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On the Consistency between Belief Revision and Belief Update

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Abstract

Belief revision and belief update are two fundamental and well-studied processes of belief change. In the present article, we introduce a consistency principle which dictates that the revision and update policies employed by a rational agent are not independent, but ought to be related in a certain coherent way. We formalize our consistency principle both axiomatically and semantically, and we establish a representation result explicitly connecting the two formalizations. Furthermore, we show that two important concrete types of belief change, namely uniform belief change and parametrized-difference belief change, serve as proof-of-concept examples for the introduced consistency principle, as they fully comply with it. Additionally, we identify an intriguing property of uniform belief change in which revision and update become indistinguishable when an epistemic input contradicts the initial state of belief, as both processes produce identical outcomes. Lastly, it is shown that, unlike parametrized-difference belief change, uniform belief change is incompatible with Parikh's notion of relevance. Consequently, building on previous results, it is demonstrated that parametrized-difference belief change is relevance-sensitive —indicating that the proposed principle of consistency is compatible with relevance— while uniform belief change is not.

1. Introduction

An intelligent agent should be capable of gathering information about the world, and modifying their state of belief in response to new evidential information. As a consequence, the agent should be able to implement *belief change*, an operation which is heavily studied in the realm of Artificial Intelligence (AI) (Fermé & Hansson, 2018). Indicatively, in the context of AI-powered *digital twins*, where virtual models of physical systems continuously receive and process data from their real-world counterparts, belief change is essential for maintaining the accuracy and reliability of the virtual models (Jiang, Yin, Li, Luo, & Kaynak, 2021; Tao, Zhang, & Zhang, 2024).

Two fundamental processes of belief change are *belief revision* and *belief update*. Belief revision (or simply revision) is the process by which a rational agent modifies their beliefs about a *static* world, in the light of new information. Belief update (or simply update), on the other hand, is the process by which a rational agent keeps their beliefs up to date with a *dynamic* (evolving) world. Therefore, in a sense, belief update concerns changes in the state of the world, whereas belief revision concerns changes in the perception (description) of the state of the world. The subsequent example provides an intuition of the conceptual difference between these two processes of belief change (Winslett, 1988).

Initially, a rational agent knows that there is either a book on the table, or a magazine on the table, but not both.

- *Belief revision: The agent is told that there is a book on the table. Thus, she concludes that there is no magazine on the table.*
- *Belief update: The agent is told that, subsequently, a book has been put on the table. The agent should not conclude that there is no magazine on the table.*

Belief revision was formalized by Alchourrón, Gärdenfors, and Makinson (1985) in a cornerstone work, in which a versatile framework was introduced, now called the *AGM framework* after the initials of its three founders. Although the difference between belief revision and belief update was first recognized by Keller and Winslett (1985) in the context of relational databases, belief update was formally captured by Katsuno and Mendelzon (1992), who, following the AGM tradition, built a concrete framework, called the *KM framework*, that models the belief-update process. Within the AGM and KM frameworks, the processes of belief revision and belief update are modelled as *binary functions* that map *belief sets* (i.e., logical theories) and *epistemic inputs* (i.e., formulae that represent new information) to modified (new) belief sets. The *rational* revision and update operators are characterized by a collection of *rationality postulates*, as well as by specific *constructive models*, which essentially are formalizations that provide "recipes" for specifying particular revision and update strategies.

A few years after the formal introduction of belief update by Katsuno and Mendelzon (1992), Peppas, Nayak, Pagnucco, Foo, Kwok, and Prokopenko (1996) studied its relation to belief revision in more depth. In that work, some interesting results pinpointed that the difference between these two types of belief change is not so much on the modification of a *fixed* belief set, but rather in the way that modifications at *different* belief sets relate to one another. This was shown by proving that, for an arbitrary but fixed belief set K, revising K is *much the same* as updating K. Furthermore, the authors of that work argued that, in the realm of belief update, the comparative plausibility of possible states of affairs is *ontological*, in the sense that it is an intrinsic, eternal property of the universe, which is dependent on the physics of the domain and independent of any cognitive attribute of an agent. This is in contrast with the process of belief revision, in the context of which the comparative plausibility of possible states of affairs retains an *epistemic* nature, by being dependent on the current beliefs of the agent.[1]

In this article, following the route of Peppas et al. (1996), we shed more light on the relationship between belief revision and belief update. Specifically, the following contributions are made:

- We introduce, both *axiomatically* and *semantically*, a principle of *consistency* between belief revision and belief update, ensuring alignment/coherence between the revision strategy and the update strategy of a *single* rational agent. This principle is established

1. Interesting links between update and revision have been discussed in other notable works as well —see indicatively the studies by Lang (2007), de Saint-Cyr and Lang (2011), del Val and Shoham (1994)— which, however, deviate from the approach followed herein.

by requiring that, from the agent's perspective, the epistemic comparative plausibility of possible worlds (governing belief revision) remains *consistent* with their ontological comparative plausibility (governing belief update). In more formal terms, our consistency principle encodes the following natural schema: If *every* possible world satisfying a theory K assigns a *certain* comparative plausibility for two states of affairs (within belief update), then this *same* comparative plausibility should also be perceived by theory K (within belief revision).

- In the presence of the aforementioned consistency principle, belief revision and belief update are *indistinguishable* in the realm of consistent *complete* belief sets (i.e., consistent theories from which every formula of the language or its negation is demonstrable), as both processes lead to *identical* outcomes.

- We show that two important concrete types of belief change —namely *uniform* belief change (Areces & Becher, 2001; Aravanis, 2020) and *parametrized-difference* belief change (Peppas & Williams, 2018; Peppas, Williams, & Antoniou, 2024; Aravanis, Peppas, & Williams, 2021; Aravanis, 2021)— respect the introduced consistency principle; in other words, these types of belief change constitute *proof of concepts* for the proposed notion of consistency.[2] What is particularly satisfying about this outcome is that both uniform and parametrized-difference belief change exhibit highly favourable properties that are essential for practical belief-change applications. In particular, they are simple and intuitive, making them easily applicable. Moreover, they are compactly specified, as a single total preorder encoding domain-dynamics is sufficient to uniquely determine an agent's modification strategies. Additionally, they inherently provide a solution to the iterated-revision problem (Peppas, 2014).

- Although uniform belief revision has already been introduced by Areces and Becher (2001), uniform belief update is defined herein. Furthermore, an interesting property of uniform belief change is identified; that is, uniform belief revision and uniform belief update by epistemic input that *contradicts* the initial state of belief are *indistinguishable*. This result is in the spirit of the main result of Peppas et al. (1996).

- Lastly, we prove that uniform belief change is *incompatible* with Parikh's notion of *relevance* (Parikh, 1999, 2011). Therefore, in view of previous results of Peppas and Williams (2016) and Aravanis (2021), it follows that parametrized-difference belief change is relevance-sensitive, whereas uniform belief change is not. Perhaps more importantly, given that parametrized-difference belief change satisfies the introduced consistency principle, it follows that the proposed concept of consistency is *compatible* with Parikh's notion of relevance.

The rest of the article is organized as follows. The next section establishes the formal background for our discussion. Thereafter, Sections 3 and 4 discuss belief revision and belief update, respectively. Section 5 presents the already established results on the

2. Throughout this article, the term "belief change" refers both to the process of belief revision *and* to the process of belief update. Hence, for example, uniform belief change accommodates uniform belief revision *and* uniform belief update.

connection between revision and update, as identified by Peppas et al. (1996). Subsequently, Section 6 introduces our consistency principle, whereas Sections 7 and 8 present uniform and parametrized-difference belief change, respectively, as proof of concepts for the introduced consistency principle. Section 9 explores the relevance-sensitivity of uniform and parametrized-difference belief change, and, as a by-product, evaluates the relevance-sensitivity of the proposed principle of consistency. The article closes with a brief concluding section.

2. Formal Preliminaries

We shall work with a propositional language $\mathcal{L}$, built over a *finite*, non-empty set $\mathcal{P}$ of atoms (propositional variables), using the standard Boolean connectives $\land$ (conjunction), $\lor$ (disjunction), $\rightarrow$ (implication), $\leftrightarrow$ (equivalence), $\neg$ (negation), and governed by *classical propositional logic*. The classical consequence relation is denoted by $\models$.

A sentence φ of $\mathcal{L}$ is *contingent* iff it is neither a tautology nor a contradiction. For a set of sentences Γ of $\mathcal{L}$, $Cn(\Gamma)$ denotes the set of all logical consequences of Γ; i.e., $Cn(\Gamma) = \{\varphi \in \mathcal{L} : \Gamma \models \varphi\}$. An agent's belief corpus shall be modelled by a *theory*, also referred to as a *belief set*. A theory K is a deductively closed set of sentences of $\mathcal{L}$, meaning that $K = Cn(K)$. The set of all theories is denoted by $\mathbb{T}$. A theory K is *complete* iff, for every sentence $\varphi \in \mathcal{L}$, either $\varphi \in K$ or $\neg\varphi \in K$. Hence, a consistent complete theory provides a maximal (complete) description —within the expressive limits of the language $\mathcal{L}$— of what is true or false in a certain state of the world. For a theory K and a sentence φ, the *expansion* of K by φ, denoted by $K + \varphi$, is defined as

$$K + \varphi = Cn\big(K \cup \{\varphi\}\big).$$

A *literal* is an atom $a \in \mathcal{P}$ or its negation. For the sake of readability, the negation of an atom a shall occasionally be represented as $\overline{a}$, instead of $\neg a$. For a set of literals Q, $|Q|$ denotes the cardinality of Q. A *possible world* (abbrev. *world*) r is a consistent set of literals, such that, for every atom $a \in \mathcal{P}$, either $a \in r$ or $\neg a \in r$. The set of all possible worlds is denoted by $\mathbb{M}$. Possible worlds shall often be represented as sequences (rather than sets) of literals. For a sentence (set of sentences) φ of $\mathcal{L}$, $[\varphi]$ is the set of worlds at which φ is true. It can be easily verified that, for any consistent complete theory $K \in \mathbb{T}$, there is a world $w \in \mathbb{M}$ such that $[K] = \{w\}$.

A *preorder* over a set V is a reflexive, transitive binary relation in V. A preorder $\preceq$ is called *total* iff, for all $r, r' \in V$, $r \preceq r'$ or $r' \preceq r$. The strict part of $\preceq$ is denoted by $\prec$; i.e., $r \prec r'$ iff $r \preceq r'$ and $r' \not\preceq r$. The indifference part of $\preceq$ is denoted by $\approx$; i.e., $r \approx r'$ iff $r \preceq r'$ and $r' \preceq r$. Furthermore, $min(V, \preceq)$ denotes the set of all $\preceq$-minimal elements of V; i.e.,

$$min(V, \preceq) = \Big\{r \in V : \text{for all } r' \in V, \text{ if } r' \preceq r, \text{ then } r \preceq r'\Big\}.$$

3. Belief Revision

In this section, we present a brief outline of the AGM framework. We first introduce its axiomatic (postulational) characterization as formulated by Alchourrón et al. (1985), and thereafter, we discuss its semantic characterization in terms of total preorders over possible worlds, as developed by Katsuno and Mendelzon (1991).

