), \\ \dfrac{\partial u}{\partial \upsilon}=-h(u_t)-\sigma u+\int_0^t \beta(t-s)u(s)ds, & (x,t)\in\Gamma_2\times(0,+\infty). \end{cases}$$

They proposed a novel analytical method to estimate the lower-order terms. By constructing auxiliary systems and employing the Sobolev Embedding theory, they derived a general decay result for the system energy $E(t)$ of the neutral viscoelastic equation. This indicates that $E(t)$ is controlled by the solution of the relevant ordinary differential equation. These results are much weaker under the condition of memory damping than those in the previous literature.

In 2021, Chan Li et al. [3] adopted the spectral method to prove that the energy decay of a linear Wentzell system is polynomial, and obtained an ideal estimate of the resolvent of the generator of the system along the imaginary axis. This enables proving that the energy of the system decays polynomially. This provides new insights into the boundary damping problem of wave equations. In 2024, Dandan Guo et al. [4] investigated a viscoelastic wave equation involving a logarithmic nonlinear source and a dynamic Wentzell boundary condition:

$$\begin{cases} u_{tt}-\Delta u+\int_0^t h(t-s)\Delta u(s)ds=|u|^{\gamma-2}\,uln|u|, & in\ \ \Omega\times(0,\infty), \\ u=0, & on\ \ \Gamma_0\times(0,\infty), \\ u_{tt}-\Delta_\Gamma u+\partial_\nu u-\int_0^t h(t-s)\partial_\nu u(s)ds=0, & on\ \ \Gamma_1\times(0,\infty), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & in\ \ \Omega. \end{cases}$$

With some assumptions made on the memory kernel function and by means of convex function theory and the Lyapunov method, they constructed the general decay estimate of the solutions, which provides a new idea for studying the wave equation by combining the memory kernel function, the logarithmic nonlinear source, and the dynamic Wentzell boundary conditions.

In [5], Chan Li et al. discussed the following equation (1.2). By combining the properties of positive definite kernels with the classical multiplier method to construct an auxiliary system[6], they finally established the long-time behaviors of wave equations stabilized by boundary memory damping and friction damping.

$$\begin{cases} \phi_{tt}(t,x)-\Delta\phi(t,x)=0, & (x,\mathrm{t})\in\Omega\times(0,+\infty), \\ \phi(t,x)=0, & (x,\mathrm{t})\in\Gamma_0\times(0,+\infty), \\ \partial_\nu\phi(t,x)-k*\phi(t,x)+\phi_t(t,x)+\phi(t,x)=0, & (x,\mathrm{t})\in\Gamma_1\times(0,+\infty). \end{cases} \tag{1.2}$$

On this basis, we relax the constraints on the memory kernel to allow it to be oscillatory, sign-changing, and non-smooth. By combining the properties of the GPDK with the classical multiplier method, an auxiliary system is constructed(for details, please refer to the literature [7-10]). Under weaker conditions, the asymptotic stability of the linear neutral integro-differential equation with a memory term is obtained.

This paper is structured into four distinct sections. The initial two sections are dedicated to the introduction and preparatory work; the latter two sections present the principal results and their corresponding proofs.

2. Preliminaries

This part is mainly the preparation for the work of this paper. Introduce some definitions, and assumptions that will be used in this paper.

We use $\langle\cdot,\cdot\rangle$ to denote inner products in $L^2(\Omega)$, and $\|\cdot\|$ to denote the corresponding norms:

$$\langle u,v\rangle=\int_\Omega u(x)v(x)dx,$$

$$\|u\|_{L^2(\Omega)}=\sqrt{\langle u,u\rangle}=\sqrt{\int_\Omega |u(x)|^2\,dx}.$$

Definition 2.1.[1] Assuming that h is a locally integrable function $L^1_{loc}(0,+\infty)$, and φ is a locally bounded and positive function $L^\infty_{loc}(0,+\infty)$ with $\varphi(t)>0$ for all $t>0$. If for all $t>0$ and any $u\in L^2_{loc}(0,+\infty;H)$, the following holds:

$$\int_0^t \varphi(s)\langle h*u(s),u(s)\rangle ds\ge 0,\qquad \forall\, t\ge 0,$$

the function h is to be a φ-positive definite kernel. Moreover, h is said to be a strongly φ-positive definite kernel if there exists two constants $\delta,N>0$ such that $h(t)-\delta e^{-Nt}$ is φ-positive definite.

