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Selfishly Prepaying in Financial Credit Networks

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Abstract

In financial credit networks, *prepayments* enable a firm to settle its debt obligations ahead of an agreed-upon due date. Prepayments have a transformative impact on the structure of networks, influencing the financial well-being (utility) of individual firms. This study investigates prepayments from both theoretical and empirical perspectives. We first establish the computational complexity of finding prepayments that maximize welfare, assuming global coordination among firms in the financial network. Subsequently, our focus shifts to understanding the strategic behavior of individual firms in the presence of prepayments. We introduce a *prepayment game* where firms strategically make prepayments, delineating the existence of pure strategy Nash equilibria and analyzing the price of anarchy (stability) within this game. Recognizing the computational challenges associated with determining Nash equilibria in prepayment games, we use a simulation-based approach, known as empirical game-theoretic analysis (EGTA). Through EGTA, we are able to find Nash equilibria among a carefully-chosen set of heuristic strategies. By examining the equilibrium behavior of firms, we outline the characteristics of high-performing strategies for strategic prepayments and establish connections between our empirical and theoretical findings.

1. Introduction

A financial system can be understood as a credit network, in which nodes represent financial institutions such as firms engaging in financial transactions, and directed edges capture bilateral financial obligations between firms, with the weight on edges denoting the amount of the corresponding debt or liability. A firm's *total assets* encompass exogenous external assets (e.g., cash or claims on entities outside the network) and its future incoming payments due within the system; *equity* measures the amount of remaining assets after payments (total assets minus liabilities) if this is positive, and zero otherwise (in which case the firm is bankrupt/insolvent). Firms employ their total assets to fulfill their liabilities by making payments to lenders. Should a firm's assets prove insufficient to cover its liabilities, the firm enters default, leading to a reduction in the value of its assets, often through costly liquidation. The extent of this decrease is denoted by *default costs*, signifying that the firm will have only a portion of its total assets available for payments. On liquidation day, "clearing" takes place, with the calculation and execution of actual payments in the financial network. These payments adhere to three fundamental principles of bankruptcy law, as outlined by Eisenberg and Noe (2001): i) *absolute priority*: firms must first settle their liabilities in full to attain positive equity; ii) *limited liability*: firms cannot pay more than their total assets; iii) *proportionality*: in the event of default, payments to lenders are made in proportion to the respective liability.

Prepayments generally refer to transactions in which a borrower opts to make repayments to lenders ahead of the debt's maturity. A common example of individual-level prepayment is observed in mortgage security markets, where some borrowers might pay off their mortgage loans before the designated maturity date (Patruno, 1994). Prepayments are also prevalent in the financial sector, notably in the commercial and industrial (C&I) business loan market, which constituted approximately two-thirds of short-term credit in the US in 2022 (Gallegati, Carraro, & Gaffeo, 2022). C&I loans provide companies with funds for working capital or capital expenditures (e.g., machinery purchases). Penalty-free prepayments, often permitted in C&I loan agreements, offer borrowing firms the flexibility to prepay without subsequent penalties[1]. Lenders universally accept such prepayments, so our work assumes such consistent acceptance (i.e., prepayments are always accepted by the lenders (Eckbo et al., 2022)). Motivations for prepayments can vary among firms in practice. Some borrowers prepay loans to lower the firm's leverage ratio (e.g., debt-to-asset ratio), while others may aim to reduce total interest costs or mitigate interest rate risk.

In our context, prepayments specify the early utilization of a firm's external assets to fulfill its liabilities before the clearing process for a given financial system. An individual firm in a financial network can have incentives to prepay with the following underlying rationale: in the event of firm bankruptcy during clearing, a firm can only employ a fraction of its original total assets to settle its liabilities due to the presence of default costs; if the firm decides to prepay before the anticipated clearing day, prior to the realization of default costs, then it can make full use of its external assets. These undiscounted prepayments could potentially prevent other firms from facing bankruptcy, yielding augmented cash flow throughout the network during clearing. In return, the augmented cash flow could benefit the firm initiating the prepayments when it flows back. Given that making prepayments leads to an immediate loss in a firm's external assets (and, consequently, in utility), strategic decisions regarding whether to prepay and whom to prepay should be contingent on whether the resulting benefit can offset the immediate loss. Our study explores the impacts of strategic prepayments in financial credit networks, addressing relevant questions through a blend of theoretical and empirical analyses.

1. Borrowers typically incur an upfront fee for the penalty-free prepayment option during the loan commitment drafting phase (Eckbo, Su, & Thorburn, 2022).

1.1 Contributions.

Our work has three main contributions:

1. We establish prepayment games, where the act of making prepayments is a strategic decision;
2. Using our game model, we offer a comprehensive theoretical understanding of the existence, (in)efficiency, and computational aspects of equilibria in prepayment games; and
3. We scrutinize the equilibrium behavior of firms in prepayment games through the application of empirical game-theoretic analysis (EGTA), involving game-theoretic reasoning via extensive simulation.

Specifically, when assuming firms have complete information of the credit network, we prove that computing a prepayment profile that maximizes the sum of total assets or total equity is **NP**-complete if there are non-trivial default costs. Then we shift our focus to understanding the strategic behavior of individual firms in the presence of prepayments. We first introduce prepayment games where firms strategically decide which lender to prepay. Then we explore the existence, quality, and computational complexity of pure-strategy Nash equilibria (PSNE) under two different utility models; total assets and equity. We observe that PSNE may not exist when firms aim to maximize their total assets in the presence of default costs, whereas they always exist when the goal is to maximize equity, irrespective of default costs. By assessing the quality of equilibria, we find that, when firms' utility is total assets, even the best PSNE can have arbitrarily bad social welfare (sum of firms' utilities) relative to the optimal subset of prepayments. In contrast, when utility is defined in terms of equity, the best PSNE is socially optimal. However, the worst PSNE remains arbitrarily inefficient. Furthermore, we emphasize the intractability of determining PSNE existence and computing the best response in the total-asset-based setting, whereas these problems are trivially solvable in the equity-based setting. The computation of the best PSNE remains **NP**-hard in the equity-based setting.

Finally, we complement our theoretical results with an empirical analysis through EGTA, with a realistic setting where firms only have local information about the network. We discover the existence of mixed Nash equilibria with prepayment strategies in the equilibrium support, demonstrating higher social welfare compared to abstaining from prepayments. This underscores the benefits of prepayments in financial systems. Additionally, we perform a systematic ablation study, unveiling the strengths and weaknesses of different prepayment strategies and providing insights for making beneficial prepayments.

1.2 Related Work.

Our model is grounded in the influential work of Eisenberg and Noe (2001) in which firms are nodes that are connected via debt contracts, with proportional payments in the case of insolvency. Subsequent work extended the basic model, incorporating features such as default costs (Rogers & Veraart, 2013), cross-ownership relations (Elliott, Golub, & Jackson, 2014; Vitali, Glattfelder, & Battiston, 2011), liabilities with various maturities

(Allouch, Jalloul, & Duncan, 2023; Kusnetsov & Veraart, 2019), and credit default swaps (Schuldenzucker, Seuken, & Battiston, 2020; Papp & Wattenhofer, 2020). Rogers and Veraart (2013) prove the existence of maximal clearing payments in the presence of default costs and provide an efficient algorithm to compute them. Schuldenzucker, Seuken, and Battiston (2017) demonstrate the intractability of finding clearing payments with credit default swaps. Ioannidis, de Keijzer, and Ventre (2022) study the clearing problem from the point of view of irrationality of solutions and strength of approximation, while Ioannidis, De Keijzer, and Ventre (2023) investigate the complexity of clearing problems in networks with derivatives and lender priorities.

A game-theoretic approach to financial networks is introduced by Bertschinger, Hoefer, and Schmand (2024). They relax the principle of proportionality (Eisenberg & Noe, 2001) and model payments under bankruptcy as a decision-making process. In particular, they propose two different payment schemes – coin-ranking and edge-ranking – and present a range of results on the existence and quality of equilibria arising in games corresponding to each payment scheme. Kanellopoulos, Kyropoulou, and Zhou (2024) extends this line of research by examining priority-proportional payment schemes. Furthermore, Papp and Wattenhofer (2020) analyze firms' incentives for redefining liabilities and donating external assets. In subsequent work, Papp and Wattenhofer (2021) investigate the impact of debt swaps in risk mitigation, while Froese, Hoefer, and Wilhelmi (2023) analyze the computational complexity of debt swapping in networks with ranking-based clearing. In recent contributions, Hoefer and Wilhelmi (2022) investigate clearing games with varying seniorities, while Bertschinger, Hoefer, Krogmann, Lenzner, Schuldenzucker, and Wilhelmi (2023) study equilibria in a game that models fire sales, analyzing the convergence of best-response dynamics. Furthermore, Hoefer, Ventre, and Wilhelmi (2023) study claims trading in financial networks and analyze the structural properties and computational aspects across different forms of claims trading. Kanellopoulos, Kyropoulou, and Zhou (2022) study edge-removal games where each bank wants to maximize its total assets by strategically removing a part of incoming edges, and provide results about the properties of resulting equilibria, while Tong, De Keijzer, and Ventre (2024b) empirically examine the impact of selfish debt cancellations on systemic risk. More recently, the same authors investigate debt transfer games, where banks can be strategic about whether or not to transfer their debt claims, and complement their theoretical study with an empirical game analysis (Kanellopoulos, Kyropoulou, & Zhou, 2023). In addition, Zhao, Polukarov, and Ventre (2023) consider the financial network setting where firms can choose different strategies to mitigate both liquidity and solvency risks, and they assess different strategic hedging portfolios and identify equilibrium conditions using empirical game-theoretic analysis, while Tong, De Keijzer, and Ventre (2024a) study the strategic donations of external assets from solvent banks to banks in distress and show that donations can indeed reduce systemic risk at equilibrium. Mayo, Grabill, and Wellman (2024) study how banks mitigate fraud risk in real-time payments by introducing an agent-based model of the payment network. They define a game between banks and fraudsters, and examine the Nash equilibria using empirical game-theoretic analysis (EGTA). To the best of our knowledge, our work is the first to study a model for financial networks with prepayment.

Extensive empirical research on financial network topologies indicates that very different structures may arise, ranging from centralized networks (Müller, 2006) to core-periphery structures (Fricke & Lux, 2015; Li & Schürhoff, 2019) and scale-free structures as in (Boss, Elsinger, Summer, & Thurner 4, 2004; Cont, Moussa, & Bastos e Santos, 2010). Additionally, numerous studies, including (Gai & Kapadia, 2010; Elliott et al., 2014; Leventides, Loukaki, & Papavassiliou, 2019), simulate default contagion within random networks. A comparable random financial network has been employed (Mayo & Wellman, 2021; Tong et al., 2024a) to simulate strategic behavior from an empirical game-theoretic perspective.

2. Preliminaries

We examine a *financial credit network* model proposed by Rogers and Veraart (2013). Specifically, we analyze a network G where nodes represent firms, and edges symbolize debt contracts. Each firm v_i initially holds non-negative *external assets* e_i, representing liquid assets received from entities external to the financial system. Moreover, firms are connected by directed edges that represent *liabilities*. The directed edge (v_i, v_j) signifies a financial relationship between firms v_i and v_j, with $l_{ij} \geq 0$ denoting the liability that firm v_i (the borrower) has to firm v_j (the lender). If no directed edge connects firms v_i and v_j, then v_i has no liability to v_j, and $l_{ij} = 0$. The graph G is irreflexive, meaning no firm can be liable to itself, resulting in a directed and positively weighted graph without self-loops. It is worth noting that both $l_{ij} > 0$ and $l_{ji} > 0$ might coexist. The total liabilities of v_i are denoted by $L_i = \sum_j l_{ij}$. Firms capable of fully meeting their obligations are considered *solvent*, while those unable to do so are labeled *in default* or *insolvent* interchangeably.