3.1 Axiomatic Characterization

A *revision operator* is a function $*$ that maps a theory K and a sentence φ to a new theory $K * \varphi$, representing the result of revising K by φ. We shall say that a revision operator $*$ is an *AGM revision operator* iff it satisfies the following postulates, known as *AGM postulates* (Alchourrón et al., 1985).

($\mathbf{K * 1}$) $K * \varphi$ is a theory.

($\mathbf{K * 2}$) $\varphi \in K * \varphi$.

($\mathbf{K * 3}$) $K * \varphi \subseteq K + \varphi$.

($\mathbf{K * 4}$) If $\neg\varphi \notin K$, then $K + \varphi \subseteq K * \varphi$.

($\mathbf{K * 5}$) $K * \varphi = \mathcal{L}$ iff φ is inconsistent.

($\mathbf{K * 6}$) If $Cn(\{\varphi\}) = Cn(\{\psi\})$, then $K * \varphi = K * \psi$.

($\mathbf{K * 7}$) $K * (\varphi \wedge \psi) \subseteq (K * \varphi) + \psi$.

($\mathbf{K * 8}$) If $\neg\psi \notin K * \varphi$, then $(K * \varphi) + \psi \subseteq K * (\varphi \wedge \psi)$.

The guiding principle behind the AGM postulates —whose rationale is discussed by Gärdenfors (1988, Section 3.3) and Peppas (2008, Section 8.3.1)— is the *economy of information*, according to which the epistemic input φ must be consistently incorporated into the initial belief set K, altering it as little as possible. Note that, in the special case where φ is *consistent* with K, it follows from postulates $(K * 3)$ and $(K * 4)$ that revision reduces to expansion, meaning that $K * \varphi = K + \varphi$.

3.2 Semantic Characterization

Katsuno and Mendelzon (1991) demonstrated that the revision operators satisfying postulates $(K * 1)$–$(K * 8)$ are precisely those that are induced by total preorders over all possible worlds.

Definition 1 (Faithful Preorder Associated with Theories, Katsuno & Mendelzon, 1991)**.** *A total preorder $\preceq_K$ over $\mathbb{M}$ is faithful to a theory K iff, for any $r, r' \in \mathbb{M}$, the following two conditions hold:*

(i) *If $r \in [K]$, then $r \preceq_K r'$.*

(ii) *If $r \in [K]$ and $r' \notin [K]$, then $r \prec_K r'$.*

Intuitively, $r \preceq_K r'$ holds precisely when the world r is at least as plausible as the world r', with respect to K.

Definition 2 (Faithful Assignment, Katsuno & Mendelzon, 1991)**.** *A faithful assignment is a function that maps each theory K of $\mathbb{T}$ to a total preorder $\preceq_K$ over $\mathbb{M}$, that is faithful to K.*

The following theorem by Katsuno and Mendelzon (1991) characterizes the class of revision operators induced by faithful assignments based on *total* preorders over worlds.

Theorem 3 (Katsuno & Mendelzon, 1991)**.** *A revision operator $*$ satisfies postulates $(K*1)$–$(K*8)$ iff there exists a faithful assignment that maps each theory K of $\mathbb{T}$ to a total preorder $\preceq_K$ over $\mathbb{M}$, such that, for any $\varphi \in \mathcal{L}$:*

$$\textbf{(F}*\textbf{)} \quad \big[K * \varphi\big] = min([\varphi], \preceq_K).$$

4. Belief Update

In this section, we present an overview of the KM framework, which formalizes belief update as defined by Katsuno and Mendelzon (1992). The KM framework was originally defined in terms of partial (thus not necessarily total) preorders over possible worlds. Yet, for the sake of comparison with belief revision, we restrict our study to update policies that are induced by *total* preorders over worlds.

4.1 Axiomatic Characterization

An *update operator* is a function $\diamond$ that maps a theory K and a sentence φ to a new theory $K \diamond \varphi$, representing the result of updating K by φ. We shall say that an update operator $\diamond$ is a *KM update operator* iff it satisfies the following postulates, known as *KM postulates* (Katsuno & Mendelzon, 1992).

$(\mathbf{K} \diamond \mathbf{1})$ $\quad K \diamond \varphi$ is a theory.

$(\mathbf{K} \diamond \mathbf{2})$ $\quad \varphi \in K \diamond \varphi$.

$(\mathbf{K} \diamond \mathbf{3})$ $\quad$ If $\varphi \in K$, then $K \diamond \varphi = K$.

$(\mathbf{K} \diamond \mathbf{4})$ $\quad K \diamond \varphi = \mathcal{L}$ iff K or φ is inconsistent.

$(\mathbf{K} \diamond \mathbf{5})$ $\quad$ If $Cn\big(\{\varphi\}\big) = Cn\big(\{\psi\}\big)$, then $K \diamond \varphi = K \diamond \psi$.

$(\mathbf{K} \diamond \mathbf{6})$ $\quad K \diamond (\varphi \wedge \psi) \subseteq (K \diamond \varphi) + \psi$.

$(\mathbf{K} \diamond \mathbf{7})$ $\quad$ If K is complete and $\neg\psi \notin K \diamond \varphi$, then $(K \diamond \varphi) + \psi \subseteq K \diamond (\varphi \wedge \psi)$.

$(\mathbf{K} \diamond \mathbf{8})$ $\quad$ If K is consistent, then $K \diamond \varphi \;=\; \bigcap_{w \in [K]} Cn(w) \diamond \varphi$.

For ease of comparison, the KM postulates have been rephrased in the AGM notation. This means that postulates $(K\diamond 1)$–$(K\diamond 8)$ are equivalent to postulates (U1)–(U9) introduced by Katsuno and Mendelzon (1992), when sentences are replaced by theories in the representation of states of belief. The details of this transformation are thoroughly examined by Peppas et al. (1996) and Peppas (1993, Section 5.5), while a discussion on the postulates of the KM framework is conducted by Peppas (2008, Section 8.8). As with the AGM postulates

$(K*1)$–$(K*8)$, postulates $(K\diamond 1)$–$(K\diamond 8)$ have been formulated in accordance with the principle of *minimal change*; hence, the initial theory K is updated in such a way that its modification is as minimal as possible.

4.2 Semantic Characterization

Katsuno and Mendelzon (1992) showcased that the update operators satisfying postulates $(K\diamond 1)$–$(K\diamond 8)$ are precisely those that are induced by total preorders over all possible worlds.

Definition 4 (Faithful Preorder Associated with Worlds, Katsuno & Mendelzon, 1992)**.** *A total preorder $\preceq_w$ over $\mathbb{M}$ is faithful to a world w iff, for any $r \in \mathbb{M}$, $w \neq r$ implies $w \prec_w r$.*

As with faithful preorders associated with theories (which govern belief revision), $r \preceq_w r'$ states that the world r is at least as plausible as the world r', relative to w.

Definition 5 (Faithful Pointwise Assignment, Katsuno & Mendelzon, 1992)**.** *A faithful pointwise assignment is a function that maps each world w of $\mathbb{M}$ to a total preorder $\preceq_w$ over $\mathbb{M}$, that is faithful to w.*

The following theorem characterizes the class of update operators induced by faithful pointwise assignments based on *total* preorders over worlds.

Theorem 6 (Katsuno & Mendelzon, 1992)**.** *An update operator $\diamond$ satisfies postulates $(K\diamond 1)$–$(K\diamond 8)$ iff there exists a faithful pointwise assignment that maps each world w of $\mathbb{M}$ to a total preorder $\preceq_w$ over $\mathbb{M}$, such that, for any theory $K \in \mathbb{T}$ and any $\varphi \in \mathcal{L}$:*

$$(\mathbf{F}\diamond) \quad \big[K \diamond \varphi\big] \;=\; \bigcup_{w\in[K]} min([\varphi], \preceq_w).$$

The semantic characterizations of revision and update, although similar, point out a major technical difference between these two processes of belief change. That is, in belief update, a whole *family* of total preorders over worlds is assigned to a single theory K (one total preorder for each K-world), whereas, in belief revision, a *single* total preorder over worlds is assigned to a theory K. This pointwise behaviour of update is due to postulate $(K\diamond 8)$, which forces that the deductive closure of every K-world —which essentially is a consistent complete theory— is modified *separately*.

Since the revision/update of inconsistent belief sets constitutes a rather limiting case, we shall focus, in the remainder of this article, on the principal case of the revision/update of *consistent* theories. Furthermore, we shall consider only *contingent* epistemic input.

5. Already Established Connections between Revision and Update

A few years after the formal introduction of belief update by Katsuno and Mendelzon (1992), Peppas et al. (1996) highlighted some interesting connections between belief revision and belief update. This section is devoted to the discussion of these connections.

We begin by presenting an interesting conceptual interpretation of the possible-worlds semantics of revision and update pinpointed in this subsequent work. According to Peppas et al. (1996), the fact that, in the realm of belief update, faithful preorders are *independent* of any cognitive attribute of an agent suggests that the adopted comparative plausibility of worlds is an intrinsic, eternal property of the universe, which is dependent on the physics of the domain; it is what we may call *ontological* comparative plausibility. On the contrary, the fact that, in the context of belief revision, faithful preorders *depend* on the current state of belief appears to suggest that the comparative plausibility of worlds in that context should be interpreted as the *agent's perception* of the ontological plausibility; this comparative plausibility is, therefore, *epistemic* in nature.

Let us now move on to the technical results of Peppas et al. (1996). We start with Theorem 7, which points out that the revision of a *fixed* theory of the language can be defined in terms of update.

Theorem 7 (Peppas et al., 1996)**.** *Let K be a theory, and let $*$ be an AGM revision operator. There exists a KM update operator $\diamond$ such that $K * \varphi = K \diamond \varphi$, for all $\varphi \in \mathcal{L}$.*

Theorem 7 refers to an arbitrary, but *fixed*, theory K, and guarantees that, for *any* AGM revision operator $*$, we can find a KM update operator $\diamond$ that behaves exactly like $*$ *at* K. It is crucial to note that Theorem 7 does *not* guarantee that this is also the case when K is not fixed; that is to say, if we allow K to vary, the KM update operator $\diamond$ will not necessarily keep on "simulating" $*$. As a matter of fact, it is proved by Peppas et al. (1996) that there is *no* AGM revision operator $*$ for which there exists a KM update operator $\diamond$, such that $K * \varphi = K \diamond \varphi$, for every theory K and all $\varphi \in \mathcal{L}$, unless the language $\mathcal{L}$ is *trivial* (i.e., it is built from a single atom).