Proposition 2.2.[1] Let h be a strongly positive definite kernel, then

$$\int_0^t \varphi(s)\|u(s)\|^2 ds\le \|u(0)\|^2+\frac{2N}{\delta}\int_0^t\langle h*u(s),u(s)\rangle ds+\frac{4}{N\delta}\int_0^t\langle h*u_t(s),u_t(s)\rangle ds,$$

for any $t\ge 0$, and $u\in L^2_{loc}(0,+\infty;H)$, where δ is the constant in **Definition2.1**.

Proposition 2.3.[6] Let $h\in([0,+\infty])$ be a positive definite kernel. Then $h(0)>0$ and

$$\|h*y(t)\|^2\le 2h(0)\int_0^t\langle h*y(\tau),y(\tau)\rangle d\tau,\qquad t\ge 0,\qquad h(0)\ge 0$$

for any $y\in L^1_{loc}(0,\infty;X)$.

(A) The kernel α: Let α be a given kernel function, α is a non-increasing continuous differentiable function satisfying:

$$\alpha(0)>0,\qquad \alpha'(t)\le 0.$$

We define the energy functional corresponding to (1.1) as

$$E(t)=\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|^2.$$

Theorem 2.4. Assuming $u_0,u_1\in H^1_0(\Omega)$, and the condition (A) holds. Then (1.1) has a unique global solution.

There exists a positive constant C independent of t such that

$$\int_0^{+\infty} E(t)dt \le CE(0),$$

and

$$E(t) \le CE(0)(t+1)^{-1}.$$

3. Main Results and Their Proofs

Lemma 3.1. Assuming $u_0, u_1 \in H_0^1(\Omega)$, and assuming that (A) holds. Then (1.1) has a unique global solution. Moreover, we have

$$E(t) \le CE(0), \qquad t \ge 0, \tag{3.1}$$

and for any $T > 0$,

$$\int_0^T \left\langle \int_0^t A(t-s)u_{tt}(s)ds, u_{tt} \right\rangle dt \le C_1 E(0). \tag{3.2}$$

Proof. It is known that the local solution can be prolonged to $[0,+\infty)$, because of the linearity of the equations. Then for any $u_0, u_1 \in H_0^1(\Omega)$, (1.1) has a unique solution on $[0,+\infty)$.

Now, let us prove (3.1), which implies $\|u_t\|$ and $\|\nabla u\|$ are bounded.

Denote $A(t) = \int_t^{+\infty} \alpha(s)ds,$ then $A'(t) = -\alpha(t)$. We have

$$\begin{aligned}\int_0^t \alpha(t-s)u_t(s)ds &= -\int_0^t A'(t-s)u_t(s)ds \\ &= \int_0^t A_s(t-s)u_t(s)ds \\ &= u_t(s)A(t-s)\Big|_0^t - \int_0^t A(t-s)u_t(s)ds \\ &= A(0)u_t - A(t)u_t(0) - \int_0^t A(t-s)u_{tt}(s)ds.\end{aligned}$$

Then the system can be transformed into

$$\left(u_t + A(0)u_t - A(t)u_t(0) - \int_0^t A(t-s)u_{tt}(s)ds\right)_t - \Delta u = 0.$$

Since initial condition, yield

$$\big(1 + A(0)\big)u_{tt} + \alpha(t)u_1 - \left(\int_0^t A(t-s)u_{tt}(s)ds\right)_t - \Delta u = 0. \tag{3.3}$$

Multiplying (3.3) by u_t and taking the inner product, we easily have

$$\langle (1+A(0))u_{tt}, u_t\rangle + \langle \alpha(t)u_1, u_t\rangle - \left\langle \left(\int_0^t A(t-s)u''(s)ds\right)_t, u_t \right\rangle - \langle \Delta u, u_t\rangle = 0. \tag{3.4}$$

Since

$$\left\langle \left(\int_0^t A(t-s)u''(s)ds\right)_t, u_t \right\rangle = \left\langle \int_0^t A(t-s)u''(s)ds, u_t \right\rangle_t - \left\langle \int_0^t A(t-s)u''(s)ds, u_{tt} \right\rangle.$$

Thus, (3.4) becomes

$$\begin{aligned}&\langle (1+A(0)u_{tt},u_t\rangle+\langle \alpha(t)u_t(0),u_t\rangle-\left\langle \int_0^t A(t-s)u''(s)ds),u_t\right\rangle_t\\&+\left\langle \int_0^t A(t-s)u''(s)ds,u_{tt}\right\rangle-\langle \Delta u,u_t\rangle=0.\end{aligned}\tag{3.5}$$