Let p_{ij} denote the actual payment made by firm v_i to firm v_j, where p_{ij} may not be equal to the liability l_{ij}, and $p_{ii} = 0$. We denote by $\mathbf{P} = (p_{ij})$ for $i, j \in [n]$ the *induced payment matrix*, where $[n] := \{1, \ldots, n\}$. The total outgoing payments of v_i are represented by $p_i = \sum_{j \in [n]} p_{ij}$. In the case of insolvency, a firm may need to liquidate its assets or make payments beyond the financial system, such as salary disbursements. Consequently, a defaulting firm can only utilize an α fraction of its external assets and incoming payments. When $\alpha = 1$, it means an absence of default costs. Payments $\mathbf{P}$ aligned with the three principles discussed in the introduction are termed *clearing payments*. Specifically, a solvent firm must fully pay all its obligations, while a firm in default can only pay partially but should repay as much of its debt as possible, based on its total assets affected by the default costs. A partial payment to a lender should be proportional to its liability to the same lender. Mathematically, the clearing payment from v_i to v_j with $i, j \in [n]$ satisfies $p_{ij} = l_{ij}$ when v_i is solvent, and $p_{ij} = \alpha \cdot \left(e_i + \sum_{j=1}^{n} x_{ji}\right) \cdot \frac{l_{ij}}{L_i}$ when v_i is in default. It is important to note that clearing payments are not necessarily unique; however, we focus on maximal clearing payments—those that point-wise maximize all corresponding payments. Such maximal clearing payments are known to exist and unique, and can be computed in polynomial time (Rogers & Veraart, 2013).

2.1 Strategic Prepayments.

Prepayments in our context are the settlement of debt obligations ahead of clearing using external assets. For a given financial network G, a single prepayment from borrower v_i to lender v_j results in a direct reduction of v_i's external assets e_i by an amount equal to the liability l_{ij} (i.e., $e_i' = e_i - l_{ij}$); meanwhile, there is an increase of l_{ij} to v_j's external assets (i.e., $e'_j = e_j + l_{ij}$), accompanied by a cancellation of the liability l_{ij}. Each firm v_i has the option to select a set of lenders to prepay, and consequently, a strategy determines the chosen set of lenders for prepayment. Assuming that all firms strategically engage in prepayments to enhance their individual financial well-being, this naturally leads to the formulation of a *prepayment game* in strategic form. Let D_i and C_i represent the sets of borrowers and creditors for v_i, respectively. Consequently, each *pure strategy* of firm v_i constitutes a subset $C'_i \subseteq C_i$ of lenders to prepay[2]. A *mixed strategy* for firm v_i is defined as a discrete probability distribution over the power set of its lenders C_i. A *pure strategy profile*, denoted by $\mathbf{s} = (s_1, \ldots, s_n)$ or $\mathbf{s} = (s_i, \mathbf{s}_{-i})$, specifies a pure strategy for each firm. Here, the pure strategy profile for all firms except v_i is denoted by $\mathbf{s}_{-i}$.

Given a prepayment profile[3], $\mathbf{s}$, and the corresponding clearing payments $\mathbf{P}$, we explore two definitions of a firm's utility. The *total assets* of firm v_i encompass the updated external assets post-prepayments, along with its incoming clearing payments, expressed as $a_i(\mathbf{P}) = e'_i + \sum_{j\in[n]} p_{ji}$, while its *equity* is defined as $E_i(\mathbf{P}) = \max\{0,\ a_i(\mathbf{P}) - L'_i\}$. Here, e'_i and L'_i represent v_i's updated external assets and total liabilities, respectively, under the strategy profile $\mathbf{s}$. Specifically, e'_i is computed as $e_i + \sum_{j\in D'_i} l_{ji} - \sum_{j\in C'_i} l_{ij}$, and L'_i is determined by $L_i - \sum_{j\in C'_i} l_{ij}$, where D'_i denotes the set of v_i's borrowers choosing to prepay v_i under $\mathbf{s}$. When $\mathbf{P}$ is clear from the context, the notation a_i and E_i will be used. The social welfare $\mathcal{SW}(\mathbf{P})$ is the sum of the firms' utilities, with the specific utility notion (total assets or equity) clarified by the context. The optimal social welfare (chosen globally, disregarding strategic considerations) is denoted by $\mathcal{OPT}$.

We consider *Nash equilibrium* (NE) as our strategic solution concept for prepayment games. A NE is a strategy profile in which no firm has an incentive to unilaterally deviate (and improve its utility); if such a profile is pure, we call it a *pure-strategy Nash equilibrium.* Let $\mathbf{P_{eq}}$ be the set of clearing payments consistent with all pure NE strategy profiles. The *Price of Anarchy* and *Price of Stability* in a game are characterized by the worst-case and best-case ratios, respectively, of the optimal social welfare over the achieved welfare at any equilibrium, considering all potential networks. Formally, PoA $= \max_G \max_{\mathbf{P}\in\mathbf{P_{eq}}} \frac{\mathcal{OPT}}{\mathcal{SW}(\mathbf{P})}$, and PoS $= \max_G \min_{\mathbf{P}\in\mathbf{P_{eq}}} \frac{\mathcal{OPT}}{\mathcal{SW}(\mathbf{P})}$, respectively. At times, a broader class of equilibria is considered, (e.g., mixed or correlated equilibria), but we focus on PSNE in this work.

2.2 An Illustrative Example.

Figure 1 presents a specific financial network that exemplifies an individual firm's incentive to prepay, illustrating some key concepts. If v_1 chooses to prepay v_2, where the liability l_{12} is 4, v_1's external assets directly decrease from 4 to 0. Simultaneously, v_2's external assets become 4, leading to the removal of the edge (v_1, v_2). The clearing payments before and after this prepayment are outlined below. In this example, we assume a default cost of $\alpha = \frac{1}{2}$. In the initial network (without prepayments), firm v_1 defaults since its potential maximum total assets of 12 fall short of total liabilities of 16. This triggers a default contagion, rendering v_2 and v_3 insolvent sequentially. Consequently, the clearing payments

2. Firms can only take *feasible* strategies in which their external assets are sufficient to fully prepay a particular selected set of lenders. That is, $e_i \geq \sum_{j\in C'_i} l_{ij}$. Fractional prepayments are not considered in this study.
3. Throughout this paper, the terms 'strategy profile' and 'prepayment profile' are used interchangeably.

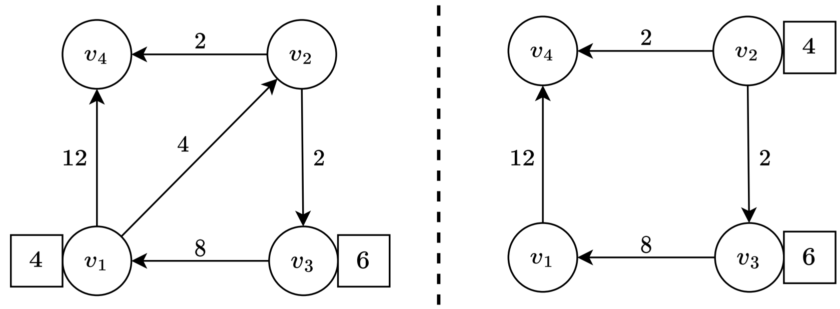


Figure 1: The figure (left) shows the initial network, while the other figure (right) shows the network after the prepayment by v_1. Nodes correspond to firms, edges are labeled with the respective liabilities, while external assets appear in a rectangle near the relevant firm.

are $p_{12} = \frac{8}{9}$, $p_{14} = \frac{24}{9}$, $p_{23} = p_{24} = \frac{2}{9}$, and $p_{31} = \frac{28}{9}$ with total assets $a_1 = \frac{64}{9}$, $a_2 = \frac{8}{9}$, $a_3 = \frac{56}{9}$, $a_4 = \frac{26}{9}$, and equities $E_1 = E_2 = E_3 = 0$, and $E_4 = \frac{26}{9}$.

However, if v_1 strategically prepays v_2, both v_2 and v_3 become solvent, avoiding asset loss due to default costs. This results in more payments on the existing edges, including those flowing back to v_1. The clearing payments in this case are $p'_{14} = 4$, $p'_{23} = p'_{24} = 2$, $p'_{31} = 8$ with $a'_1 = 8$, $a'_2 = 4$, $a'_3 = 8$, $a'_4 = 6$, and equities $E'_1 = E'_2 = E'_3 = 0$, and $E'_4 = 6$, respectively. These values can be easily verified by running the clearing algorithm of Rogers and Veraart (2013). We can conclude that v_1 is better off after the prepayment in terms of total assets, i.e., $a'_1 > a_1$, although it remains insolvent with zero equity.

3. Computing Optimal Prepayments

We start with the centralized setting, where as motivation one can imagine a financial regulator that possesses the authority to govern each firm. We present computational hardness results associated with achieving specific objectives related to the financial well-being of the entire system. With this in mind, we define three decision problems. The first is Partition, a well-known **NP**-hard problem, whilst the other three decision problems correspond to the restriction of exact cover by 3-sets (RXC3), and the maximization of total assets and equity in financial networks.

Decision Problem 1 (Partition)**.** *Given an instance $\mathcal{I}$ that consists of a set X of positive integers $\{x_1, x_2, \dots, x_k\}$, determine whether there exists a subset X' of X such that $\sum_{i \in X'} x_i = \sum_{i \notin X'} x_i = \frac{1}{2} \sum_{i \in X} x_i$.*

Decision Problem 2 (Restricted Exact Cover by 3-Sets (RXC3))**.** *Given an instance $\mathcal{I}$ that consists of a set X with $|X| = 3k$ for $k \in \mathbb{N}$ and a collection C of subsets of X where each such subset contains exactly three elements, and each element in X appears in exactly three subsets in C, determine whether there exists a subset $C' \subseteq C$ of size k that contains each element of X exactly once.*

Decision Problem 3 (Max-Sum-Total-Assets). *Given a financial credit network G and a positive constant μ, determine whether there exists a prepayment profile such that the sum of total assets is greater than or equal to μ.*

Decision Problem 4 (Max-Sum-Equity). *Given a financial credit network G and a positive constant μ, determine whether there exists a prepayment profile such that the total equity is greater than or equal to μ.*

Theorem 1. Max-Sum-Total-Assets *is* **NP**-*complete.*

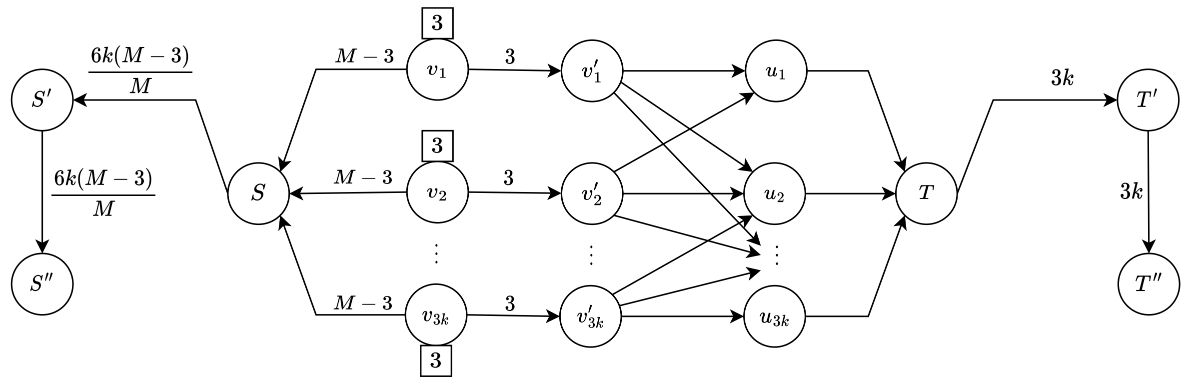


Figure 2: Illustrating the reduction from the proof of Theorem 1. All edges with missing labels correspond to liability 1, and $M > 9k$.