Next, let K be an arbitrary, but *fixed*, theory, and let $\{\preceq_w\}_{w \in \mathbb{M}}$ be a family of total preorders over worlds. Peppas et al. (1996) considered the following two constraints on $\{\preceq_w\}_{w \in \mathbb{M}}$, modulo K.[3]

(L1) For all $w \in [K]$, all $r \in [K]$ and all $r' \notin [K]$, $r \prec_w r'$.

(L2) For all $w, w' \in [K]$ and all $r, r' \notin [K]$, $r \preceq_w r'$ iff $r \preceq_{w'} r'$.

Condition (L1) ensures that any world inside $[K]$ is *strictly more plausible* than any world outside $[K]$, with respect to each K-world w. Condition (L2) states that any two total preorders $\preceq_w$ and $\preceq_{w'}$ are *identical* when restricted to the worlds outside $[K]$.

On those premises, Peppas et al. (1996) proved Theorem 8, which states that, given a *fixed* theory K and a KM update operator $\diamond$ whose corresponding faithful preorders respect (L1) and (L2) *at* K, we can find an AGM revision operator $*$ that is equivalent to $\diamond$ *at* K.

Theorem 8 (Peppas et al., 1996)**.** *Let K be a theory, let $\diamond$ be a KM update operator, and let $\{\preceq_w\}_{w \in \mathbb{M}}$ be the family of total preorders over worlds that corresponds to $\diamond$, via condition (F$\diamond$). If the family $\{\preceq_w\}_{w \in \mathbb{M}}$ satisfies conditions (L1) and (L2) at K, then there exists an AGM revision operator $*$ such that $K * \varphi = K \diamond \varphi$, for all $\varphi \in \mathcal{L}$.*

3. To be precise, Peppas et al. (1996) formulated conditions (L1) and (L2) in terms of the constructive model for belief change introduced by Grove (1988), known as the *system of spheres*. However, these conditions can be easily translated in terms of faithful preorders, as presented herein.

Similar to Theorem 7, Theorem 8 does *not* guarantee that $*$ will keep on "simulating" $\diamond$, if we allow K to vary. For example, if we consider a theory K' which differs from K, what Theorem 8 ensures is that there exists an AGM revision operator $*'$, such that $K' *' \varphi = K' \diamond \varphi$, for all $\varphi \in \mathcal{L}$. However, $*'$ may not necessarily be the same as $*$.

Example 9 below will help up clarify how, given a *fixed* theory of the language, KM update operators can be constructed from AGM revision operators and vice versa.

Example 9 (KM Update Operators from AGM Revision Operators and Vice Versa)**.** *Assume that $\mathcal{P}$ contains exactly two atoms; thus, the set of possible worlds $\mathbb{M}$ contains exactly four possible world. Let $\mathbb{M} = \{w_1, w_2, w_3, w_4\}$ and let K be a theory such that $[K] = \{w_1, w_2\}$.*

Suppose that an AGM revision operator $*$ assigns (via (F$*$)) at K a faithful preorder $\preceq_K$ such that $w_1 \approx_K w_2 \prec_K w_3 \prec_K w_4$. Then, Theorem 7 ensures that there exists a KM update operator $\diamond$, such that $K * \varphi = K \diamond \varphi$, for all $\varphi \in \mathcal{L}$. It can be easily verified that such a KM update operator is a KM update operator $\diamond$ that assigns (via (F$\diamond$)) at w_1 a faithful preorder $\preceq_{w_1}$ such that $w_1 \prec_{w_1} w_2 \prec_{w_1} w_3 \prec_{w_1} w_4$, and at w_2 a faithful preorder $\preceq_{w_2}$ such that $w_2 \prec_{w_2} w_1 \prec_{w_2} w_3 \prec_{w_2} w_4$.

Now, let $\diamond$ be a KM update operator that assigns at w_1 a faithful preorder $\preceq_{w_1}$ such that $w_1 \prec_{w_1} w_2 \prec_{w_1} w_4 \prec_{w_1} w_3$, and at w_2 a faithful preorder $\preceq_{w_2}$ such that $w_2 \prec_{w_2} w_1 \prec_{w_2} w_4 \prec_{w_2} w_3$. Observe that the family $\{\preceq_w\}_{w \in [K]}$ of faithful preorders satisfies conditions (L1) and (L2). Then, Theorem 8 ensures that there exists an AGM revision operator $*$, such that $K * \varphi = K \diamond \varphi$, for all $\varphi \in \mathcal{L}$. Such an AGM revision operator is an AGM revision operator $*$ that assigns at K a faithful preorder $\preceq_K$ such that $w_1 \approx_K w_2 \prec_K w_4 \prec_K w_3$.

6. Extending the Connections: A Principle of Consistency

Consider a rational agent who uses an AGM revision operator $*$ to revise their beliefs, and a KM update operator $\diamond$ to update their beliefs. Since the operators $*$ and $\diamond$ are employed by a *single* rational agent, they *ought to be related* in a way that ensures a kind of *consistency* in the choices that the agent makes across different types of belief change. In this section, we introduce a consistency principle that guarantees exactly that scheme, by establishing an alignment between the revision strategy and the update strategy of the agent.

Before formally presenting the alluded consistency principle, let us first present a simple belief-change scenario that illustrates its intuition. Assume that K is a theory, for which there exist two distinct possible worlds w_1, w_2, such that $[K] = \{w_1, w_2\}$. Moreover, let $\{\preceq_K\}_{K \in \mathbb{T}}$, $\{\preceq_w\}_{w \in \mathbb{M}}$ be the families of total preorders over worlds that correspond to $*$, $\diamond$, via (F$*$), (F$\diamond$), respectively, and let r, r' be two distinct possible worlds, different from w_1 and w_2. Suppose that, from the perspective of w_1, as well as from the perspective of w_2, r is at least as plausible as r'; in symbols, $r \preceq_{w_1} r'$ and $r \preceq_{w_2} r'$. Since, as far as the agent knows at K, the real world can be either w_1 or w_2, and in both states of affairs r is at least as plausible as r', it seems unreasonable for the agent to assign at theory K a faithful preorder $\preceq_K$ that reverses the plausibility of r and r' — namely, a faithful preorder $\preceq_K$ such that $r' \prec_K r$. An abstract representation of the preceding scenario is depicted in Figure 1.

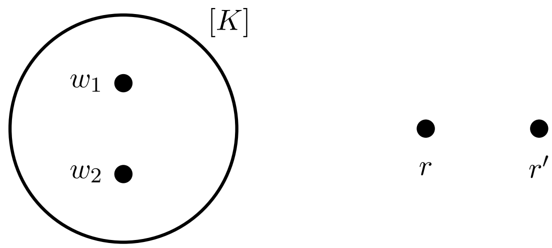

Figure 1: A theory K such that $[K] = \{w_1, w_2\}$, and two possible worlds r, r' for which $r \preceq_{w_1} r'$ and $r \preceq_{w_2} r'$ — the $\preceq_{w_1}$-plausibility (resp., $\preceq_{w_2}$-plausibility) of r, r' is inversely proportional to their Euclidean distance from w_1 (resp., w_2). According to the proposed consistency principle, it should be the case that $r \preceq_K r'$, where the $\preceq_K$-plausibility of r, r' is inversely proportional to their Euclidean distance from the centre of the circle representing $[K]$.

6.1 Semantic Constraints

The proposed consistency principle is formally encoded into the following two constraints on the families of preorders $\{\preceq_K\}_{K \in \mathbb{T}}$ and $\{\preceq_w\}_{w \in \mathbb{M}}$, relative to an arbitrary theory K.

(C1) If, for every $w \in [K]$, $r \preceq_w r'$, then $r \preceq_K r'$.

(C2) If, for every $w \in [K]$, $r \prec_w r'$, then $r \prec_K r'$.

Conditions (C1) and (C2) essentially generalize the aforementioned belief-change scenario, serving as reasonable and intuitive constraints on total preorders over worlds that relate update policies to revision policies of a *single* rational agent. Condition (C1) states that, if a world r is at least as plausible as a world r' with respect to *every* K-world, then r should be at least as plausible as r' with respect to theory K. Condition (C2) strengthens condition (C1) by stating that, if a world r is strictly more plausible than a world r' with respect to *every* K-world, then r should be strictly more plausible than r' with respect to theory K.

In view of the conceptual interpretation of the difference between revision and update provided by Peppas et al. (1996) and discussed in Section 5, the rationale behind conditions (C1) and (C2) can be expressed as follows: If the *ontological* comparative plausibility of two worlds r, r' is the same with respect to *every* world satisfying a theory K for a rational agent, then the agent should retain this comparative plausibility of r, r', with respect to theory K. In this way, the *epistemic* comparative plausibility of worlds, as employed by a *single* rational agent, is, in a sense, *consistent* with the ontological comparative plausibility of worlds.

Example 10, presented below, illustrates how the update policy of a rational agent affects their revision policy, in the presence of conditions (C1) and (C2).

Example 10 (Belief Change under Conditions (C1) & (C2))**.** *Consider a rational agent who employs a KM update operator $\diamond$ for belief updating, and an AGM revision operator $*$ for belief revision. Suppose that the operations of $\diamond$ and $*$ align with the dictates of conditions*

(C1) and (C2). Assume that $\mathcal{P}$ contains exactly three atoms; thus, the set of possible worlds $\mathbb{M}$ contains exactly eight possible worlds. Let $\mathbb{M} = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$ and let K be a theory such that $[K] = \{w_1, w_2\}$. Suppose that the KM update operator $\diamond$ assigns (via (F$\diamond$)) at w_1 a faithful preorder $\preceq_{w_1}$, such that

$$w_1 \prec_{w_1} w_2 \prec_{w_1} w_3 \prec_{w_1} w_4 \prec_{w_1} w_5 \prec_{w_1} w_6 \prec_{w_1} w_7 \prec_{w_1} w_8,$$

and at w_2 a faithful preorder $\preceq_{w_2}$, such that

$$w_2 \prec_{w_2} w_1 \prec_{w_2} w_3 \prec_{w_2} w_4 \prec_{w_2} w_5 \prec_{w_2} w_6 \prec_{w_2} w_7 \prec_{w_2} w_8.$$

Then, conditions (C1) and (C2) uniquely determine the following faithful preorder $\preceq_K$ that the AGM revision operator $*$ assigns (via (F$*$)) at K:

$$w_1 \approx_K w_2 \prec_K w_3 \prec_K w_4 \prec_K w_5 \prec_K w_6 \prec_K w_7 \prec_K w_8.$$

Consequently, in the presence of conditions (C1) and (C2), the operators $\diamond$ and $*$ function in a unified manner, providing a stable mechanism for modifying (revising/updating) theory K in response to any epistemic input within a given domain.