Then, integrating (3.5) over the time variable t on the interval $[0,T]$ for any $T>0$, we obtain

$$\begin{aligned}&\int_0^T\langle (1+A(0)u_{tt},u_t\rangle dt+\int_0^T\langle \alpha(t)u_t(0),u_t\rangle dt-\int_0^T\left\langle \int_0^t A(t-s)u_{tt}(s)ds,u_t\right\rangle_t dt\\&+\int_0^T\left\langle \int_0^t A(t-s)u_{tt}(s)ds,u_{tt}\right\rangle dt-\int_0^T\langle \Delta u,u_t\rangle dt=0.\end{aligned}\tag{3.6}$$

Using the integration by parts, the first term of (3.6) become

$$\int_0^T\langle (1+A(0)u_{tt},u_t\rangle dt=\left\langle \left(1+A(0)\right)u_t,u_t\right\rangle\Big|_0^T-\int_0^T\langle (1+A(0))u_t,u_{tt}\rangle dt,$$

hence, we get

$$\int_0^T\langle (1+A(0))u_{tt},u_t\rangle dt=\frac{1}{2}(1+A(0))\|u_t(T)\|^2-\frac{1}{2}(1+A(0))\|u_1\|^2.$$

Similarly, we then estimate the last term in (3.6) as follows:

$$-\int_0^T\langle \Delta u,u_t\rangle dt=\frac{1}{2}\|\nabla u(T)\|^2-\frac{1}{2}\|\nabla u_0\|^2.$$

Substituting the above two equations into (3.6), we get

$$\begin{aligned}&\frac{1+A(0)}{2}\|u_t(T)\|^2+\frac{1}{2}\|\nabla u(T)\|^2+\int_0^T\left\langle \int_0^t A(t-s)u_{tt}(s)ds,u_{tt}\right\rangle dt\\&=\frac{1+A(0)}{2}\|u_1\|^2+\frac{1}{2}\|\nabla u_0\|^2-\int_0^T\langle \alpha(t)u_1,u_t\rangle dt+\left\langle \int_0^T A(t-s)u_{tt}(s)ds,u_t(T)\right\rangle,\end{aligned}\tag{3.7}$$

where, using Proposition 2.3 and Young inequality, we deduce for any $\mu>0$

$$\begin{aligned}\left\langle \int_0^t A(t-s)u_{tt}(s)ds,u_t\right\rangle&\le\frac{1}{4\mu}\left\|\int_0^t A(t-s)u_{tt}(s)ds\right\|^2+\mu\|u_t(T)\|^2\\&\le\frac{A(0)}{2\mu}\int_0^T\left\langle \int_0^t A(t-s)u_{tt}(s)ds,u_{tt}\right\rangle dt+\mu\|u_t(T)\|^2,\end{aligned}\tag{3.8}$$

and

$$-\int_0^T\langle \alpha(t)u_1,u_t\rangle dt\le A(0)\left\|u_1\right\|^2+\frac{1}{4}\int_0^T\alpha(t)\left\|u_t\right\|^2dt.\tag{3.9}$$

according to the integrability of $\alpha(t)$.

Then substituting (3.8)-(3.9), into (3.7) and taking $\mu=\frac{1+2A(0)}{4}$ yield that for any $T>0$,

$$\frac{1}{4}\| u_t(T)\|^2 + \frac{1}{1+2A(0)}\int_0^T \left\langle \int_0^t A(t-s)u_{tt}(s)ds, u_{tt}\right\rangle dt + \frac{1}{2}\| \nabla u(T)\|^2 \\ \leq \frac{1+5A(0)}{4}\| u_1\|^2 + \frac{1}{2}\| \nabla u_0\|^2 + \frac{1}{4}\int_0^T \alpha(t)\|u_t\|^2 dt. \tag{3.10}$$

(3.10) implies

$$\| u_t(T)\|^2 \leq (1+5A(0))\| u_1\|^2 + 2\| \nabla u_0\|^2 + \int_0^T \alpha(t)\|u_t\|^2 dt,$$

then, an application of the Gronwall inequality gives that, for $T > 0$,

$$\| u_t(T)\|^2 \leq \left((1+5A(0))\| u_1\|^2 + 2\| \nabla u_0\|^2\right)e^{A(0)}.$$

Thus, we know $\| u_t\|^2$ is bounded and there is a constant $C_2 > 0$ such that

$$\| u_t\|^2 \leq C_2 E(0).$$

This with (3.10) will lead us to obtain (3.1) and (3.2) according to the integrability of $\alpha(t)$. This completes the proof.