Proof. We first prove Max-Sum-Total-Assets belongs to **NP**. We can run the maximal cleaning payments algorithm to verify if one proposed prepayment profile achieves the sum of total assets at least value μ, which is in polynomial time (Rogers & Veraart, 2013).

We then show that Max-Sum-Total-Assets is **NP**-hard, and the following proof relies on a reduction from the **NP**-complete RXC3 problem. Given an instance $\mathcal{I}$ of RXC3, we construct an instance $\mathcal{I}'$ as follows. We add firm u_i for each element i of X, firms v_i and v'_i for each subset i in C, as well as another six firms S, S', S'', T, T' and T''. Each firm v_i, corresponding to set $\{x, y, z\} \in C$, has external assets $e_i = 3$ and liability 1 to corresponding v'_i, as well as liability $M-3$ to firm S where M is an arbitrarily large number with $M > 9k$. Each v'_i has liability 1 to the three firms u_x, u_y, and u_z corresponding to the three elements $x, y, z \in X$. Firm S and S' owes $\frac{6k(M-3)}{M+3}$ to S' and S'' respectively. Furthermore, each u_i has a liability 1 to T, while T and T' owe $3k$ to firm T' and T'' respectively; see also Figure 2. Note that this construction requires polynomial time. Clearly, only v_i for $i \in [3k]$ is eligible to make prepayments.

Now, it suffices to show that the sum of total assets at least $39k - \frac{18k}{M}$ can be achieved if and only if instance $\mathcal{I}$ is a 'yes'-instance for problem RXC3.

For the backward direction we work as follows. Let instance $\mathcal{I}$ be a 'yes'-instance for RXC3 and let C' be the solution to $\mathcal{I}$. We claim that $\mathcal{I}'$ admits a solution with the sum of total assets at least $39k - \frac{18k}{M}$. Indeed, it suffices to let all v_i's with $i \in C'$ prepay the corresponding v'_i, while all other v_i's make payments as normal on the clearing day.

This choice makes all v_i' with $i \in C'$ exactly solvent so that each u_i receives at least 1 from v_i's, which further ensures firms T and T' exactly solvent. That is, $a_T = a_{T'} = a_{T''} = 3k$. Additionally, the total payments from v_i's to S equals to $\frac{(9k-3k)(M-3)}{M}$, leading to $a_S = \frac{6k(M-3)}{M}$. This implies that firm S' is exactly solvent with $p_{S',S''} = \frac{6k(M-3)}{M}$ and $a_{S'} = a_{S''} = \frac{6k(M-3)}{M}$. The sum of total assets for v_i's and v_i''s is $\sum_i a_{v_i} = 9k - 3k = 6k$, and $\sum_{i\in C} a_{v_i'} = 6k \cdot \frac{3}{M} + 3k$, respectively. Meanwhile, the sum of total assets for u_i's would exactly equals to $6k \cdot \frac{3}{M} + 3k$, comprising of the payments $3k$ from v_i' with $i \in C'$ and $6k \cdot \frac{3}{M}$ from v_i' with $i \in C \setminus C'$. The detailed total assets for all firms are also presented as the second column in Table 1.

For the forward direction, it suffices to show that any collection of prepayments that generates the sum of total assets at least $39k - \frac{18k}{M}$ can lead to a solution for instance $\mathcal{I}$. Let η be the number of firms v_i who prepays the corresponding v_i'. We perform case analysis study for the each of the following cases: $\eta = k$, $\eta < k$, and $\eta > k$. Further, in Table 1 we present the detailed total assets for firms for the three cases.

Note that the total payments contributed from the v_i's to their neighbors is always $3 \cdot 3k - 3 \cdot \eta = 9k - 3\eta$, resulting in $\sum_{i\in C} a'_{v_i} = 9k - 3\eta$, $a'_S = (9k - 3\eta) \cdot \frac{M-3}{M}$, and $\sum_{i\in C} a'_{v_i'} = (9k - 3\eta) \cdot \frac{3}{M} + 3\eta$, respectively, with $\left(\sum_{i\in C} a'_{v_i} + a'_S + \sum_{i\in C} a'_{v_i'}\right) = 18k - 3\eta$.

For $\eta < k$, or equivalently $\eta \le k - 1$ since $\eta \in \mathbb{N}$, the total assets of firm S is $(9k - 3\eta) \cdot \frac{M-3}{M} > l_{S,S'} = l_{S',S''} = \frac{6k(M-3)}{M}$, allowing firms S and S' to fully pay their liability, resulting in $a'_{S'} = a'_{S''} = \frac{6k(M-3)}{M}$. On the other hand, $\sum_{i\in C} a'_{v_i'} = (9k-3\eta) \cdot \frac{3}{M} + 3\eta < 3k$ as $\eta \le k-1$ and $M > 9k$, directly leading to $\sum_{i\in C} a'_{u_i'} = a'_T = a'_{T'} = a'_{T''} = (9k-3\eta) \cdot \frac{3}{M} + 3\eta$ at most. Therefore, the sum of total assets is as follows:

$$
\begin{aligned}
\mathcal{F} &= \left(\sum\nolimits_{i\in C} a'_{v_i} + a'_S + \sum\nolimits_{i\in C} a'_{v_i'}\right) + \left(a'_{S'} + a'_{S''}\right) + \left(\sum\nolimits_{i\in C} a'_{u_i'} + a'_T + a'_{T'} + a'_{T''}\right) \\
&= (18k - 3\eta) + 2 \cdot \frac{6k(M-3)}{M} + 4 \cdot \left((9k - 3\eta) \cdot \frac{3}{M} + 3\eta\right) \\
&= 18k + \frac{(12M + 72)k}{M} + \left(9 - \frac{36}{M}\right) \cdot \eta \\
&\le 18k + \frac{(12M + 72)k}{M} + \left(9 - \frac{36}{M}\right) \cdot (k-1) \\
&= 39k + \left(\frac{36k + 36}{M} - 9\right) \\
&< 39k - \frac{18k}{M},
\end{aligned}
$$

where the last inequality holds by the assumption of $M > 9k > 6k + 4$ where $k > 1$.

For $\eta > k$, considering the total capacity of liabilities from v_i''s to T being $3k$, it follows that $a'_T \le 3k$, and similarly, $a'_T = a'_{T'} = a'_{T''} \le 3k$. Furthermore, $a'_S = (9k - 3\eta) \cdot \frac{M-3}{M} < \frac{6k(M-3)}{M}$ implies that firms S and S' are in default with $p'_{S,S'} = p'_{S',S''} = (9k - 3\eta) \cdot \frac{M-3}{M}$ and $a'_S = a'_{S'} = a'_{S''} = (9k - 3\eta) \cdot \frac{M-3}{M}$. Additionally, we have $\sum_{i\in C} a'_{u_i} \le \sum_{i\in C} a'_{v_i'} = (9k - 3\eta) \cdot \frac{3}{M} + 3\eta$ because all the u_i's total assets come from the payments from v_i''s.

Therefore, the sum of total assets is,

$$\begin{aligned}
\mathcal{F} &= \left(\sum\nolimits_{i\in C} a'_{v_i} + a'_S + \sum\nolimits_{i\in C} a'_{v'_i}\right) + \left(a'_{S'} + a'_{S''}\right) + \sum\nolimits_{i\in C} a'_{u'_i} + \left(a'_T + a'_{T'} + a'_{T''}\right) \\
&< (18k - 3\eta) + 2\cdot(9k-3\eta)\cdot\frac{M-3}{M} + (9k-3\eta)\cdot\frac{3}{M} + 3\eta + 3\cdot 3k \\
&= 27k + (9k-3\eta)\cdot\left(1 + \frac{M-3}{M}\right) \\
&= 39k - \frac{18k}{M}.
\end{aligned}$$

It remains to argue about the case $\eta = k$. If these k firms v'_i's can cover all u_i's, this then exactly coincides with the 'yes'-instance above with the sum of total assets of $39k - \frac{18k}{M}$, and we obtain a solution to RXC3; a contradiction. The proof is complete.

	$\eta = k$	$\eta < k$	$\eta > k$
$\sum_i a_{v_i}$	$6k$	$9k - 3\eta$	$9k - 3\eta$
$\sum_i a_{v'_i}$	$6k \cdot \frac{3}{M} + 3k$	$(9k-3\eta)\cdot\frac{3}{M} + 3\eta$	$(9k-3\eta)\cdot\frac{3}{M} + 3\eta$
$\sum_i a_{u_i}$	$6k \cdot \frac{3}{M} + 3k$	$\leq (9k-3\eta)\cdot\frac{3}{M} + 3\eta$	$(9k-3\eta)\cdot\frac{3}{M} + 3\eta$
a_S	$6k\cdot\frac{3}{M}$	$(9k-3\eta)\cdot\frac{M-3}{M}$	$(9k-3\eta)\cdot\frac{M-3}{M}$
$a_{S'}$	$6k\cdot\frac{3}{M}$	$6k\cdot\frac{3}{M}$	$(9k-3\eta)\cdot\frac{M-3}{M}$
$a_{S''}$	$6k\cdot\frac{3}{M}$	$6k\cdot\frac{3}{M}$	$(9k-3\eta)\cdot\frac{M-3}{M}$
a_T	$3k$	$\leq (9k-3\eta)\cdot\frac{3}{M} + 3\eta$	$3k$
$a_{T'}$	$3k$	$\leq (9k-3\eta)\cdot\frac{3}{M} + 3\eta$	$3k$
$a_{T''}$	$3k$	$\leq (9k-3\eta)\cdot\frac{3}{M} + 3\eta$	$3k$
$\mathcal{F}$	$39k - \frac{18k}{M}$	$< 39k - \frac{18k}{M}$	$< 39k - \frac{18k}{M}$

Table 1: The total assets for firms under various realizations of η.

□

Note that this result holds for any $\alpha \in [0,1]$. Next, we consider the objective of maximizing total equity.

Lemma 1. ((Schuldenzucker et al., 2020)). *In the networks without default costs, the total equity always equals the sum of external assets, that is,* $\sum_i E_i = \sum_i e_i$.

An implication of Lemma 1 is that any prepayment profile maximizes the total equity. The situation changes significantly, however, in the presence of non-trivial default costs. We formally present these remarks in the following result.

Theorem 2. Max-Sum-Equity *is*

a) trivial when $\alpha = 1$*;*

b) **NP***-complete when* $\alpha \in [0, 1)$*.*

Proof. Regarding case a), the statement straightforward holds by Lemma 1 that any prepayment profile maximizes total equity.

With respect to case b), we first prove Max-Sum-Equity belongs to **NP**. Clearly, we can run the maximal clearing payments algorithm to verify if one proposed prepayment profile achieves the total equity at least value μ, which is in polynomial time (Rogers & Veraart, 2013). We then prove Max-Sum-Equity is **NP**-hard, and this is established by reduction from Partition.

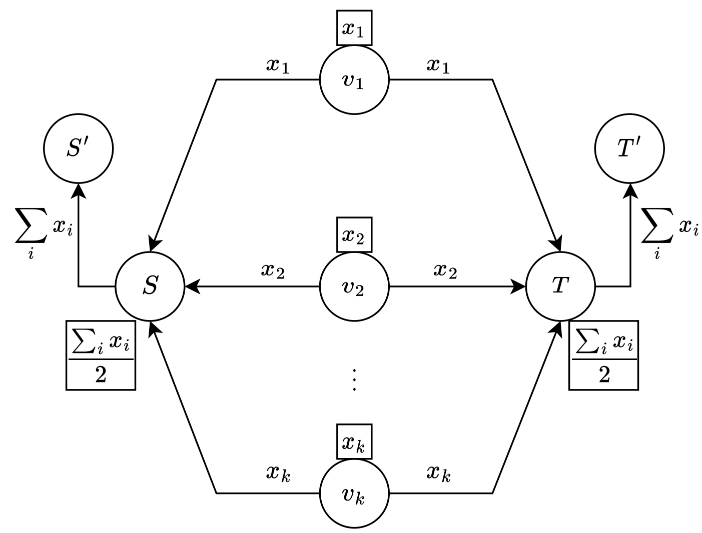


Figure 3: Illustrating the reduction from the proof of Theorem 2.