Although Example 10 might appear abstract, it is intended to model real-world scenarios where precise control of belief-change operations is crucial. In many technical applications —such as automated planning, robotics, or safety-critical systems— understanding the exact mechanics of both belief revision and belief update is essential. The formal structure of this example demonstrates how the proposed consistency principle ensures reliability and coherence under complex conditions.

We conclude this subsection by noting that the core intuition behind conditions (C1) and (C2) parallels the Pareto conditions in Social Choice Theory, according to which "if every individual prefers a certain option to another, then so should the resulting societal preference order" (Arrow, 1963). In our context, this principle is translated as follows: "If every K-world employs a certain comparative plausibility for two states of affairs (within belief update), then this comparative plausibility should be perceived by theory K as well (within belief revision)". Furthermore, we observe that conditions capturing the intuition behind (C1) and (C2) also appear in notable works of Belief Merging (Konieczny & Pérez, 2002, Section 3), (Konieczny & Pérez, 1998, Section 4), as well as Belief Revision (Peppas & Williams, 2016, Section 4).

6.2 Complete Belief Sets

Let us now present some notable results concerning consistent *complete* belief sets, which, despite their rigidity, offer a high degree of intuitiveness and conceptual simplicity. While such belief structures may seem extreme, they do have approximate real-world counterparts — an indicative example is databases operating under the Closed World Assumption (CWA), where any fact not explicitly stored is assumed to be false. Thus, in the case of complete belief sets, conditions (C1) and (C2) reduce to a single natural condition, as shown in the following proposition.

Proposition 11. *Let $*$, $\diamond$ be an AGM revision operator and a KM update operator, respectively, and let $\{\preceq_K\}_{K\in\mathbb{T}}$, $\{\preceq_w\}_{w\in\mathbb{M}}$ be the families of total preorders over worlds that correspond to $*$, $\diamond$, via (F$*$), (F$\diamond$), respectively. Moreover, let K be a complete theory such that, for a world $w \in \mathbb{M}$, $[K] = \{w\}$. Then, conditions (C1) and (C2) are equivalent to condition (CC).*

$$\textbf{(CC)} \quad r \preceq_w r' \quad \textit{iff} \quad r \preceq_K r'.$$

Proof. First we prove that conditions (C1) and (C2) entail condition (CC). To that end, assume that conditions (C1) and (C2) are satisfied. We show that condition (CC) is also satisfied. The left-to-right implication of (CC) follows directly from (C1). For the right-to-left implication of (CC), let r, r' be two worlds of $\mathbb{M}$ and suppose that $r \preceq_K r'$. Assume, towards contradiction, that $r \not\preceq_w r'$. Then, since $\preceq_w$ is a total preorder (i.e., either $r \preceq_w r'$ or $r' \preceq_w r$), it follows that $r' \prec_w r$. This again entails, from condition (C2), that $r' \prec_K r$, which contradicts our initial assumption. Hence, we have shown that (C1) and (C2) entail (CC). The converse, namely that (CC) entails (C1) and (C2), is obvious. ■

Condition (CC) essentially says that, if a complete theory K is such that $[K] = \{w\}$, then the faithful preorders $\preceq_w$ and $\preceq_K$ are *identical*.

Corollary 12 below is an immediate consequence of Proposition 11, and states that, in the realm of complete belief sets and in the presence of conditions (C1) and (C2), revision and update are as a matter of fact *indistinguishable*.

Corollary 12. *Let $*$, $\diamond$ be an AGM revision operator and a KM update operator, respectively, and let $\{\preceq_K\}_{K\in\mathbb{T}}$, $\{\preceq_w\}_{w\in\mathbb{M}}$ be the families of total preorders over worlds that correspond to $*$, $\diamond$, via (F$*$), (F$\diamond$), respectively. Moreover, let K be a complete theory. If $\{\preceq_K\}_{K\in\mathbb{T}}$ and $\{\preceq_w\}_{w\in\mathbb{M}}$ satisfy conditions (C1) and (C2), then $K * \varphi = K \diamond \varphi$, for all $\varphi \in \mathcal{L}$.*

Hence, under conditions (C1) and (C2), the complete certainty of ("opinionated") agents results in the indistinguishability between revision and update — a rather strict property whose desirability depends on the specific context. Of course, in the general case of incomplete belief sets, the proposed consistency principle establishes a coherent relationship between belief revision and belief update, but it does *not* inherently lead to their equivalence.

The next concrete, real-world scenario examines the revision and update of a complete theory of the language.

Example 13 (Complete Theories under Conditions (C1) & (C2))**.** *Consider a room with a table, a magazine and a book. A rational agent looks through an open door at the room, and sees that both the magazine and the book are on the table. Supposing that the atom b represents the fact that "the book is on the table", and that the atom m represents the fact that "the magazine is on the table", the beliefs of the agent reflecting the initial state of the world can be expressed by the theory*

$$K = Cn\big(\{b, m\}\big),$$

which essentially provides a complete description of the environmental configuration.

Now, the agent orders a robot to place the book on the floor. To determine the new state of the world following the robot's successful action, the agent uses a KM update operator $\diamond$, and updates theory K by the sentence $\varphi = \neg b$, yielding the posterior theory $K \diamond \varphi = Cn\big(\{\neg b, m\}\big)$.

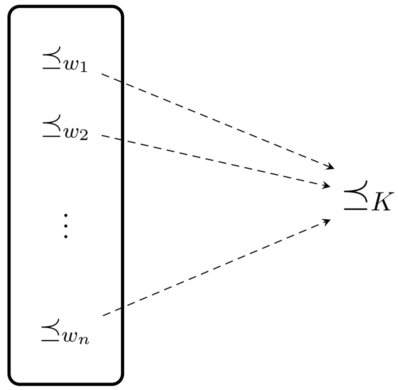

Figure 2: Given that $[K] = \{w_1, w_2, \ldots, w_n\}$, in view of conditions (C1) and (C2), the faithful preorders $\preceq_{w_1}, \preceq_{w_2}, \ldots, \preceq_{w_n}$ provide guidelines for the structure of the faithful preorder $\preceq_K$.

According to Corollary 12 and in view of conditions (C1) and (C2), the AGM revision operator $*$ that the agent uses to respond to new information is such that $K * \varphi = Cn(\{\neg b, m\}) = K \diamond \varphi$.

Interestingly, in the absence of conditions (C1) and (C2), the agent of Example 13 could have selected an AGM revision operator $*'$ (distinct from $*$) such that $K *' \varphi = Cn(\{\neg b, \neg m\})$, implying that, in the revised state of belief, the agent concludes the magazine is on the floor ($\neg m$) — even though the KM update operator $\diamond$ yields a different outcome, namely $K \diamond \varphi = Cn(\{\neg b, m\})$. In other words, without the consistency principle encoded in (C1) and (C2), nothing would prevent an arbitrary divergence between the outcomes of belief revision and belief update. By contrast, the consistency principle ensures that both operators remain aligned, thereby providing a unified approach to belief change.

6.3 Fill In the Gaps

Conditions (C1) and (C2) could be regarded as *domain-independent* constraints, according to which, given a theory K such that $[K] = \{w_1, w_2, \ldots, w_n\}$, the faithful preorders $\preceq_{w_1}, \preceq_{w_2}, \ldots, \preceq_{w_n}$ provide information about the structure of the faithful preorder $\preceq_K$ (see Figure 2). Example 10 demonstrated that there exist belief-change scenarios where conditions (C1) and (C2) can even *uniquely* determine the faithful preorder $\preceq_K$ from $\preceq_{w_1}, \preceq_{w_2}, \ldots, \preceq_{w_n}$. This is also the case for *any complete* theory of the language, as shown in Proposition 11 of the previous subsection. However, in the general case, conditions (C1) and (C2) do *not* suffice to uniquely specify $\preceq_K$. For such unique specification, the agent needs to *explicitly* provide all the missing information. For reducing the amount of information that needs to explicitly be provided, a collection of (intuitive) *default rules* could be adopted, which may be domain-specific and *automatically* "fill in the gaps". Consider, for example, the following directive:

> *If r is more plausible than r' for the majority of the K-worlds, then $r \preceq_K r'$; otherwise, $r \approx_K r'$.*

Informally, the above indicative rule expresses the idea that, if the comparative plausibility of two worlds r, r' is predominant among the K-worlds, then it is this comparative plausibility that will be assigned at K. Otherwise, r and r' should be considered equally plausible relative to K. This directive, along with others of a similar spirit, can provide information on the structure of the faithful preorder $\preceq_K$, and may save the agent from supplying vast amounts of information. Thus, such default rules ensure that the belief-change process remains manageable and practically applicable.[4]

6.4 Axiomatic Constraints

In this subsection, we develop the axiomatic characterization of the proposed consistency principle, by formulating the postulational counterparts of the semantic conditions (C1) and (C2). To that end, consider the following postulates (PC1) and (PC2), which relate (constrain) the KM update operator $\diamond$ and the AGM revision operator $*$ of a *single* rational agent.

(PC1) If, for every $w \in [K]$, $\neg\varphi \notin Cn(w) \diamond (\varphi \vee \psi)$, then $\neg\varphi \notin K * (\varphi \vee \psi)$.

(PC2) If, for every $w \in [K]$, $\neg\psi \in Cn(w) \diamond (\varphi \vee \psi)$, then $\neg\psi \in K * (\varphi \vee \psi)$.

On that basis, Theorem 14 is a *representation result* that establishes the equivalence between postulates (PC1) and (PC2) and constraints (C1) and (C2), respectively.

Theorem 14. *Let $*$, $\diamond$ be an AGM revision operator and a KM update operator, respectively, and let $\{\preceq_K\}_{K\in\mathbb{T}}$, $\{\preceq_w\}_{w\in\mathbb{M}}$ be the families of total preorders over worlds that correspond to $*$, $\diamond$, via (F$*$), (F$\diamond$), respectively. Then, $*$, $\diamond$ satisfy postulates (PC1) and (PC2) iff $\{\preceq_K\}_{K\in\mathbb{T}}$, $\{\preceq_w\}_{w\in\mathbb{M}}$ satisfy conditions (C1) and (C2), respectively.*

Proof. Within the context of this proof, we shall sometimes abuse notation and treat a possible world as a sentence, namely, the *conjunction* of all its literals, leaving it to the context to resolve any ambiguity; for example, for two possible worlds $r, r' \in \mathbb{M}$, we may write sentences of $\mathcal{L}$ such as $\neg r$ or $r \vee r'$.