Lemma 3.2. Set (A) holds and $u_0, u_1 \in H_0^1(\Omega)$. Then

$$\int_0^{+\infty} \| u_t\|^2 dt \leq CE(0).$$

Proof. Notice Proposition 2.2, for any $T > 0$

$$\int_0^T \| u_t\|^2 dt \leq \| u_1\|^2 + \frac{2N}{\delta}\int_0^T \left\langle \int_0^t A(t-s)u_t(s)ds, u_t\right\rangle dt \\ + \frac{4}{N\delta}\int_0^T \left\langle \int_0^t A(t-s)u_{tt}(s)ds, u_{tt}\right\rangle dt.$$

We have obtained the estimate of

$$\int_0^T \left\langle \int_0^t A(t-s)u_{tt}(s)ds, u_{tt}\right\rangle dt$$

in (3.2). Now let us estimate

$$\int_0^T \left\langle \int_0^t A(t-s)u_t(s)ds, u_t\right\rangle dt.$$

The system (1.1) can be rewritten as

$$u_{tt} + \alpha(0)u_t(t) + \int_0^t \alpha'(t-s)u_t(s)ds - \Delta u = 0.$$

Integrating the above equation over the time variable t on the interval $[0,T]$, we obtain

$$u_t \Big|_0^T + \alpha(0)u \Big|_0^T + \int_0^T \int_0^t \alpha'(t-s)u_t(s)dsdt - \int_0^T \Delta u dt = 0, \tag{3.11}$$

Where

$$\begin{aligned}\int_0^T\int_0^t \alpha'(t-s)u_t(s)dsdt &= \int_0^T\int_s^T \alpha'(t-s)u_t(s)dtds\\ &= \int_0^T u_t(s)\int_s^T \alpha'(t-s)dtds\\ &= \int_0^T \alpha(T-s)u_t(s)ds - \int_0^T \alpha(0)u_t(s)ds.\end{aligned}$$

Then, (3.11) transformed into

$$\begin{aligned}&u_t(T)-u_1+\alpha(0)(u(T)-u_0)+\int_0^T \alpha(T-s)u_t(s)ds\\ &-\alpha(0)\int_0^T u_t(s)ds-\int_0^T \Delta u dt = 0.\end{aligned} \tag{3.12}$$

Denote $U=\int_0^t u(s)ds$, then it is easy to verify that

$$U_t = u, \qquad U_t(0)=u_0$$

From this, (3.12) can be transformed into

$$U_{tt}+\int_0^T \alpha(T-s)U_{tt}(s)ds-\Delta U = u_1. \tag{3.13}$$

Furthermore,

$$\begin{aligned}\left(\int_0^t \alpha(t-s)U_t(s)ds\right)_t &= \alpha(0)U_t(t)+\int_0^t \alpha'(t-s)U_t(s)ds\\ &= \alpha(0)U_t(t)-\int_0^t \alpha_s(t-s)U_t(s)ds\\ &= \alpha(0)U_t(t)-\alpha(t-s)U_t(s)\Big|_0^t+\int_0^t \alpha(t-s)U_{tt}(s)ds\\ &= \alpha(t)U_t(0)+\int_0^t \alpha(t-s)U_{tt}(s)ds,\end{aligned}$$

therefore, (3.13) can be transformed into a equation with the same form as (1.1),

$$\left(U_t+\int_0^t \alpha(t-s)U_t(s)ds\right)_t-\Delta U = u_1+\alpha(t)u_0,$$

Then the system (1.1) can be transformed as

$$\begin{cases}\left(U_t+\int_0^t \alpha(t-s)U_t(s)ds\right)_t-\Delta U = u_1+\alpha(t)u_0, & \text{in}\Omega\times(0,+\infty),\\ U\text{=}0, & \text{on}\Gamma\times(0,+\infty),\\ U(x,0)\text{=}0,\ U_t(x,0)\text{=}u_0(x), & x\in\times\Omega.\end{cases} \tag{3.14}$$

It can be seen the system (3.14) has the same structure as (1.1) except for the right-hand items. Then using the same argument (taking U_t as the multiplier), we have

$$\frac{1}{4}\| U_t(T)\|^2+\frac{1}{1+2A(0)}\int_0^T\left\langle\int_0^t A(t-s)U_{tt}(s)ds,U_{tt}\right\rangle dt+\frac{1}{2}\| \nabla U(T)\|^2$$
$$=\frac{1+A(0)}{4}\| u_0\|^2-\int_0^T\alpha(t)\langle u_0,U_t\rangle dt.$$