Starting from $\mathcal{I}$, we create an instance $\mathcal{I}'$ as follows. First, we add firms v_i for each element $x_i \in X$ and we allocate an external asset of x_i to v_i. We also include four additional firms S, S', T and T' and allocate an external asset of $\frac{1}{2}\sum_{i\in X} x_i$ to both S and T. Each firm v_i has a liability of x_i to S and T, while S and T have a liability of $\sum_{i\in X} x_i$ to S' and T' respectively; see also Figure 3. Clearly, the reduction requires polynomial time.

Observe that all v_i's are insolvent with zero equity, while firms S' and T' are solvent anyway. Due to default costs, the total payments from all v_i's to each of S and T is $\alpha\left(\frac{1}{2}\sum_{i\in X} x_i\right) < \frac{1}{2}\sum_{i\in X} x_i$, thus both S and T are in default. Hence, the sum of equities, due to firms S' and T', is $\alpha\sum_{i\in X} x_i + \alpha^2\sum_{i\in X} x_i$.

To minimize default costs through prepayments, we should make sure that all payments can reach S and T directly so that both firms can be (exactly) solvent, thereby further avoiding default costs.

Now, we will show that in instance $\mathcal{I}'$, a total equity of at least $2\sum_{i\in X} x_i$ can be achieved if and only if instance $\mathcal{I}$ of Partition is a 'yes'-instance. If $\mathcal{I}$ is indeed a 'yes'-instance, i.e., there exists a subset $X' \subseteq X$ with $\sum_{i\in X'} x_i = \frac{1}{2}\sum_{i\in X} x_i$, each v_i with $x_i \in X'$ directly prepays to S, while each v_i with $x_i \notin X'$ makes prepayment to T. Consequently, the total payments from all v_i's to each of S and T exactly amount to $\frac{1}{2}\sum_{i\in X} x_i$. This, in turn, ensures that S and T are solvent (albeit with equity of 0), resulting in $E_{S'} = E_{T'} = \sum_{i \in X} x_i$ and a total equity of $2 \sum_{i \in X} x_i$.

Note that, by Lemma 1, it is not hard to see that the total equity cannot be more than $2 \sum_{i \in X} x_i$ and this value can be obtained only when there are no equities lost. Hence, if it holds that at least one firm with positive total assets makes payments after insolvency, then the total equity is strictly less than $2 \sum_{i \in X} e_i$. We will exploit this property to show that the total equity would be strictly less than $2 \sum_{i \in X} x_i$ if instance $\mathcal{I}$ is a 'no'-instance.

Let X_S and X_T be the set of v_i's who decide to prepay S and T, respectively. If $X/(X_S \cup X_T) \neq \emptyset$, then each $v_i \in X/(X_S \cup X_T)$ would make prepayment after in default; since such a v_i has strictly positive total assets, we obtain that the total equity is below $2 \sum_{i \in X} x_i$. Otherwise, if $X/(X_S \cup X_T) = \emptyset$, then $X_S \cup X_T = X$. Since instance $\mathcal{I}$ admits no solution X' with $\sum_{i \in X'} x_i = \frac{1}{2} \sum_{i \in X} x_i$, we can assume, w.l.o.g, that $\sum_{i \in X_S} x_i > \frac{1}{2} \sum_{i \in X} x_i > \sum_{i \in X_T} x_i$, which implies that the total incoming payment in T is strictly less than $\frac{1}{2} \sum_{i \in X} x_i$ and firm T is in default; again, the total equity will be strictly less than $2 \sum_{i \in X} x_i$. The proof is complete. □

Note that the hardness of decision problems 3 and 4 immediately imply that the corresponding optimization problems are computationally intractable unless $\mathbf{NP} = \mathbf{P}$.

4. Prepayment Games

We turn our attention to the scenario where firms strategically determine prepayments to maximize their utility. Specifically, we investigate two variations of prepayment games, in which the utility of a firm is defined by total assets and equity respectively. For each variant, we examine the existence, quality, and computational complexity of finding PSNE. In both the total asset and equity setting, we assume that the firms know the financial network G.

4.1 Maximizing Total Assets

We first consider prepayment games where firms attempt to maximize their total assets and investigate the existence of PSNE. The following result establishes the existence of a unique PSNE in prepayment games without default costs.

Theorem 3. *In the total assets setting, for $\alpha = 1$, the prepayment profile where no firm prepays is the unique PSNE.*

Proof. Consider a financial network and two strategy profiles $\mathbf{s}$, $\tilde{\mathbf{s}}$ such that in $\mathbf{s}$ no firm prepays, while in $\tilde{\mathbf{s}}$ firm v_i is the only firm that prepays. Let v_j be the firm that v_i prepays under $\tilde{\mathbf{s}}$. Clearly, the prepayment from v_i to v_j reduces v_i's external asset by l_{ij}. Given the topology of the network, and since there are no default costs, the augmented external assets of v_j can lead to, at most, l_{ij} payments back to v_i. This implies, that for v_i it holds $a_i(\mathbf{P_s}) \geq a_i(\mathbf{P_{\tilde{s}}})$ where $\mathbf{P_s}$ and $\mathbf{P_{\tilde{s}}}$ are corresponding clearing payments under $\mathbf{s}$ and $\tilde{\mathbf{s}}$ respectively. Hence, firm v_i cannot be better off by deviating to prepaying, and then $\mathbf{s}$ is a PSNE.

For the uniqueness, we prove that any other payment profile $\bar{\mathbf{s}}$ is not a PSNE. Without loss of generality, we assume that a firm v_i prepays one of its lenders v_j under $\bar{\mathbf{s}}$, then we show that v_i always has an incentive to deviate to not-prepaying v_j. Starting from $\bar{\mathbf{s}}$, canceling the prepayment to v_j, firm v_i increases its external assets by l_{ij}. On the other hand, we denote by Δ the absolute value of decreased incoming payments to v_i triggered by the decrease of l_{ij} in v_j's external assets as prepayment cancellation, and we show that $\Delta < l_{ij}$. In fact, as v_i prepays v_j under $\bar{\mathbf{s}}$, this implies $e_i \geq l_{ij} > 0$. Then the payment from v_i to v_j after the prepayment cancellation must be a positive term, denoted by p. This further implies that v_j loses assets of amount $l_{ij} - p$, since v_i recovers the prepayment. Again, given the topology of the network, and since there are no default costs, a decrease of $(l_{ij} - p)$ in v_j's assets can lead to, at most, $(l_{ij} - p)$ decreased payments flowing back to v_i, that is $\Delta < (l_{ij} - p)$. Therefore, v_i can increase at least by $(l_{ij} - \Delta) > 0$ its total assets by deviating to not prepaying v_j, $\bar{\mathbf{s}}$ is not a PSNE, and we conclude. □

The existence of pure equilibria, however, is not guaranteed when non-trivial default costs apply.

Theorem 4. *In the total assets setting, there is a game such that for all $\alpha \in (0, 1)$ there is no PSNE.*

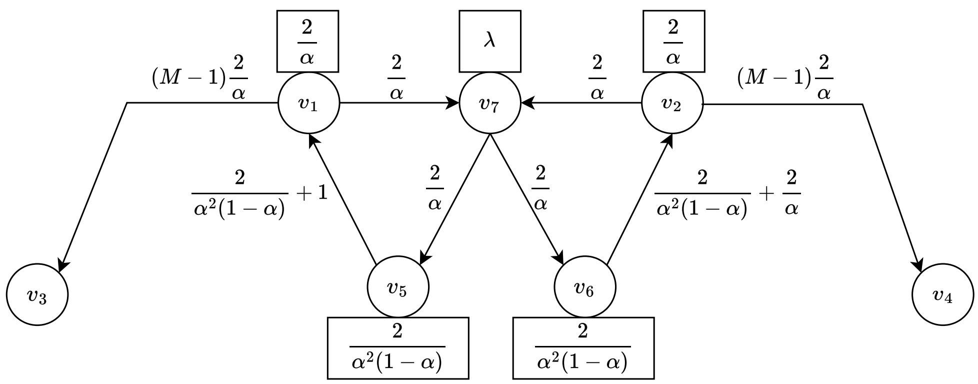


Figure 4: A prepayment game with default costs $\alpha \in (0, 1)$ that does not admit PSNE when firms maximize their total assets, with $M > \frac{6}{\alpha(1-\alpha)}$.

Proof. Fix the default costs $\alpha \in (0, 1)$ and let λ be a constant that is strictly less than $\frac{2}{\alpha} - \left(4 + \alpha + \alpha^2 + \frac{2+2\alpha}{\alpha(1-\alpha)}\right) \cdot \frac{1}{M}$. Consider the financial network in Figure 4. Clearly, firms v_3 and v_4 would be solvent, while v_1 and v_2 would be in default in any case, thus it suffices to focus on the firms v_5, v_6, and v_7 in the following. Note that all firms except for v_1 and v_2 are ineligible to make prepayments since their external assets are strictly less than each of their liabilities. Furthermore, the liability $(M - 1)\frac{2}{\alpha}$ between v_1 and v_3 is larger than v_1's external assets $\frac{2}{\alpha}$ so that v_1 can only strategize about whether prepay to v_7 or not. Similarly, v_2 can only decide if it will prepay to v_7 or not. Therefore, it suffices to consider the possible prepayments on edges (v_1, v_7) and (v_2, v_7).

Next, we articulate the above interaction as a normal-form game with two players, firms v_1 and v_2, with two pure strategies each, either to make prepayments or not. The payoff matrix for this game is provided in Table 2. In what follows, first we show the utility values for the following pure strategy profiles: a) neither v_1 nor v_2 make a prepayment, b) v_1 makes a prepays but v_2 does not, c) both firms prepay, and d) v_2 makes a prepays but v_1 does not. Thereupon, we prove that this game has no PSNE.

We first consider the clearing state of the original network, that is firms v_1 and v_2 play the strategy profile a). In this case, the possible total assets that v_1 can have is at most $\left(\frac{2}{\alpha}+\frac{2}{\alpha^2(1-\alpha)}+1\right)$, and can be obtained only when v_1 gets fully paid by v_5. On the other hand, v_1 has at least total assets $\frac{2}{\alpha}+\frac{2}{\alpha(1-\alpha)}$, so $\frac{2}{\alpha}+\frac{2}{\alpha(1-\alpha)} \le a_1^a \le \frac{2}{\alpha}+\frac{2}{\alpha^2(1-\alpha)}+1$. Considering the symmetry of network topology one can verify that if we let λ be $\frac{2}{\alpha} - \left(4+\alpha+\alpha^2+\frac{2+2\alpha}{\alpha(1-\alpha)}\right)\cdot\frac{1}{M}$, then the cleaning payments are as follows: $p'_{75}=p'_{76}=1$, $p'_{51}=\frac{2}{\alpha^2(1-\alpha)}+1$, $p'_{17}=\frac{1}{M}\cdot\left(\frac{2}{\alpha(1-\alpha)}+\alpha+2\right)$, $p'_{62}=\frac{2}{\alpha(1-\alpha)}+\alpha$, and, $p'_{27}=\frac{1}{M}\cdot\left(\frac{2}{(1-\alpha)}+\alpha^2+2\right)$, respectively. Consequently, firm v_5 is exactly solvent, and firms v_6 and v_7 are in default. Furthermore, utilizing our assumption of $\lambda < \frac{2}{\alpha} - \left(4+\alpha+\alpha^2+\frac{2+2\alpha}{\alpha(1-\alpha)}\right)\cdot\frac{1}{M}$ we have that $p_{75}=p_{76}<1$. This results in v_5 being in default with $a_1^a < \frac{2}{\alpha}+\alpha\cdot l_{51} = \frac{2}{\alpha}+\frac{2}{\alpha(1-\alpha)}+\alpha$. Again, according to the symmetry of network topology, we can obtain $a_1^a = a_2^a < \frac{2}{\alpha}+\frac{2}{\alpha(1-\alpha)}+\alpha$, which corresponds to the top-left cell in Table 2.