For the left-to-right implication, assume that $*$, $\diamond$ satisfy postulate (PC1). We show that $\{\preceq_K\}_{K\in\mathbb{T}}$, $\{\preceq_w\}_{w\in\mathbb{M}}$ satisfy condition (C1). Let r, r' be two worlds of $\mathbb{M}$, let K be a theory, and suppose that, for every $w \in [K]$, $r \preceq_w r'$. Clearly then, for every $w \in [K]$, $r \in min([r \vee r'], \preceq_w)$. Hence, from condition (F$\diamond$), we have that, for every $w \in [K]$, $\neg r \notin Cn(w) \diamond (r \vee r')$. Therefore, from condition (PC1), it follows that $\neg r \notin K * (r \vee r')$. Consequently, from condition (F$*$), we derive that $r \preceq_K r'$, as desired.

Next, assume that $*$, $\diamond$ satisfy postulate (PC2). We show that $\{\preceq_K\}_{K\in\mathbb{T}}$, $\{\preceq_w\}_{w\in\mathbb{M}}$ satisfy condition (C2). Let r, r' be two worlds of $\mathbb{M}$, let K be a theory, and suppose that, for every $w \in [K]$, $r \prec_w r'$. Clearly then, for every $w \in [K]$, $min([r \vee r'], \preceq_w) = \{r\}$. Hence, from condition (F$\diamond$), we have that, for every $w \in [K]$, $\big[Cn(w) \diamond (r \vee r')\big] = \{r\}$; thus, $\neg r' \in Cn(w) \diamond (r \vee r')$. Therefore, from condition (PC2), it follows that $\neg r' \in K * (r \vee r')$. Consequently, from condition (F$*$), we derive that $r \prec_K r'$, as desired.

4. For a propositional language $\mathcal{L}$ with n atoms, there exist 2^n possible worlds in $\mathbb{M}$ and $2^{(2^n)}$ theories in $\mathbb{T}$. Clearly then, the specification of faithful preorders over possible worlds is a laborious task.

For the right-to-left implication, assume that $\{\preceq_K\}_{K\in\mathbb{T}}$, $\{\preceq_w\}_{w\in\mathbb{M}}$ satisfy condition (C1). We show that $*$, $\diamond$ satisfy postulate (PC1). Let K be a theory, and let φ, ψ be any two sentences of $\mathcal{L}$, such that, for every $w \in [K]$, $\neg\varphi \notin Cn(w) \diamond (\varphi \vee \psi)$. From $\neg\varphi \notin Cn(w) \diamond (\varphi \vee \psi)$, we derive that there is a world $r \in \big[Cn(w) \diamond (\varphi \vee \psi)\big]$, such that $r \models \varphi$. Therefore, from condition (F$\diamond$), we have that, for every $w \in [K]$, $r \preceq_w r'$, for all $r' \in [\varphi \vee \psi]$. Hence, from condition (C1), it follows that $r \preceq_K r'$, for all $r' \in [\varphi \vee \psi]$. This again entails, from condition (F$*$), that $r \in \big[K * (\varphi \vee \psi)\big]$, and since $r \models \varphi$, we derive that $\neg\varphi \notin K * (\varphi \vee \psi)$, as desired.

Next, assume that $\{\preceq_K\}_{K\in\mathbb{T}}$, $\{\preceq_w\}_{w\in\mathbb{M}}$ satisfy condition (C2). We show that $*$, $\diamond$ satisfy postulate (PC2). Let K be a theory, and let φ, ψ be any two sentences of $\mathcal{L}$, such that, for every $w \in [K]$, $\neg\psi \in Cn(w) \diamond (\varphi \vee \psi)$. If ψ is inconsistent, then postulate $(K*1)$ immediately entails that $\neg\psi \in K * (\varphi \vee \psi)$, as desired. Suppose, therefore, that ψ is consistent. From $\neg\psi \in Cn(w) \diamond (\varphi \vee \psi)$, we derive that every world in $\big[Cn(w) \diamond (\varphi \vee \psi)\big]$ entails $\neg\psi$. Therefore, from condition (F$\diamond$), we have that, for every world $r \in [\varphi \vee \psi]$ such that $r \in [\neg\psi]$, $r \prec_w r'$, for all $r' \in [\psi]$.[5] Hence, from condition (C2), it follows that, for every world $r \in [\varphi \vee \psi]$ such that $r \in [\neg\psi]$, $r \prec_K r'$, for all $r' \in [\psi]$. This again entails, from condition (F$*$), that every world in $\big[K * (\varphi \vee \psi)\big]$ entails $\neg\psi$; thus, $\neg\psi \in K * (\varphi \vee \psi)$, as desired. ■

We conclude this section by emphasizing that, while the proposed approach focuses on single-shot belief change, an agent may perform belief revision at one moment, belief update at another, or vice versa — perhaps even simultaneously. Even when only a single operation is executed at a given instance, a consistency principle remains crucial because it ensures that the underlying belief-change strategies remain coherent across both modes. This coherence is essential not only for smooth transitions between different types of belief change, but also for supporting counter-factual and hypothetical reasoning. For example, if an agent were to consider what the outcome would have been had they applied the alternative operation, a consistency principle will guarantee that such counter-factual assessments align with a unified belief stance. Ultimately, this unified approach can prevent arbitrary differences in outcomes and provide a robust basis for potential extensions to iterated belief-change processes.

7. Uniform Belief Change

Having defined a consistency principle that relates in a coherent way revision and update policies of a single rational agent, we proceed to the introduction of a type of belief change, named *uniform belief change*, which, as it will be shown, constitutes a *proof of concept* for the introduced notion of consistency. The process of uniform belief revision has already been developed by Areces and Becher (2001), whereas the process of uniform belief update shall be defined in this section. We start with the presentation of the former process, as defined by Areces and Becher (2001).[6]

5. Since ψ is consistent, the set $[\psi]$ is non-empty.
6. Areces and Becher named the revision operators that implement uniform belief revision "iterable AGM revision functions", as the authors focus on the favourable properties of these operators related to iterated revision (Peppas, 2014). In this article, we adopt the term "uniform revision operators" which reflects the homogeneous way the operators act on belief sets.

Let $\preccurlyeq$ be an arbitrary, but *fixed*, total preorder over the set of all possible worlds $\mathbb{M}$. Intuitively, the total preorder $\preccurlyeq$ —which is *belief-set-independent*— encodes the (state-independent) comparative plausibility of worlds, which in turn reflects the dynamics of a domain, as shaped, for example, by a body of pre-established background knowledge, such as objective laws of physics, legal or moral codes. In view of condition (UR) presented below, the total preorder $\preccurlyeq$ can generate a whole *family* $\{\preceq_K\}_{K\in\mathbb{T}}$ of total preorders over worlds, that is, one preorder for each theory of $\mathbb{T}$. Essentially, for *each* theory $K \in \mathbb{T}$, condition (UR) sets the structure of a faithful preorder $\preceq_K$ outside $[K]$ identical to $\preccurlyeq$.

$$\textbf{(UR)} \quad \text{For any } r, r' \notin [K], \quad r \preceq_K r' \quad \text{iff} \quad r \preccurlyeq r'.$$

The specified family $\{\preceq_K\}_{K\in\mathbb{T}}$ of total preorders over worlds induces, through condition (F$*$) of Theorem 3, an AGM revision operator. An AGM revision operator so constructed is called a *uniform revision operator* (Areces & Becher, 2001). Example 15 below, adapted from Aravanis (2020, Example 4.2), illustrates the construction of a uniform revision operator.

Example 15 (Uniform Belief Revision, Aravanis, 2020)**.** *Let $\mathcal{P} = \{a, b, c\}$ and let $\preccurlyeq$ be a total preorder over $\mathbb{M}$, such that*

$$\begin{array}{c} abc \\ \overline{a}\overline{b}c \end{array} \prec\!\!\cdot \begin{array}{c} \overline{a}b\overline{c} \\ a\overline{b}\overline{c} \end{array} \prec\!\!\cdot \begin{array}{c} ab\overline{c} \\ \overline{a}\overline{b}\overline{c} \end{array} \prec\!\!\cdot \begin{array}{c} \overline{a}bc \\ a\overline{b}c \end{array}$$

where $\prec\!\!\cdot$ denotes the strict part of $\preccurlyeq$. The total preorder $\preccurlyeq$ essentially encodes the dynamics of the domain, as shaped, for example, by a body of background knowledge. Now, let K, H be two theories such that $K = Cn\big(\{b, \neg c\}\big)$ and $H = Cn\big(\{a \leftrightarrow b, c\}\big)$. Clearly, $[K] = \big\{\{a, b, \neg c\}, \{\neg a, b, \neg c\}\big\}$ and $[H] = \big\{\{a, b, c\}, \{\neg a, \neg b, c\}\big\}$. Then, condition (UR) entails that the faithful preorders $\preceq_K$ and $\preceq_H$, induced from $\preccurlyeq$, are as follows:

$$\begin{array}{c} ab\overline{c} \\ \overline{a}b\overline{c} \end{array} \prec_K \begin{array}{c} abc \\ \overline{a}\overline{b}c \end{array} \prec_K a\overline{b}\overline{c} \prec_K \overline{a}\overline{b}\overline{c} \prec_K \begin{array}{c} \overline{a}bc \\ a\overline{b}c \end{array}$$

$$\begin{array}{c} abc \\ \overline{a}\overline{b}c \end{array} \prec_H \begin{array}{c} \overline{a}b\overline{c} \\ a\overline{b}\overline{c} \end{array} \prec_H \begin{array}{c} ab\overline{c} \\ \overline{a}\overline{b}\overline{c} \end{array} \prec_H \begin{array}{c} \overline{a}bc \\ a\overline{b}c \end{array}$$

The family $\{\preceq_K\}_{K\in\mathbb{T}}$ of total preorders over worlds, derived from $\preccurlyeq$ as above, uniquely determines (via condition (F$*$)) a particular uniform revision operator $*$. This operator can serve as a systematic tool for revising belief sets by epistemic inputs within the specific domain.

Condition (UR) defines uniform belief revision, as introduced by Areces and Becher (2001). Herein, we shall define the counterpart of uniform belief revision in the realm of belief update, namely, the process of uniform belief update. This can be achieved through condition (UU) shown subsequently, which, for *any* world $w \in \mathbb{M}$, sets the structure of a faithful preorder $\preceq_w$ outside $\{w\}$ identical to $\preccurlyeq$. With the aid of condition (UU), the initial total preorder $\preccurlyeq$ can generate not only a family $\{\preceq_K\}_{K\in\mathbb{T}}$ of total preorders over worlds, associated with theories, but also a family $\{\preceq_w\}_{w\in\mathbb{M}}$ of total preorders over worlds, associated with worlds.