Since $U_t(t)=u(t),U_{tt}(t)=u_t(t)$,

$$\frac{1}{4}\| U_t(T)\|^2+\frac{1}{1+2A(0)}\int_0^T\left\langle\int_0^t A(t-s)u_t(s)ds,u_t\right\rangle dt+\frac{1}{2}\| \nabla U(T)\|^2$$
$$=\frac{1+A(0)}{4}\| u_0\|^2-\int_0^T\alpha(t)\langle u_0,u\rangle dt.$$

Notice the integrability of $\alpha(t)$, (A) and the Theorem 3.1, we get

$$\int_0^T\alpha(t)\langle u_0,u\rangle dt\le C_3E(0),$$

this gives

$$\int_0^T\left\langle\int_0^t A(t-s)u_t(s)ds,u_t\right\rangle dt\le C_4E(0).$$

Thus by Proposition 2.2 we have for any $T>0$,

$$\int_0^T\| u_t\|^2dt\le CE(0).$$

So we complete the proof.

Lemma 3.3. Let (A) holds. We arrive at

$$\int_0^T\| \nabla u\|^2dt\le CE(0),\qquad \forall T\ge t.$$

Proof. Multiply the equation (1.1) by u and take the inner product, we deduce that

$$\langle u_{tt},u\rangle+\left\langle\left(\int_0^t\alpha(t-s)u_t(s)ds\right)_t,u\right\rangle-\langle\Delta u,u\rangle=0. \tag{3.15}$$

eventually become

$$\langle u_{tt},u\rangle+\left\langle\int_0^t\alpha(t-s)u_t(s)ds,u\right\rangle_t-\left\langle\int_0^t\alpha(t-s)u_t(s)ds,u_t\right\rangle-\langle\Delta u,u\rangle=0. \tag{3.16}$$

Integrating (3.16) over the time variable t on the interval [0,T], we easily have

$$\int_0^T\langle u_{tt},u\rangle dt+\int_0^T\left\langle\int_0^t\alpha(t-s)u_t(s)ds,u\right\rangle_t dt-\int_0^T\left\langle\int_0^t\alpha(t-s)u_t(s)ds,u_t\right\rangle dt-\int_0^T\langle\Delta u,u\rangle dt=0,$$

using integration by parts, yield:

$$\langle u_t(T),u(T)\rangle-\langle u_1,u_0\rangle-\int_0^T\| u_t\|^2dt+\left\langle\int_0^t\alpha(t-s)u_t(s)ds,u\right\rangle$$
$$-\int_0^T\left\langle\int_0^t\alpha(t-s)u_t(s)ds,u_t\right\rangle dt+\int_0^T\| \nabla u\|^2dt=0,$$

That is

$$\int_0^T \| \nabla u\|^2 dt = -\langle u_t(T), u(T)\rangle + \langle u_1, u_0\rangle + \int_0^T \| u_t\|^2 dt$$
$$-\left\langle \int_0^t \alpha(t-s)u_t(s)ds, u\right\rangle + \int_0^T \left\langle \int_0^t \alpha(t-s)u_t(s)ds, u_t\right\rangle dt.$$

Where

$$\left\langle \int_0^t \alpha(t-s)u_t(s)ds, u\right\rangle \le C_5\left(\left\|\int_0^t \alpha(t-s)u_t(s)ds\right\|^2 + \|u\|^2\right) \le C_6 E(0),$$

and

$$\int_0^T \left\langle \int_0^t \alpha(t-s)u_t(s)ds, u_t\right\rangle dt \le C_7 \int_0^T \left(\left\|\int_0^t \alpha(t-s)u_t(s)ds\right\|^2 + \|u_t\|^2\right) dt$$
$$\le C_8 \int_0^T \|u_t\|^2\, dt.$$

Noting that $\alpha(t)$ and $\| u_t\|^2$ is integrable, we can conclude that

$$\int_0^T \| \nabla u\|^2 dt \le CE(0), \qquad \forall T > 0.$$

Thus, the proof of Lemma 3.3 is complete.

4. Proof of Theorem 2.4

The work completed thus far lays the groundwork for proving the uniform decay conclusion of Theorem 2.4.

Assume that $u_0, u_1 \in H_0^1(\Omega)$, the energy functional $E(t)$ consists of terms $\| u_t\|^2$ and $\| \nabla u\|^2$. Lemmas 3.1, 3.2, and 3.3 respectively analyze the properties of these two types of terms, for any $T > 0$,

$$\int_0^T \| u_t\|^2 dt \le CE(0),$$

$$\int_0^T \| \nabla u\|^2 dt \le CE(0),$$

due to the arbitrariness of T ,we obtain the integral estimate of the energy as follows:

$$\int_0^{+\infty} E(t) dt \le CE(0).$$

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