Now consider the case b), where v_1 makes a prepayment to v_7 but v_2 does not. After prepayment by v_1 to v_7, the total asset of v_7 equals to $\left(\frac{2}{\alpha}+\lambda\right)+p_{27}$. First, we will use the following claim regarding the efficiency of firm v_7 in case b).

Claim 1. *Consider the financial network as provided by Figure 4 with default costs* $\alpha \in (0,1)$ *and* $M > \frac{6}{\alpha(1-\alpha)}$. *If firm* v_1 *prepays and firm* v_2 *does not, then firm* v_7 *is in default.*

Proof. We implement straightforward computations for the utility firm v_7. That is,

$$
\begin{aligned}
a_7^b &= \left(\frac{2}{\alpha}+\lambda\right)+p_{27} \\
&= \frac{2}{\alpha}+\lambda+\alpha\cdot a_2^b\cdot\frac{1}{M} \\
&\le \frac{2}{\alpha}+\lambda+\alpha\cdot\left(\frac{2}{\alpha}+\frac{2}{\alpha^2(1-\alpha)}+\frac{2}{\alpha}\right)\cdot\frac{1}{M} \\
&< \frac{2}{\alpha}+\frac{2}{\alpha}-\left(4+\alpha+\alpha^2+\frac{2+2\alpha}{\alpha(1-\alpha)}\right)\cdot\frac{1}{M}+\left(4+\frac{2}{\alpha(1-\alpha)}\right)\cdot\frac{1}{M} \\
&= \frac{4}{\alpha}-\left(\alpha+\alpha^2+\frac{2\alpha}{\alpha(1-\alpha)}\right)\cdot\frac{1}{M} \\
&< \frac{4}{\alpha},
\end{aligned}
$$

where the weak inequality holds since $a_2^b \le \frac{2}{\alpha}+\frac{2}{\alpha^2(1-\alpha)}+\frac{2}{\alpha}$, while the first inequality holds by the assumption of $\lambda < \frac{2}{\alpha}-\left(4+\alpha+\alpha^2+\frac{2+2\alpha}{\alpha(1-\alpha)}\right)\cdot\frac{1}{M}$. Therefore, we can conclude that v_7 is in default with $\frac{2}{\alpha} < a_7^b < \frac{4}{\alpha}$. □

Since v_7 is in default, for the clearing payments we have $1 < p_{75}^b = p_{76}^b < \frac{2}{\alpha}$ resulting in v_5 being solvent with v_6 still being in default. Hence, we can obtain that $p_{51}^b = \frac{2}{\alpha^2(1-\alpha)}+1$

and $p^b_{62} < \alpha \cdot l_{62} = \frac{2}{\alpha(1-\alpha)} + 2$ with $a^b_1 = \frac{2}{\alpha^2(1-\alpha)} + 1$ and $a^b_2 < \frac{2}{\alpha} + \frac{2}{\alpha(1-\alpha)} + 2$. This corresponds to the bottom-left cell in Table 2.

In case c) where both v_1 and v_2 make prepayment to v_7. The total asset of v_7 exactly equals $\frac{4}{\alpha}$ and makes v_7 exactly solvent, resulting in both v_5 and v_6 being solvent as well. Thus, we have $a^c_1 = l_{51} = \frac{2}{\alpha^2(1-\alpha)} + 1$ and $a^c_2 = l_{62} = \frac{2}{\alpha^2(1-\alpha)} + \frac{2}{\alpha}$. This corresponds to the bottom-right cell in Table 2.

Finally, consider the case d) where v_2 makes prepayment to v_7 and v_1 does not. Due to the symmetry of the network, we can derive that $a^d_7 < \frac{4}{a}$, as in case b). Therefore, for the clearing payments it holds that $1 < p^d_{75} = p^d_{76} < \frac{2}{\alpha}$ with v_5 being solvent but v_6 being in default. Thus, the total asset of v_1 is $a^d_1 = \frac{2}{\alpha} + \frac{2}{\alpha^2(1-\alpha)} + 1$, while $a^d_2 < \alpha \cdot l_{62} = \frac{2}{\alpha(1-\alpha)} + 2$. This corresponds to the top-right cell in Table 2.

Now, from Table 2 it is not hard to see that $a^b_1 > a^a_1$, $a^c_2 > a^b_2$, $a^d_1 > a^c_1$, and $a^a_2 > a^d_2$. Thus the best response dynamics for this game imply that for each pure strategy profile exactly one firm has incentives to deviate in order to improve it's utility. Hence, the game has no PSNE.

□

v_1 \ v_2	NO Prepayment	Prepayment
NO Prepayment	$\frac{2}{\alpha} + \frac{2}{\alpha(1-\alpha)} \leq a^a_1 < \frac{2}{\alpha} + \frac{2}{\alpha(1-\alpha)} + \alpha, \quad a^a_2 = a^a_1$	$a^d_1 = \frac{2}{\alpha} + \frac{2}{\alpha^2(1-\alpha)} + 1, \quad a^d_2 < \frac{2}{\alpha(1-\alpha)} + 2$
Prepayment	$a^b_1 = \frac{2}{\alpha^2(1-\alpha)} + 1, \quad a^b_2 < \frac{2}{\alpha} + \frac{2}{\alpha(1-\alpha)} + 2$	$a^c_1 = \frac{2}{\alpha^2(1-\alpha)} + 1, \quad a^c_2 = \frac{2}{\alpha^2(1-\alpha)} + \frac{2}{\alpha}$

Table 2: The table of utilities for the financial network in Figure 4.

Next, we investigate the quality of pure equilibria and observe that even the best pure equilibria can be highly undesirable in terms of social welfare.

Theorem 5. *In the total assets setting, for all $\alpha \in [0, 1]$, the Price of Stability is unbounded.*

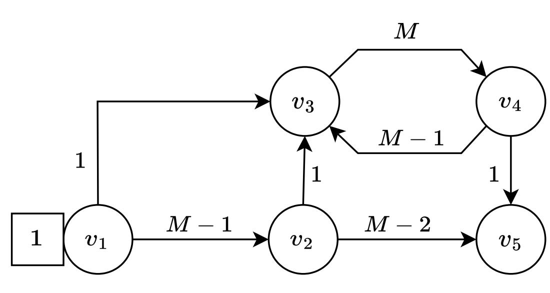


Figure 5: A prepayment game with unbounded Price of Stability when firms maximize their total assets.

Proof. Consider the financial network G in Figure 5, where M is an arbitrarily large constant. Note that v_1 is the only firm that can make a prepayment in G and can only strategize about the prepayment to v_3. Since there are no incoming edges for v_1, any prepayment will decrease its utility, so not prepaying is a dominant strategy for v_1. Moreover, firm v_1 would always have positive total assets of 1 by keeping payment both v_2 and v_3 on clearing day. So, the prepayment game corresponding to the financial network G has a unique NE.

By assuming no default costs, we will show an upper bound of $SW < 5$ on the sum of total assets in the network; default costs $\alpha \in [0, 1)$ can only result in smaller payments. Indeed, when $\alpha = 1$, the clearing state at the unique equilibrium are

$$p_{12} = 1 - \tfrac{1}{M},\ p_{13} = \tfrac{1}{M},\ p_{23} = \tfrac{1}{M},\ p_{25} = 1 - \tfrac{2}{M},\ p_{34} = 2,\ p_{43} = 2 - \tfrac{2}{M},\ \text{and } p_{45} = \tfrac{2}{M},$$

resulting in $\mathcal{SW} = \left(4 + \frac{5}{M}\right) < 5$. However, if v_1 makes prepayment to v_3, then the cycle between v_3 and v_4 would be saturated, resulting in $a'_3 = a'_4 = M$, $a'_5 = 1$, $a'_1 = a'_2 = 0$ and a social welfare equal to $2M + 1$, regardless of α; so, we have $\mathcal{OPT} \geq 2M + 1$.

□

We conclude with our results on computational complexity for the setting with default costs. These findings are quite negative and mean that computing pure equilibria with total assets is not tractable, which essentially serves as a foundational motivation for the subsequent exploration of EGTA in Section 5.

Theorem 6. *In the total assets setting, for all $\alpha \in (0, 1)$, the following problems are* **NP**-*hard:*

a) verifying if a given strategy profile is a PSNE even when PSNE are guaranteed to exist;

b) computing the best response strategy;

c) deciding if there exists a PSNE.

Proof. a) **Hardness of verifying if a given strategy profile is a PSNE, when pure equilibria are guaranteed to exist.** Our proof follows a reduction from the Partition problem. In more detail, starting from $\mathcal{I}$, we create an instance $\mathcal{I}'$ as follows. We first add firms v_i for each element x_i. Furthermore, we include three additional firms S, T and R, and allocate external assets of $\frac{\sum_i x_i}{2}$ and $\frac{\alpha \sum_i x_i}{2(1-\alpha)} - \frac{\alpha^2 \sum_i x_i}{2(1-\alpha)(2M-1)} + \epsilon$ to S and T respectively, where $M > \frac{2\sum_i x_i}{1-\alpha}$, and ϵ is an arbitrarily small with $0 < \epsilon < \frac{\alpha^2}{2(1-\alpha)(2M-1)}$. Firm S owes $(M-1)\sum_i x_i$ to firm R, and owes x_i to v_i, $\forall i \in [k]$. Additionally, each firm v_i has a liability of x_i to firm T, while, finally, firm T has a liability of $\frac{\sum_i x_i}{2(1-\alpha)}$ to S, see also Figure 6. Clearly, the reduction requires polynomial time.

Although both firms S and T have positive external assets, only firm S in the network is eligible to make prepayment since firm T's external assets cannot fully cover the liability l_{TS}, i.e.,

$$e_T = \frac{\alpha \sum_i x_i}{2(1-\alpha)} - \frac{\alpha^2 \sum_i x_i}{2(1-\alpha)(2M-1)} + \epsilon < \frac{\alpha \sum_i x_i}{2(1-\alpha)} < \frac{\sum_i x_i}{2(1-\alpha)} = l_{T,S}.$$

Thus, this guarantees the existence of a PSNE in $\mathcal{I}'$. We will show that we can verify a PSNE with $a_S = \frac{\sum_i x_i}{2(1-\alpha)}$, if and only if there is a solution to instance $\mathcal{I}$.

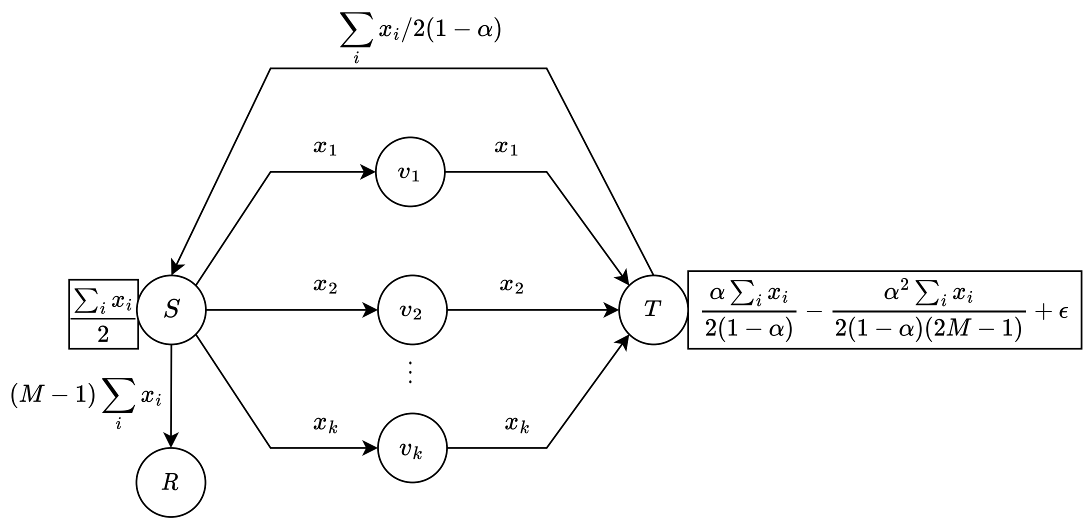


Figure 6: The instance arising from the reduction used in Theorem 6b.