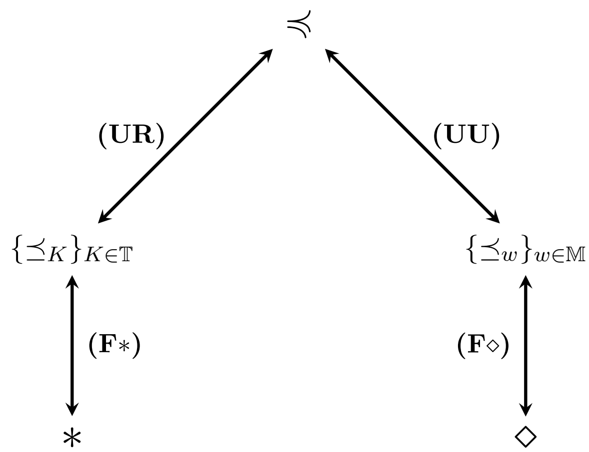

Figure 3: An arbitrary, but fixed, total preorder $\preccurlyeq$ over $\mathbb{M}$ that induces a family $\{\preceq_K\}_{K\in\mathbb{T}}$ of total preorders over worlds associated with theories, and a family $\{\preceq_w\}_{w\in\mathbb{M}}$ of total preorders over worlds associated with worlds. The families $\{\preceq_K\}_{K\in\mathbb{T}}$ and $\{\preceq_w\}_{w\in\mathbb{M}}$ induce in turn a uniform revision operator $*$ and a uniform update operator $\diamond$, respectively.

(UU) For any r, r' distinct from w, $\quad r \preceq_w r' \quad$ iff $\quad r \preccurlyeq r'$.

Likewise, the specified family $\{\preceq_w\}_{w\in\mathbb{M}}$ of total preorders over worlds induces, through condition (F$\diamond$) of Theorem 6, a KM update operator. A KM update operator so constructed shall be called a *uniform update operator*. A concrete illustration of the construction of such operators is provided in Example 19 at the end of this section.

Figure 3 visualizes how a uniform revision operator and a uniform update operator can be constructed from a *single* fixed total preorder $\preccurlyeq$ over possible worlds.

In this context, Theorem 16 proves that uniform belief change *respects* the consistency principle encoded into postulates (PC1) and (PC2).

Theorem 16. *Let $\preccurlyeq$ be a total preorder over $\mathbb{M}$, and let $*$, $\diamond$ be the uniform revision operator and the uniform update operator induced from $\preccurlyeq$, respectively. Then, $*$ and $\diamond$ satisfy postulates (PC1) and (PC2).*

Proof. Let $\{\preceq_K\}_{K\in\mathbb{T}}$, $\{\preceq_w\}_{w\in\mathbb{M}}$ be the families of total preorders over worlds that correspond to $*$, $\diamond$, via (F$*$), (F$\diamond$), respectively. It suffices to show that $\{\preceq_K\}_{K\in\mathbb{T}}$ and $\{\preceq_w\}_{w\in\mathbb{M}}$ satisfy conditions (C1) and (C2).

For (C1), let K be an arbitrary theory, and let r, r' be two worlds of $\mathbb{M}$ such that, for every $w \in [K]$, $r \preceq_w r'$. First, observe that it cannot be the case that $r' \in [K]$. For otherwise, we would have that $r \preceq_{r'} r'$, a fact which contradicts the faithfulness of $\preceq_{r'}$. Therefore, we assume that $r' \notin [K]$. Furthermore, observe that, if $r \in [K]$, then $r \prec_K r'$, in view of the faithfulness of $\preceq_K$. Hence, if $r \in [K]$, then condition (C1) is trivially satisfied. Therefore, we also assume that $r \notin [K]$. Now, conditions (UU) and (UR) entail that, for every $w \in [K]$, $r \preceq_w r'$ iff $r \preceq_K r'$, a fact which in turn entails that $r \preceq_K r'$, as desired.

With a totally symmetric argument, we can show that $\{\preceq_K\}_{K\in\mathbb{T}}$ and $\{\preceq_w\}_{w\in\mathbb{M}}$ satisfy condition (C2) as well. ■

We now turn to the identification of some interesting properties of uniform belief change. We begin first with the observation that uniform belief change —in fact condition (UU)— entails condition (L2) of Section 5. This is proved in the next proposition.

Proposition 17. *Let $\preccurlyeq$ be a total preorder over $\mathbb{M}$, and let $\{\preceq_w\}_{w\in\mathbb{M}}$ be the family of total preorders over worlds that is induced from $\preccurlyeq$, via condition (UU). Then, $\{\preceq_w\}_{w\in\mathbb{M}}$ satisfies condition (L2) at an arbitrary, but fixed, theory K.*

Proof. Let K be an arbitrary, but fixed, theory. Since $\{\preceq_w\}_{w\in\mathbb{M}}$ satisfies condition (UU), it follows that, for any *distinct* worlds $w, w', r, r' \in \mathbb{M}$, $r \preceq_w r'$ iff $r \preceq_{w'} r'$. Therefore, for all $w, w' \in [K]$ and all $r, r' \notin [K]$, we have that $r \preceq_w r'$ iff $r \preceq_{w'} r'$. Thus, $\{\preceq_w\}_{w\in\mathbb{M}}$ satisfies condition (L2) at K, as desired. ■

The fact that uniform belief change can generate condition (L2), which as a matter of fact constrains non-K-worlds, makes us wonder whether it is possible to formulate, for that belief-change type, a result in the spirit of Theorem 8 of Peppas et al. (1996) (Section 5), when we restrict ourselves to the principal case where the epistemic input φ *contradicts* the initial belief set K (in which case all φ-worlds are non-K-worlds, as $[K] \cap [\varphi] = \varnothing$). This is indeed feasible, as Theorem 18 points out, which proves that uniform belief revision and uniform belief update by epistemic input that *contradicts* the initial state of belief are *indistinguishable*, in the sense that both processes lead to identical outcomes.

Theorem 18. *Let $\preccurlyeq$ be a total preorder over $\mathbb{M}$, and let $*$, $\diamond$ be the uniform revision operator and the uniform update operator induced from $\preccurlyeq$, respectively. Moreover, let K be a theory, and let φ be any sentence of $\mathcal{L}$ such that $\neg\varphi \in K$. Then, $K * \varphi = K \diamond \varphi$.*

Proof. Let $\{\preceq_K\}_{K\in\mathbb{T}}$, $\{\preceq_w\}_{w\in\mathbb{M}}$ be the families of total preorders over worlds that correspond to $*$, $\diamond$, via (F$*$), (F$\diamond$), respectively. By definition, $\{\preceq_K\}_{K\in\mathbb{T}}$ and $\{\preceq_w\}_{w\in\mathbb{M}}$ satisfy conditions (UR) and (UU), respectively. Since $\neg\varphi \in K$, it follows that $[K] \cap [\varphi] = \varnothing$. However, we also derive from conditions (UR) and (UU) that, for each $w \in [K]$ and any $r, r' \notin [K]$, $r \preceq_K r'$ iff $r \preceq_w r'$. Therefore, we conclude from conditions (F$*$) and (F$\diamond$) that $min([\varphi], \preceq_K) = \bigcup_{w\in[K]} min([\varphi], \preceq_w)$. Hence, $[K * \varphi] = [K \diamond \varphi]$, thus, $K * \varphi = K \diamond \varphi$, as desired. ■

It is important to note that Theorem 18 proves the equivalence of operators $*$ and $\diamond$ for *every* theory of the language, contrary to Theorem 8 of Peppas et al. (1996), which proves such an equivalence for a *fixed* theory.

As stated, Theorem 18 refers to epistemic input that contradicts the initial state of belief. It turns out that, in case the epistemic input is *consistent* with the initial belief set, uniform belief revision does *not always* lead to the same result as uniform belief update. This is illustrated in the subsequent example.

Example 19 (Uniform Belief Change)**.** *Assume that $\mathcal{P}$ contains exactly two atoms; thus, the set of possible worlds $\mathbb{M}$ contains exactly four possible worlds. Let $\mathbb{M} = \{w_1, w_2, w_3, w_4\}$ and let $\preccurlyeq$ be a total preorder over $\mathbb{M}$, such that $w_2 \prec\!\cdot\, w_3 \prec\!\cdot\, w_4 \prec\!\cdot\, w_1$. Moreover, let K be* *a theory such that* $[K] = \{w_1, w_2\}$*, and let* φ *be a sentence of* $\mathcal{L}$ *such that* $[\varphi] = \{w_1, w_3\}$*; observe that* φ *is consistent with* K*, as* $[K] \cap [\varphi] \neq \varnothing$*. Then,* $\preccurlyeq$ *induces (via (UR) and (F$*$)) a uniform revision operator* $*$ *that assigns at* K *a faithful preorder* $\preceq_K$ *such that* $w_1 \approx_K w_2 \prec_K w_3 \prec_K w_4$*. Furthermore,* $\preccurlyeq$ *induces (via (UU) and (F$\diamond$)) a uniform update operator* $\diamond$ *that assigns at* w_1 *a faithful preorder* $\preceq_{w_1}$ *such that* $w_1 \prec_{w_1} w_2 \prec_{w_1} w_3 \prec_{w_1} w_4$*, and at* w_2 *a faithful preorder* $\preceq_{w_2}$ *such that* $w_2 \prec_{w_2} w_3 \prec_{w_2} w_4 \prec_{w_2} w_1$*. Then, we derive from condition (F$*$) that* $[K * \varphi] = [K] \cap [\varphi] = \{w_1\}$*, whereas we derive from condition (F$\diamond$) that* $[K \diamond \varphi] = \{w_1, w_3\}$*. Clearly then,* $K * \varphi \neq K \diamond \varphi$*.*

The reason why uniform belief revision and uniform belief update are distinguishable in the case of an epistemic input that is consistent with the initial belief set is that condition (UU) *violates* condition (L1) of Section 5. This is immediately evident from Example 19, in which the faithful preorder $\preceq_{w_2}$ that the uniform update operator $\diamond$ assigns at the world w_2 violates condition (L1).

8. Parametrized-Difference Belief Change

This section is devoted to the presentation of a second type of belief change, known as *parametrized-difference belief change*, which also constitutes a *proof of concept* for the introduced consistency principle, encoded into postulates (PC1) and (PC2). The process of parametrized-difference belief revision has been defined by Peppas and Williams (2018) and Peppas et al. (2024), whereas the process of parametrized-difference belief update has been defined by Aravanis (2021). We present these two processes in what follows.

Let $\trianglelefteq$ be a total preorder over the set $\mathcal{P}$ of all atoms. Intuitively, the total preorder $\trianglelefteq$ should be seen as an encoding of the comparative epistemic value of the atoms; the more epistemic entrenched (and thus more resistant to change) an atom is, the higher it appears in $\trianglelefteq$. For a set of atoms $\mathcal{S}$ and an atom q, by $\mathcal{S}_q$ we denote the set $\mathcal{S}_q = \{p \in \mathcal{S} : p \trianglelefteq q\}$. The definition of $\trianglelefteq$ can then be extended to *sets* of atoms, as demonstrated in Definition 20.

Definition 20 (Total Preorder over Sets of Atoms, Peppas et al., 2024)**.** *Let* $\trianglelefteq$ *be a total preorder over the set of atoms* $\mathcal{P}$*. For any two sets of atoms* $\mathcal{S}$*,* $\mathcal{S}'$*, we define* $\mathcal{S} \trianglelefteq \mathcal{S}'$ *iff one of the following three conditions holds (*$\vartriangleleft$ *denotes the strict part of* $\trianglelefteq$*):*

- **(i)** $|\mathcal{S}| < |\mathcal{S}'|$.
- **(ii)** $|\mathcal{S}| = |\mathcal{S}'|$*, and for all* $q \in \mathcal{P}$*,* $|\mathcal{S}_q| = |\mathcal{S}'_q|$.
- **(iii)** $|\mathcal{S}| = |\mathcal{S}'|$*, and for some* $q \in \mathcal{P}$*,* $|\mathcal{S}_q| > |\mathcal{S}'_q|$*, and for all* $p \vartriangleleft q$*,* $|\mathcal{S}_p| = |\mathcal{S}'_p|$.

In the above definition, condition (i) is straightforward; condition (ii) states that $\mathcal{S}$ and $\mathcal{S}'$ are *lexicographically indistinguishable* (with respect to $\trianglelefteq$); condition (iii) states that $\mathcal{S}$ *lexicographically precedes* $\mathcal{S}'$ (with respect to $\trianglelefteq$). It turns out that the extended $\trianglelefteq$ of Definition 20 is a total preorder over $2^{\mathcal{P}}$. The intended interpretation of the extended total preorder $\trianglelefteq$, defined over $2^{\mathcal{P}}$, is the same as that of a total preorder defined over $\mathcal{P}$; namely, $\mathcal{S} \trianglelefteq \mathcal{S}'$ states that a change of *all* atoms of $\mathcal{S}'$ is less plausible than a change of *all* atoms of $\mathcal{S}$.

Furthermore, let us denote by *Diff* (w, r) the *difference* between two possible worlds w and r of $\mathbb{M}$, which is defined as the set of atoms over which w and r disagree; in symbols,

$$\mathit{Diff}\,(w,r)=\Big((w\setminus r)\cup(r\setminus w)\Big)\cap\mathcal{P}.$$

In this context, the next condition (PDR) can specify a whole *family* $\{\preceq_K\}_{K\in\mathbb{T}}$ of total preorders over worlds, that is, one total preorder for each theory of the language. Essentially, condition (PDR) builds $\{\preceq_K\}_{K\in\mathbb{T}}$ by means of the difference between worlds, parametrized by $\trianglelefteq$. The specified family $\{\preceq_K\}_{K\in\mathbb{T}}$ induces in turn, through condition (F$*$), an AGM revision operator, which is called *parametrized-difference revision operator* or *PD revision operator* for short (Peppas et al., 2024).

(PDR) $r\preceq_K r'$ iff there is a $w\in[K]$, such that, for all $w'\in[K]$, $\mathit{Diff}\,(w,r)\trianglelefteq\mathit{Diff}\,(w',r')$.

As with uniform belief change, the total preorder $\trianglelefteq$ can generate not only a family $\{\preceq_K\}_{K\in\mathbb{T}}$ of total preorders over worlds, associated with theories, but also a family $\{\preceq_w\}_{w\in\mathbb{M}}$ of total preorders over worlds, associated with worlds. This is accomplished by means of condition (PDU), which, similar to (PDR), builds $\{\preceq_w\}_{w\in\mathbb{M}}$ by means of the difference between worlds, parametrized by $\trianglelefteq$. The specified family $\{\preceq_w\}_{w\in\mathbb{M}}$ induces in turn, through condition (F$\diamond$), a KM update operator, which is called *parametrized-difference update operator* or *PD update operator* for short (Aravanis, 2021).

(PDU) $r\preceq_w r'$ iff $\mathit{Diff}\,(w,r)\trianglelefteq\mathit{Diff}\,(w,r')$.

As pointed out by Peppas et al. (2024) and Aravanis (2021), in the special case where all atoms of $\mathcal{P}$ are $\trianglelefteq$-equivalent (i.e., $\trianglelefteq\,=\mathcal{P}\times\mathcal{P}$), conditions (PDR) and (F$*$) induce Dalal's revision operator (Dalal, 1988), whereas conditions (PDU) and (F$\diamond$) induce Forbus' update operator (Forbus, 1989) — both these well-known operators are built through a *Hamming-based* difference between worlds.

Figure 4 illustrates how a PD revision operator and a PD update operator can be constructed from a *single* fixed total preorder $\trianglelefteq$ over atoms (cf. Figure 3). A concrete example that presents the construction of such operators is Example 22 subsequently; for further examples, the interested reader is referred to the works of Peppas et al. (2024), Aravanis et al. (2021), and Aravanis (2021).

As it is proven in Theorem 21, parametrized-difference belief change *respects* the consistency principle encoded into postulates (PC1) and (PC2).

Theorem 21. *Let $\trianglelefteq$ be a total preorder over $\mathcal{P}$, and let $*$, $\diamond$ be the PD revision operator and the PD update operator induced from $\trianglelefteq$, respectively. Then, $*$ and $\diamond$ satisfy postulates (PC1) and (PC2).*

Proof. Let $\{\preceq_K\}_{K\in\mathbb{T}}$, $\{\preceq_w\}_{w\in\mathbb{M}}$ be the families of total preorders over worlds that correspond to $*$, $\diamond$, via (F$*$), (F$\diamond$), respectively. It suffices to show that $\{\preceq_K\}_{K\in\mathbb{T}}$ and $\{\preceq_w\}_{w\in\mathbb{M}}$ satisfy conditions (C1) and (C2).

For (C1), given a world $w\in\mathbb{M}$, denote by W the theory for which $[W]=\{w\}$. First, it follows from conditions (PDR) and (PDU) that, for any world $w\in\mathbb{M}$, $\preceq_w\,=\,\preceq_W$. Next, let K be any theory, and suppose that, for every $w\in[K]$ and any $r,r'\in\mathbb{M}$, it holds that $r\preceq_w r'$. Then, we have that $r\preceq_W r'$. However, it was shown in Theorem 4 of Peppas and

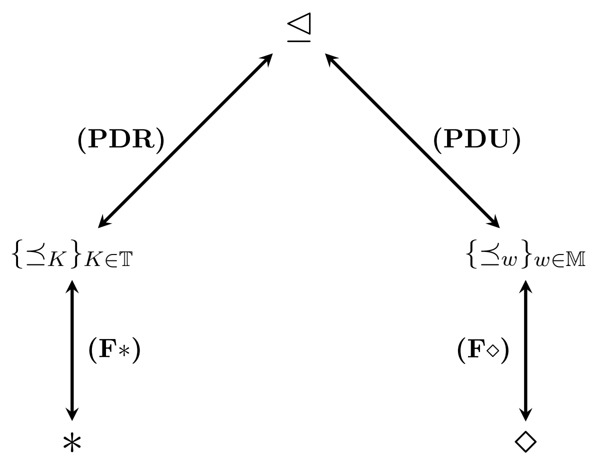

Figure 4: An arbitrary, but fixed, total preorder $\trianglelefteq$ over $\mathcal{P}$ that induces a family $\{\preceq_K\}_{K\in\mathbb{T}}$ of total preorders over worlds associated with theories, and a family $\{\preceq_w\}_{w\in\mathbb{M}}$ of total preorders over worlds associated with worlds. The families $\{\preceq_K\}_{K\in\mathbb{T}}$ and $\{\preceq_w\}_{w\in\mathbb{M}}$ induce in turn a PD revision operator $*$ and a PD update operator $\diamond$, respectively.

Williams (2016) that, for any two theories H, T and any worlds $r, r' \in \mathbb{M}$, if $r \preceq_H r'$ and $r \preceq_T r'$, then $r \preceq_{H\cap T} r'$. This again entails that if, for every $w \in [K]$, $r \preceq_W r'$, then $r \preceq_K r'$. Combining the above, we derive that $r \preceq_K r'$. Hence, we have shown that $\{\preceq_K\}_{K\in\mathbb{T}}$ and $\{\preceq_w\}_{w\in\mathbb{M}}$ satisfy condition (C1), as desired.

With a totally symmetric argument, we can show that $\{\preceq_K\}_{K\in\mathbb{T}}$ and $\{\preceq_w\}_{w\in\mathbb{M}}$ satisfy condition (C2) as well. ■

It turns out that a result analogous to Theorem 18 of Section 7 *cannot* be obtained in the case of parametrized-difference belief change. This is evident from the next example, which shows that, for a belief set and an epistemic input, a PD revision operator and a PD update operator, both induced from the same total preorder over atoms, lead to different outcomes.

Example 22 (Parametrized-Difference Belief Change)**.** *Let $\mathcal{P} = \{a, b, c, d\}$ and let $\trianglelefteq$ be a total preorder over $\mathcal{P}$ such that $\trianglelefteq = \mathcal{P}\times\mathcal{P}$ (i.e., all atoms have equal epistemic value). Moreover, let K be a theory such that $[K] = \{abcd, \overline{a}\overline{b}\overline{c}\overline{d}\}$, and let φ be a sentence of $\mathcal{L}$ such that $[\varphi] = \{\overline{a}bcd, a\overline{b}\overline{c}d\}$; observe that φ contradicts K, as $[K]\cap[\varphi] = \varnothing$.*

It is not hard to verify that, for all $w' \in [K]$, Diff $\left(abcd, \overline{a}bcd\right) \vartriangleleft$ Diff $\left(w', a\overline{b}\overline{c}d\right)$. Therefore, $\trianglelefteq$ induces (via (PDR)) a family $\{\preceq_K\}_{K\in\mathbb{T}}$ of total preorders over worlds, such that $\overline{a}bcd \prec_K a\overline{b}\overline{c}d$. Furthermore, observe that Diff $\left(abcd, \overline{a}bcd\right) \vartriangleleft$ Diff $\left(abcd, a\overline{b}\overline{c}d\right)$ and Diff $\left(\overline{a}\overline{b}\overline{c}\overline{d}, a\overline{b}\overline{c}d\right) \vartriangleleft$ Diff $\left(\overline{a}\overline{b}\overline{c}\overline{d}, \overline{a}bcd\right)$. Therefore, $\trianglelefteq$ induces (via (PDU)) a family $\{\preceq_w\}_{w\in\mathbb{M}}$ of total preorders over worlds, such that $\overline{a}bcd \prec_{abcd} a\overline{b}\overline{c}d$ and $a\overline{b}\overline{c}d \prec_{\overline{a}\overline{b}\overline{c}\overline{d}} \overline{a}bcd$.