First, we prove that if $\mathcal{I}$ is a 'yes'-instance, then firm S obtains $a_S = \frac{\sum_i x_i}{2(1-\alpha)}$. Indeed, consider the subset $X' \subseteq X$ in $\mathcal{I}$ with $\sum_{i \in X'} x_i = \frac{\sum_{i \in X} x_i}{2}$ and let firm S prepay firms v_i, where $i \in X'$. Note that firm S will be in default anyway due to the arbitrary large liability to R, while every v_i for $i \in X'$ will be exactly solvent. Therefore, under this profile, we have the following clearing payments: $p_{v_i,T} = x_i, \forall i \in X'$, $\sum_{i \notin X'} p_{v_i,T} = \frac{\alpha^2 \sum_{i \notin X'} x_i}{2(1-\alpha)(2M-1)}$, $p_{T,S} = \frac{\sum_{i \notin X'} x_i}{2(1-\alpha)}$, $p_{S,G} = \alpha \frac{(M-1)\sum_{i \notin X'} x_i}{(1-\alpha)(2M-1)}$, and $\sum_{i \notin X'} p_{S,v_i} = \frac{\alpha \sum_i x_i}{2(1-\alpha)(2M-1)}$, respectively. With this at hand, we obtain $a_S = \frac{\sum_i x_i}{2(1-\alpha)}$ as desired; note also that firm T is solvent with $a_T = \frac{\sum_i x_i}{2(1-\alpha)} + \epsilon$ under the clearing payments.

Next, for the forward direction, we will prove the contrapositive statement, that is, if $\mathcal{I}$ is a 'no'-instance, then S's total assets in $\mathcal{I}'$ are $a_S \neq \frac{\sum_i x_i}{2(1-\alpha)}$. First we will prove the following result.

Lemma 2. *Consider the instance $\mathcal{I}'$ as provided in the proof of Theorem 6. If $\mathcal{I}$ is a 'no'-instance, then S's total assets in $\mathcal{I}'$ are strictly less than $\frac{\sum_i x_i}{2(1-\alpha)}$.*

Proof. Consider any subset $X' \subseteq X$ in $\mathcal{I}$ and the corresponding setting in $\mathcal{I}'$. Here, S makes prepayments to firms v_i with $i \in X'$ while maintaining payments to firms v_i until the liquidation day corresponding to $i \notin X'$. Let $\chi = \sum_{i \in X'} x_i$, then we have $0 \leq \chi < \frac{1}{2}\sum_i x_i$ since the external asset of S is at most $\frac{1}{2}\sum_i x_i$[4]. Furthermore, since the integrity of x_i, we then have $0 \leq \chi \leq \frac{1}{2}\sum_i x_i - \frac{1}{2}$, and $\frac{1}{2} \leq \sum_{i \notin X'} x_i = \sum_i x_i - \chi \leq \sum_i x_i$, respectively. The total assets of firm S are

$$a_S = \frac{1}{2}\sum_i x_i - \chi + p_{T,S}.$$

4. Observe that, If $\chi = \frac{1}{2}\sum_i x_i$, then this would straightforward coincide with the 'yes'-instance.

Next, we will show that T would be in default, and then we prove that $a_S < \frac{\sum_i x_i}{2(1-\alpha)}$. For that, first we compute the proportion ρ of the total liabilities of S to v_i's, with $i \notin X'$, to its total liability is

$$\rho = \frac{\sum_{i \notin X'} x_i}{(M-1)\sum_i x_i + \sum_{i \notin X'} x_i} = \frac{\sum_i x_i - \chi}{(M-1)\sum_i x_i + \sum_i x_i - \chi} \leq \frac{\sum_i x_i}{(M-1)\sum_i x_i + \sum_i x_i} = \frac{1}{M}.$$

where the inequality holds because ρ is an decreasing function with respect to χ. Since firm S is necessarily in default, it holds that

$$\begin{aligned}
\sum\nolimits_{i \notin X'} p_{S,v_i} &= \rho \cdot a_S \cdot \alpha \\
&= \rho \left(\tfrac{1}{2} \sum_i x_i - \chi + p_{T,S} \right) \alpha \\
&\leq \rho \left(\tfrac{1}{2} \sum\nolimits_i x_i - \chi + \tfrac{1}{2(1-\alpha)} \sum\nolimits_i x_i \right) \alpha \\
&= \rho \left(\tfrac{(2-\alpha)}{2(1-\alpha)} \sum_i x_i - \chi \right) \alpha \\
&\leq \rho \tfrac{\alpha(2-\alpha)}{2(1-\alpha)} \sum\nolimits_i x_i \\
&< \tfrac{1}{M(1-\alpha)} \sum\nolimits_i x_i \\
&< \tfrac{1}{2},
\end{aligned}$$

where the first strict inequality holds since $\frac{\alpha(2-\alpha)}{2} < \frac{(2-\alpha)}{2} < 1$ and $\rho < \frac{1}{M}$, while the last strict inequality is derived by $\frac{2}{1-\alpha} \sum_i x_i$. Consequently, v_i with $i \notin X'$ is insolvent since $\sum_{i \notin X'} p_{S,v_i} < \frac{1}{2} < \sum_{i \notin X'} x_i$ where $\sum_{i \notin X'} x_i > \frac{1}{2}$ as mentioned before. Then, the total assets of firm T are

$$\begin{aligned}
a_T &= \alpha \sum\nolimits_{i \notin X'} p_{v_i,S} + \sum\nolimits_{i \in X'} p_{v_i,S} + \left(\tfrac{\alpha \sum_i x_i}{2(1-\alpha)} - \tfrac{\alpha^2 \sum_i x_i}{2(1-\alpha)(2M-1)} + \epsilon \right) \\
&< \tfrac{\alpha}{2} + \chi + \tfrac{\alpha}{2(1-\alpha)} \sum\nolimits_i x_i \\
&< \tfrac{1}{2} + \tfrac{1}{2} \sum\nolimits_i x_i - \tfrac{1}{2} + \tfrac{\alpha}{2(1-\alpha)} \sum\nolimits_i x_i \\
&= \tfrac{1}{2(1-\alpha)} \sum\nolimits_i x_i,
\end{aligned}$$

where the first strict inequality is derived by $\sum_{i \notin X'} p_{v_i,S} < \frac{1}{2}$ and $\left(\epsilon - \frac{\alpha^2 \sum_i x_i}{2(1-\alpha)(2M-1)} \right) < 0$, while the last strict inequality holds since $0 \leq \chi \leq \frac{1}{2} \sum_i x_i - \frac{1}{2}$. Thus, we then conclude that firm T is in default, as desired.

Now, we move into the last step of the proof and show that $a_S < \frac{\sum_i x_i}{2(1-\alpha)}$. Indeed, as T is in default, we have $a_S = \frac{1}{2} \sum_i x_i - \chi + p_{T,S} < \frac{1}{2} \sum_i x_i + p_{T,S} < \frac{1}{2} \sum_i x_i + l_{T,S} \cdot \alpha = \frac{1}{2} \sum_i x_i + \frac{\sum_i x_i}{2(1-\alpha)} \alpha = \frac{\sum_i x_i}{2(1-\alpha)}$. This completes the proof. □

With Lemma 2 at hand, we have a stronger result for the contrapositive statement. Thus, the forward direction of the if and only if statement holds.

b) **Hardness of computing the best response.** The proof of this claim follows by the previous proof. Indeed, since firm S is the only firm that can strategize, we have that finding the best response of S is equivalent to verifying if a strategy profile is a PSNE or not.

c) **Hardness of deciding if there exists a PSNE.** The proof is based on Theorem 6b and Theorem 4 by simply connecting the corresponding gadgets. In particular, we directly connect firm T to firm w_7 with a amount of liability of ϵ and we also set $\lambda = \frac{2}{\alpha} - \left(4 + \alpha + \alpha^2 + \frac{2+2\alpha}{\alpha(1-\alpha)}\right) \cdot \frac{1}{M}$; see also Figure 7. We denote by G_w the subnetwork induced by all w_i's and all their liabilities, while $G \setminus G_w$ is the subnetwork used also in the proof of Theorem 6b. Note that there is a single directed edge connecting $G \setminus G_w$ and G_w, i.e., the liability from firm T to w_7. Hence, any deviation by some w_i does not affect the utility of firm S. Furthermore, despite firm T being qualified to prepay w_7, it would not engage in such prepayment at equilibrium because there is no directed path from G_w back to T, rendering such a prepayment ineffective in increasing T's total assets, while incurring a cost of ϵ.

By the proof of Theorem 4, we know that subnetwork G_w cannot reach a NE if $a_{w_7} < \frac{2}{\alpha} - \left(4 + \alpha + \alpha^2 + \frac{2+2\alpha}{\alpha(1-\alpha)}\right) \cdot \frac{1}{M}$, i.e., if $p_{T,w_7} < \epsilon$.

Additionally, by Theorem 6b firm S's best response leads to $a_S = \frac{2-\alpha}{2(1-\alpha)} \sum_{i \in X} x_i$ and $a_T = \frac{1}{2(1-\alpha)} \cdot \sum_{i \in X} x_i + \epsilon$, thereby $p_{T,w_7} = \epsilon$ if and only if there is solution to the instance of Partition. By the discussion above, since $p_{T,w_7} = \epsilon$ if and only if T is exactly solvent, this suffices for our claim to hold.

□

4.2 Maximizing Equity

We now turn our attention to prepayment games where utility is defined by equity. In contrast to the negative results considering total assets, we illustrate that prepayment games with equity-based utilities possess several desirable properties. To begin, we establish the existence of PSNE, even in the presence of default costs.

Theorem 7. *In the equity setting, for all $\alpha \in [0, 1]$, any prepayment profile is a PSNE.*

Proof. Consider a fixed prepayment profile, $\mathbf{s}_{-i}$, for all firms except v_i. We assume that firm v_i can unilaterally increase its equity by deviating from strategy s_i to s'_i, where $s_i \neq s'_i$, so $E'_i > E_i \geq 0$. Further, firm v_i remains solvent, capable of settling all its debts under the strategy profile $\mathbf{s}' = (s'_i, \mathbf{s}_{-i})$. Recall that C_i represents the set of v_i's lenders, and let $C'_i(s_i)$ and $C'_i(s'_i)$ be the set of selected lenders prepaid by v_i under s_i and s'_i, respectively.

We commence with a financial network represented by G', comprising nodes $\{v_1, \dots, v_n\}$ and edges reflecting the strategy profile $\mathbf{s}'$. We then introduce a liability l_{ij}, which is a new financial commitment that firm v_i owes to firm v_j. Further, for each lender in $C'_i(s'_i) \setminus C'_i(s_i)$, we increase firm v_i's external assets and decrease firm v_j's external assets by the amount of l_{ij}. In parallel, we eliminate the liability l_{ij}, reduce firm v_i's external assets, and increase firm v_j's external assets by l_{ij} for lenders in $C'_i(s_i) \setminus C'_i(s'_i)$. This step-by-step process ensures the solvency of firm v_i at each juncture, without altering the payments firm v_j receives in C_i, which consistently equates to the liability l_{ij}. As a result of these adjustments, the

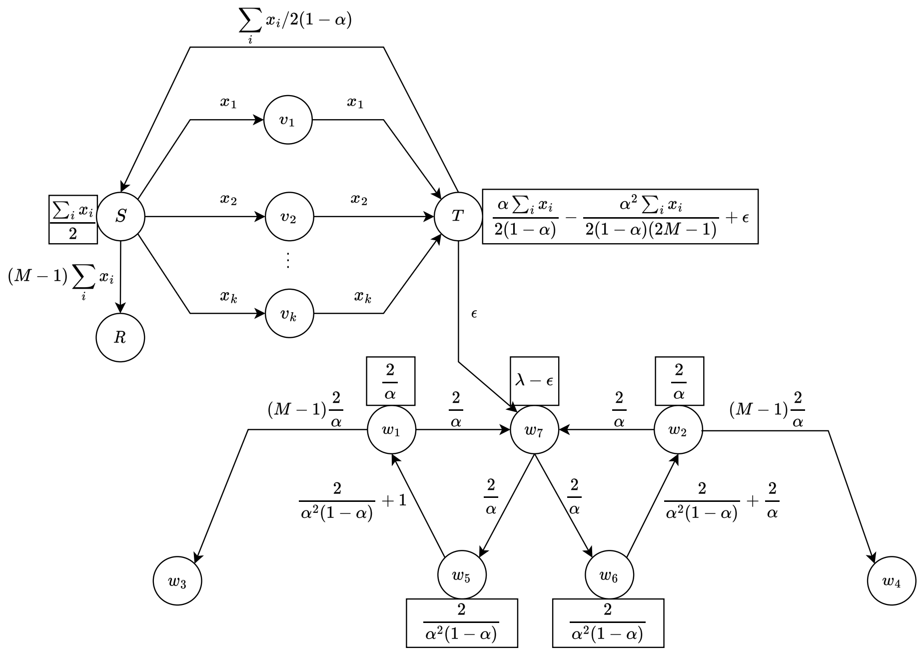


Figure 7: An example of the reduction used in the proof of Theorem 6c where $\lambda = \frac{2}{\alpha} - \left(4 + \alpha + \alpha^2 + \frac{2+2\alpha}{\alpha(1-\alpha)}\right) \cdot \frac{1}{M}$.

financial network G' is gradually transformed back to G, corresponding to the strategy profile $\mathbf{s} = (s_i, \mathbf{s}_{-i})$. This meticulous process guarantees that the equity of firm v_i remains unchanged, thereby preserving the flow of payments within the rest of the network, including those returning to firm v_i. Thus, $E_i' = E_i$, which is a contradiction. Note that this argument holds under any pure strategy profile $\mathbf{s}$, implying that any pure strategy profile is a PSNE. The proof is complete. □

According to Theorem 7, the strategy profile with the maximum total equity is a PSNE, which implies that:

Corollary 1. *In the equity setting, for all $\alpha \in [0, 1]$, the Price of Stability is* 1.

Although the best NE consistently yields the global optima, we then show that the worst pure equilibria can deviate significantly from the optimal scenario in terms of total equity.

Theorem 8. *In the equity setting, for all $\alpha \in [0, 1)$, the Price of Anarchy is unbounded.*

Proof. Consider the network as shown in Figure 8. Clearly, only v_1 can strategize prepayments. Since v_1 is always in default with zero equity, any strategy selected by v_1 would admit a NE. When v_1 decides not to make any prepayments, corresponding to the original network, all firms except for v_4 are in default and the clearing payments are

$$p_{12} = p_{14} = \alpha, \;\; p_{23} = \frac{M(M-1)\alpha}{M-(M-1)\alpha^2}, \;\; p_{32} = \frac{(M-1)^2\alpha^2}{M-(M-1)\alpha^2}, \text{ and } p_{34} = \frac{(M-1)\alpha^2}{M-(M-1)\alpha^2}$$

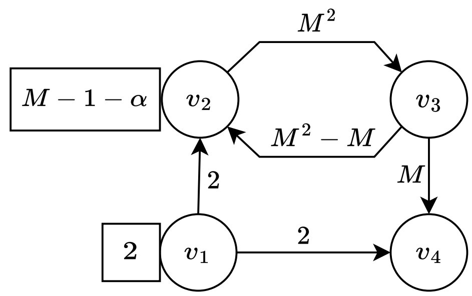


Figure 8: A prepayment game with default costs $\alpha \in [0, 1)$ with unbounded Price of Anarchy when players maximize their equities.

with $E_1 = E_2 = E_3 = 0$ and $E_4 = \frac{(M-1)\alpha^2}{M-(M-1)\alpha^2}$, resulting in a social welfare of $\frac{(M-1)\alpha^2}{M-(M-1)\alpha^2}$. Further, when v_1 prepays v_4, then there would be no cash flows on the edges between v_1, v_2 and v_3, that is $p'_{12} = p'_{23} = p'_{32} = p'_{34} = 0$. On the other hand, v_4 archives 2 units of additional external assets, resulting in a social welfare of 2. Lastly, if v_1 prepays v_2, then v_2, v_3, and v_4 would be solvent with $E_2 = 1$, $E_3 = 0$, and $E_4 = M$, then we obtain a social welfare of $M + 1$. The claim follows since M is arbitrarily large.

□

Regarding the computational aspects, Theorem 7 ensures that it is trivial to compute the best response for a given strategy profile and determine whether pure equilibria exist or not. Nevertheless, the problem of computing the best PSNE remains **NP**-hard – a corollary of Theorem 2.

Corollary 2. *In the equity setting, for all $\alpha \in [0, 1)$, computing the best PSNE is* **NP***-hard.*

Proof. Consider the financial credit network as presented in Figure 3. According to Lemma 1, the total equity is bounded by $2\sum_{i\in X} x_i$, whereas Theorem 7 asserts that any pure strategy profile is a PSNE when firms maximize equity. Hence, the payment profiles achieving total equity of $2\sum_{i\in X} x_i$ are precisely the best PSNE. This implies that the decision problem of determining if there is a prepayment profile with total equity of $2\sum_{i\in X} x_i$ corresponds exactly to finding the best pure equilibrium. Finally, by the proof of Theorem 2, we know that this decision problem is equivalent to the question if there is a solution to Partition, so we conclude. □

5. Empirical Game-Theoretic Analysis

We now present our empirical analysis of prepayments on synthetic networks. Given the computational challenges in determining NE in prepayment games involving multiple firms, as indicated in Theorem 6, we employ EGTA, a process that engages in game-theoretic reasoning through extensive simulation[5].

5. The source code and data used for the simulations and experiments of this paper are publicly available at https://github.com/haozhou-egta/prepayment-game.

5.1 Method

Our approach begins by proposing a set of heuristic prepayment strategies, which, in contrast to the theoretical results in Section 4, *rely solely on local information* that would definitely be available to firms, such as claims (expected incoming payments), liabilities, and external assets. This is *realistic* and mirrors real-world financial scenarios where firms typically lack awareness of the external assets and liabilities of other market participants. Consequently, each strategy becomes a mapping from the local information to the set of lenders to prepay. Additionally, each strategy is inherently designed to adhere to feasibility constraints. The individual strategies are outlined below:

No Operation ($\mathbf{h_1}$): Make no prepayment.

Random ($\mathbf{h_2}$): Uniformly prepay a subset of lenders.

Max Claim ($\mathbf{h_3}$): Prepay the lender who has the highest claim to itself.

Max Claim Greedy ($\mathbf{h_4}$): Prepay all lenders, ordered by the claims, until the external assets are exhausted.

Heuristic Belief ($\mathbf{h_5}$): Estimate the external assets of lenders[6]. If the sum of the prepayment to a lender and its asset estimate is larger than the claim from the lender *and* this does not hold under their estimated payment without prepayment, then prepay the lender.

Random with Solvency Check ($\mathbf{h_6}$): If the liability is greater than the sum of claims and the external assets, employ the random strategy h_2. Otherwise, take no-operation strategy h_1.

Max Claim with Solvency Check ($\mathbf{h_7}$): If the liability is greater than the sum of claims and the external assets, employ the max-claim-greedy strategy h_3. Otherwise, take no-operation strategy h_1.

Max Claim Greedy with Solvency Check ($\mathbf{h_8}$): If the liability is greater than the sum of claims and the external assets, employ the max-claim-greedy strategy h_4. Otherwise, take no-operation strategy h_1.

Heuristic Belief with Solvency Check ($\mathbf{h_9}$): If the liability is greater than the sum of claims and the external assets, employ the heuristic-belief strategy h_5. Otherwise, take no-operation strategy h_1.

Among these strategies, the "no-operation" strategy and the "random" strategy correspond to scenarios without prepayment and those involving random behaviors, respectively. The "max-claim" strategy and the "heuristic-belief" strategy consider whether a firm could benefit from prepayments. In essence, a firm should only engage in prepayments to lenders

6. A firm can estimate the external assets of other firms (e.g., based on annual financial reports). This estimate is given by imposing a noise sampled from a normal distribution with zero mean and standard deviation equal to 5 on the external assets. The estimated payment without prepayment is obtained by discounting the liability with a discount factor, viewed as a hyperparameter.

within a debt circle that leads back to itself, preventing the insolvency of lenders. Due to the limited local observation of individual firms, it is challenging to identify lenders on this cycle (i.e., which lenders to prepay). Therefore, heuristics become crucial in making such decisions.

The "max-claim" strategy opts to prepay the lender with the highest claim (i.e., the firm owing the most). In contrast, the "heuristic-belief" strategy estimates other firms' external assets (e.g., based on annual financial reports) and prepays only lenders with insufficient estimated external assets to cover the claim. Essentially, the "heuristic-belief" strategy estimates whether prepayments would make a difference.

We then refine these strategies (h_2-h_5) and introduce strategies h_6-h_9 by incorporating an additional condition that checks the solvency of the firm engaging in prepayment. Since a firm can only observe the claims rather than the actual incoming payments, which could be less than the claims, a firm may still turn out to be insolvent even if the solvency check passes. However, if the solvency check fails, a firm will undoubtedly be insolvent. Adding a solvency check is rooted in the fact that prepayments, when made by a firm believing itself to be insolvent, are not discounted by α. Consequently, engaging in prepayments under these circumstances injects more cash flow into the network, potentially benefiting the entire financial system and the firm itself.

Taking an ex-ante perspective, we model the prepayment game as a symmetric game, where firms share the same strategy set and utility function. The utility function is based on total assets rather than equity, as every profile serves as a PSNE with equity as the utility function, as per Theorem 7. In this model, a firm aims to maximize its expected utility with respect to a network generator that defines the joint distribution of liabilities and external assets. The details about the network generator are shown in Table 3.

In a symmetric game, it is sufficient to outline a strategy profile by the number of players employing each strategy. In our EGTA approach, given a strategy profile, expected payoffs are calculated by averaging the payoffs over individual instances sampled from the network generator. For each individual instance, the payoffs are determined by the clearing of a corresponding stable network—meaning no firm would make further prepayments to others. This stability is achieved by executing the strategies in the profile simultaneously and continuously.

As the number of firms and strategies in a game grows, conducting a thorough game-theoretic analysis becomes computationally impractical. To address this challenge, we employ an aggregation technique, known as deviation-preserving reduction (DPR) (Wiedenbeck & Wellman, 2012), to approximate complex many-player games as simpler ones with fewer participants. DPR has been widely applied in the analysis of various AI applications, including for financial systems (Wah, Wright, & Wellman, 2017; Wang & Wellman, 2017; Wright & Wellman, 2018). In DPR, each player views itself as controlling a single player in the full game, but views the profile of opponent strategies in the reduced game as an aggregation of all other players in the full game. Formally, denote a normal-form symmetric game $\mathcal{G} = (N, (S), (u))$ as a tuple of a finite set of players N; a non-empty shared set of strategies S; and a utility function u. A reduced game $DPR_k = (k, S, u')$, where k is the number of players in the reduced game, S is the strategy set same as the original game, and the utility function of the reduced game

$$u'(s, \langle c_1 \times s_1, \ldots, c_s \times s, \ldots \rangle) = u\left(s, \left\langle \tfrac{N-1}{k-1} c_1 \times s_1, \ldots, \left[\tfrac{N-1}{k-1}(c_s - 1) + 1\right] \times s, \ldots \right\rangle\right)$$

wherein c_j is the number of players choosing s_j. With player reduction, (Wiedenbeck & Wellman, 2012) proved that a profile is the symmetric NE of the reduced game if and only if its corresponding profile in the full game is a symmetric NE. In this study, we simplify prepayment games with $N = 10$ firms and 9 strategies to games with $k = 4$ firms and 9 strategies. Note that we identify symmetric NE of the reduced game, in lieu of being able to find NE of the original game.

5.2 Game Setup and Parameters

We study prepayment games under various configurations of credit-network generators. A network generator creates a credit-network instance by sampling a configuration from the distribution prescribed in Table 3, yielding various network typologies, different scales of external assets (i.e., uniform distributions over scale intervals), and liabilities. We also sample a subset of firms with *shocks*, assigned by the lower bound of the external-asset distribution as their external assets. Shocks model situations where the financial condition of a firm is poor.

We consider two types of utility functions: total assets and equity. With each utility function, a pure-strategy profile in the reduced game will be evaluated by averaging the payoffs over 1000 instances sampled from the generator. Once the reduced game has been fully evaluated, we enumerate all pure-strategy profiles to search for PSNE and apply replicator dynamics (RD)[7] to approximate mixed-strategy NE.

Parameters	Values
Number of Firms	10
Number of Edges	$U(0, 9)$ for each firm
Number of Strategies	9 for each firm
Utility Function	{Total Asset, Equity}
External Assets	$\{U(0, 40),\ U(40, 70),\ U(70, 120)\}$
Liabilities	$U(0, x)$, for $x \in \{10, 20, 35\}$
Shocks	$U(0, 5)$

Table 3: Prepayment game parameters.

5.3 Experimental Results

We begin by calculating the NE for the prepayment games generated by the network generator with external assets ranging from 40 to 70. Our findings reveal the existence of a mixed-strategy NE, wherein all firms decide between the random strategy h_1 and the heuristic-belief with solvency check strategy h_9. As we varied the default cost discount α,

7. Given that the solution from replicator dynamics may depend on the initial state, we consistently initialize with a uniform distribution over all pure strategies.

we observed a reduction in the probability mass on h_9 as α approaches 1. This trend is attributed to the diminishing benefits of prepayments as α nears 1. As α becomes closer to 1, firms are less inclined to choose prepayment, as the immediate loss incurred outweighs the damage caused by insolvency. The explanation of this immediate loss will be provided later in this section. In the extreme case where α equals 1, we observed that refraining from prepayment becomes a PSNE, and the probability of playing h_9 drops to zero, as per Theorem 3.

Figure 9 illustrates the equilibrium measures across varying αs. The considered measures encompass PoA, PoS, *Effect of Anarchy* (EoA), and *Effect of Stability* (EoS), as defined in (Kanellopoulos et al., 2022). EoA and EoS represent the ratio of the initial-state social welfare to the social welfare of the worst, and best, respectively, NE. They serve as metrics for evaluating the performance of NE in comparison to making no prepayment. Upon examination of Figure 9, we initially observed that both EoA and EoS are below 1. This indicates that making prepayments enhances social welfare compared to refraining from prepayments. Subsequently, all curves converge to a horizontal line with a value of 1 as α increases. This convergence is attributed to the diminishing benefits of making prepayments. Ultimately, abstaining from prepayment becomes the social optimum.

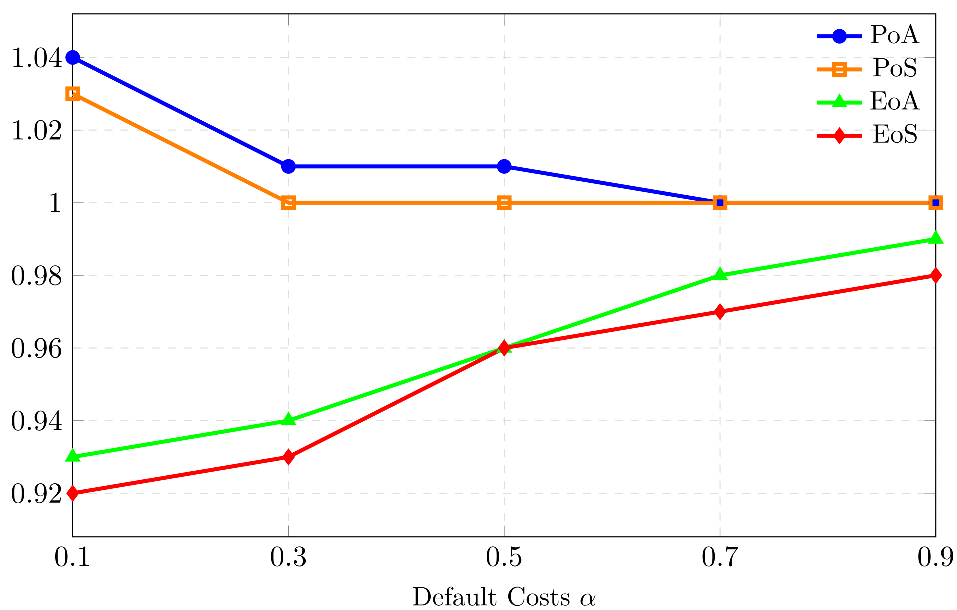
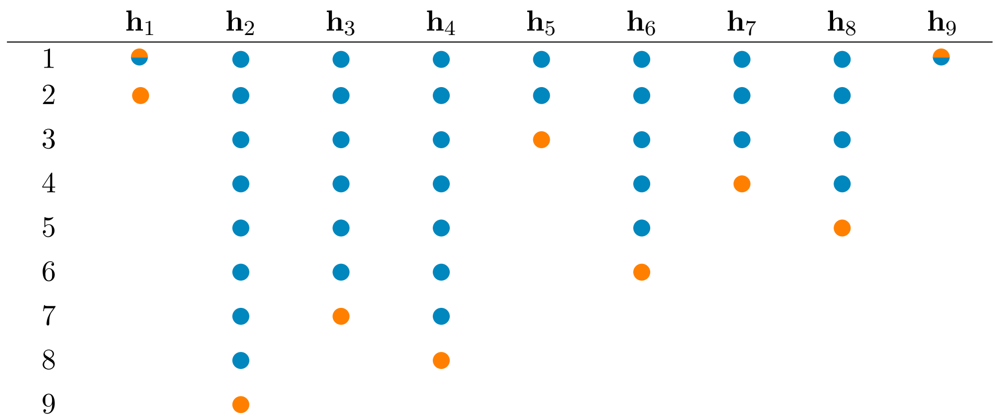


Figure 9: Sensitivity analysis on equilibrium measures with respect to default costs. PoA(S) and EoA(S) are computed based on the ex-ante symmetric game, with each entry of the payoff matrix representing the average payoff simulated across 1000 generated games.

Through our experiments, we observed that opting not to make prepayments emerges as a robust strategy when compared to other heuristic prepayment approaches. This strength stems from the fact that making prepayments incurs an immediate loss in a firm's external assets, consequently impacting its utility. The viability of prepayments depends on their substantial influence on the well-being of firms within a debt circle that includes the concerned firm, wherein the immediate loss could be offset by the resulting benefit. Given the stringent conditions required for prepayments to be effective (e.g., the existence of debt circles, prepayments making a significant difference to lenders, and the benefit of a favorable prepayment outweighing the costs of all other non-beneficial ones), choosing not to make prepayments can easily outperform simplistic heuristics.

To assess the relative effectiveness of all strategies except h_1, we conduct an ablation study with $\alpha = 0.5$, as depicted in Figure 10. Specifically, we iteratively eliminate equilibrium strategies, starting from h_9, from the current strategy set and recompute NE. In Figure 10, the orange dots signify NE, the blue dots denote strategies outside the NE support, and the absence of a dot indicates the removal of a strategy. Each row in the figure represents a game obtained by removing the NE strategy (pure NE exists in row 2 to row 9) from the preceding row. The earlier a strategy appears in an NE, the stronger its strategic position.



$$\mathbf{h}_9, \mathbf{h}_1 \succ \mathbf{h}_5 \succ \mathbf{h}_7 \succ \mathbf{h}_8 \succ \mathbf{h}_6 \succ \mathbf{h}_3 \succ \mathbf{h}_4 \succ \mathbf{h}_2$$

Figure 10: An ablation study for the performance of strategies. The orange circles represent NE and the blue ones indicate strategies outside the NE support. Gradient colors identify the mixed strategies. Each row represents a game given by removing NE strategy from the previous row. The strength of strategies is ordered.

Our initial observation underscores the significance of heuristics in determining which lender to prepay, as evidenced by the superior performance of h_3, h_4, and h_5 compared to the random strategy h_2. This trend persists even when applying the solvency check, as strategies h_7, h_8, and h_9 all outperform the random strategy h_6. Upon comparing the performance of h_3 and h_4, we discerned that making one prepayment at a time might be more advantageous than making multiple ones, primarily due to the potential loss incurred by non-beneficial prepayments. Additionally, it becomes apparent that strategies h_7 and h_8 do not participate in an NE if h_9 remains in the strategy set. This underscores the importance of considering the estimated external assets of other firms in the decision-making process.

Moreover, we observed that strategies incorporating the solvency check outperform those without. This finding aligns with intuition, as only insolvent firms stand to increase their utilities through prepayments. The solvency check effectively prevents a solvent firm from engaging in prepayments, a scenario that would otherwise result in an immediate loss of external assets for the firm.

Finally, we made more aggressive adjustments to the configurations of network generators, enabling either insufficient or excessive external assets. In both scenarios, we observed a lack of prepayment. When external assets are insufficient, no firm has the capacity to engage in prepayments. Similarly, when the entire network is solvent with an ample amount of external assets, firms have no incentive to make prepayments.

To sum up, the crux of making advantageous prepayments lies in the fact that prepayments made by insolvent firms are not discounted by α. This implies that a greater amount of money flows into the network, potentially benefiting other firms. The payoff from prepayments occurs when there is a path for the money to circulate back, covering the immediate loss. Consequently, making prepayments is meaningful only if the firm engaging in prepayment is insolvent, and appropriate debt circles exist.

6. Conclusion and Discussion

We examine the characteristics and impacts of prepayments in financial credit networks through both theoretical analysis and empirical study. We evaluate the computational complexity of identifying prepayments that maximize welfare from a centralized standpoint. Additionally, we delve into the behavior of individual firms from a game-theoretic perspective, outlining the strengths and weaknesses of different heuristics. Our findings indicate that prepayments can be advantageous for financial systems, particularly in situations where the insolvency of firms happens.

Building upon the existing model, our analysis leaves a number of compelling questions unresolved. For instance, there remains a need for a conclusive answer regarding the existence of PSNE when firms seek to maximize their total assets under the condition $\alpha = 0$. Additionally, our discrete treatment of prepayments, wherein firms either fully prepay or refrain entirely, prompts an interesting trajectory for further investigation. Extending the prepayment game model to encompass fractional prepayments will make our analysis more realistic. Furthermore, while our theoretical analysis utilizes the PSNE solution concept, an unexplored avenue entails the examination of prepayment games under the mixed Nash equilibria. The limitations and structural assumptions inherent in our analysis give rise to additional avenues for future exploration. In particular, since in our model firms act in a solitary manner, a trajectory for future work involves exploring scenarios where a firm enhances its utility by incentivizing another firm to prepay. This injects the prepayment games into the area of Cooperative game theory and gives rise to many intriguing research questions such as the outcome of the interaction and the formation of coalitions. Taking a different point of view, there are many intriguing scenarios when we study prepayment games through the lens of a central authority (e.g. regulator, mediator, etc). For instance, it is worth investigating methodologies and centralized algorithms in order to improve the efficiency of financial networks.

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