The families $\{\preceq_K\}_{K\in\mathbb{T}}$ and $\{\preceq_w\}_{w\in\mathbb{M}}$ induce in turn a PD revision operator $*$ and a PD update operator $\diamond$, via (F$*$) and (F$\diamond$), respectively.[7] *Then, it follows from condition (F$*$)*

7. Since $\trianglelefteq = \mathcal{P}\times\mathcal{P}$, $*$ is Dalal's revision operator and $\diamond$ is Forbus' update operator.

that $[K * \varphi] = \{\overline{a}bcd\}$*, while from condition (F*$\diamond$*) that* $[K \diamond \varphi] = \{\overline{a}bcd, a\overline{b}\overline{c}d\}$*. Therefore,* $K * \varphi \neq K \diamond \varphi$*.*

We close this section by noting that uniform belief change and parametrized-difference belief change are two *mutually exclusive* types of belief change. This is pointed out in Theorem 23 subsequently, a part of which has been proven by Aravanis (2020).

Theorem 23. *The class of uniform revision operators is disjoint from the class of PD revision operators. Likewise, the class of uniform update operators is disjoint from the class of PD update operators.*

Proof. The fact that the classes of uniform revision operators and PD revision operators are disjoint has been proven in Theorem 6.4 of Aravanis (2020). The proof that the classes of uniform update operators and PD update operators are also disjoint is symmetric to the proof of Theorem 6.4 of Aravanis (2020); we present it subsequently for the sake of completeness.

It suffices to show that no KM update operator exists whose corresponding family of faithful preorders, associated with worlds, satisfies both conditions (PDU) and (UU). Consider the *non-trivial* case where there are at least two atoms in $\mathcal{P}$; hence, the set $\mathbb{M}$ contains at least four possible worlds. Let $\diamond$ be a KM update operator such that its corresponding family $\{\preceq_w\}_{w \in \mathbb{M}}$ of faithful preorders satisfies (PDU). Let w, w' be two possible worlds of $\mathbb{M}$, such that w' contains the negation of all literals of w; in symbols, $w' = \{\neg l : l \in w\}$. Furthermore, let r, r' be two possible worlds of $\mathbb{M}$, such that $r, r' \notin \{w, w'\}$. From the relation of w and w', it follows that $\mathit{Diff}\,(w, w') = \mathcal{P}$, and moreover, that $\mathit{Diff}\,(w, r) = \mathit{Diff}\,(w, w') \setminus \mathit{Diff}\,(w', r) = \mathcal{P} \setminus \mathit{Diff}\,(w', r)$ and $\mathit{Diff}\,(w, r') = \mathit{Diff}\,(w, w') \setminus \mathit{Diff}\,(w', r') = \mathcal{P} \setminus \mathit{Diff}\,(w', r')$. Combining the above, we derive that $\left|\mathit{Diff}\,(w, r)\right| < \left|\mathit{Diff}\,(w, r')\right|$ iff $\left|\mathit{Diff}\,(w', r')\right| < \left|\mathit{Diff}\,(w', r)\right|$. Therefore, from condition (i) of Definition 20, it follows that $\mathit{Diff}\,(w, r) \lhd \mathit{Diff}\,(w, r')$ iff $\mathit{Diff}\,(w', r') \lhd \mathit{Diff}\,(w', r)$. Hence, from condition (PDU), we derive that $r \prec_w r'$ iff $r' \prec_{w'} r$. This, however, violates condition (UU), which requires that the comparative plausibility of r, r' relative to w and w' remains the same. ∎

Theorem 23 should not be regarded as a mere technical contribution. In conjunction with Theorems 16 and 21, it points out that the introduced principle of consistency is respected by a *wider range* of belief-change types, as opposed to the (hypothetical) scenario where uniform belief change and parametrized-difference belief change overlap — something Theorem 23 disproves.

9. Relevance-Sensitive Belief Change

In this section, we explore uniform belief change and parametrized-difference belief change —previously introduced as proof-of-concept models for postulates (PC1) and (PC2)— in relation to *relevance*, a fundamental notion in belief change with both conceptual and computational significance that has been extensively studied; see, indicatively, the works of Gärdenfors (1990), Parikh (1999, 2011), Peppas, Williams, Chopra, and Foo (2015), Aravanis (2021), Kourousias and Makinson (2007), Makinson (2009), Hansson and Wassermann (2002), Wassermann (2001b, 2001a), Kern-Isberner and Brewka (2017), Delgrande and Peppas (2018).

Although the ultimate objective of this section is to elucidate the connection between the proposed consistency principle and relevance, we begin by examining the relevance sensitivity of uniform belief change and parametrized-difference belief change, as this analysis will serve as a foundation for our main objective.

In what follows, we shall consider a well-known notion of relevance, often referred to as the *language-splitting model*, which is encoded into a simple and intuitive axiom, introduced by Parikh (1999, 2011). The main intuition of Parikh's axiom is that, if a belief set K can be expressed in two *syntactically disjoint* compartments (representing distinct subject matters), then the revision of K by an epistemic input φ affects *only* the part of K that is syntactically relevant to φ. In a later work, Peppas et al. (2015) pointed out two different interpretations of Parikh's axiom, namely, its *weak* and its *strong* version, which are both plausible depending on the context. Herein, we shall focus on the weak version of Parikh's axiom, as it is more general and intuitive.

Let us first introduce some additional notation and terminology. For a sentence x of $\mathcal{L}$, $\mathcal{L}_x$ denotes the (unique) *minimal* language within which x can be expressed. The *complement* language of $\mathcal{L}_x$ is denoted by $\overline{\mathcal{L}_x}$; that is, $\overline{\mathcal{L}_x}$ is the language built from the atoms that do *not* appear in $\mathcal{L}_x$ (if there are no atoms that do not appear in $\mathcal{L}_x$, then $\overline{\mathcal{L}_x}$ is empty). Moreover, given a *complete* theory K such that, for a world $w \in \mathbb{M}$, $[K] = \{w\}$, $\mathit{Diff}\,(K, r)$ denotes the *difference* between theory K and a possible world r, which is defined as $\mathit{Diff}\,(K, r) = \mathit{Diff}\,(w, r) = \big((w \setminus r) \cup (r \setminus w)\big) \cap \mathcal{P}$ (refer to the definition of $\mathit{Diff}\,(w, r)$ in Section 8).

We now present the weak version of Parikh's axiom for belief revision, which is encoded into the next postulate (RR). Postulate (RR) states that, if a belief set K can be expressed in two syntactically disjoint compartments $Cn\big(\{x\}\big)$ and $Cn\big(\{y\}\big)$, then the revision of K by an epistemic input that can be asserted within the (sub)language $\mathcal{L}_x$ should *not* affect anything outside $\mathcal{L}_x$.

(RR) If $K = Cn\big(\{x, y\}\big)$, $\mathcal{L}_x \cap \mathcal{L}_y = \varnothing$, and $\mathcal{L}_\varphi \subseteq \mathcal{L}_x$, then $(K * \varphi) \cap \overline{\mathcal{L}_x} = K \cap \overline{\mathcal{L}_x}$.

Peppas et al. (2015) formulated the semantic characterization of postulate (RR), which, for a *complete* theory K, turns out to be the following constraint (SRR) on faithful preorders over worlds.[8] According to condition (SRR), the less a world differs from a complete theory K in terms of atoms, the more $\preceq_K$-plausible it is.

(SRR) If $\mathit{Diff}\,(K, r) \subset \mathit{Diff}\,(K, r')$, then $r \prec_K r'$.

Aravanis (2021) identified that there exist KM update operators that yield to *counter-intuitive results*. In light of these shortcomings, it was shown that a translation of Parikh's axiom (RR) into the realm of belief update is sufficient to prevent the unreasonable behaviour of KM update operators. The translation of (RR) in the context of belief update is encoded into the following postulate (RU), which is presented subsequently along with its semantic characterization (SRU). Postulate (RU) resembles postulate (RR), but it applies to complete theories rather than arbitrary theories. Likewise, condition (SRU) parallels condition (SRR),

8. For a semantic characterization of postulate (RR) in the general case of possibly incomplete theories, the interested reader is referred to Section 7 of Peppas et al. (2015); this characterization was extended to all popular constructive models for belief revision by Aravanis, Peppas, and Williams (2019).

but it constrains faithful preorders associated with worlds, rather than those associated with theories (Aravanis, 2021).

(RU) If K is complete, then $(K \diamond \varphi) \cap \overline{\mathcal{L}_\varphi} = K \cap \overline{\mathcal{L}_\varphi}$.

(SRU) If $\mathit{Diff}\,(w, r) \subset \mathit{Diff}\,(w, r')$, then $r \prec_w r'$.

Having presented relevance-sensitive belief change (revision and update), we turn to the study of uniform belief change and parametrized-difference belief change with respect to relevance. We begin with uniform belief change, which, as Theorem 24 shows, is *incompatible* with Parikh's notion of relevance.[9]

Theorem 24. *There is no uniform revision operator that satisfies postulate (RR), and there is no uniform update operator that satisfies postulate (RU).*

Proof. Let $*$ be an arbitrary uniform revision operator, which by definition satisfies condition (UR). Assume, towards contradiction, that $*$ satisfies postulate (RR). Let w, w' be two possible worlds of $\mathbb{M}$, such that $w = \{\neg l : l \in w'\}$. Therefore, $\mathit{Diff}\,(w, w') = \mathcal{P}$. Moreover, let K, H be two complete theories, such that $[K] = \{w\}$ and $[H] = \{w'\}$, and let r, r' be two possible worlds of $\mathbb{M}$, such that $r, r' \notin [K] \cup [H]$ and $\mathit{Diff}\,(w, r) \subset \mathit{Diff}\,(w, r')$. From the relation of w and w', it follows that $\mathit{Diff}\,(w, r) = \mathit{Diff}\,(w, w') \setminus \mathit{Diff}\,(w', r) = \mathcal{P} \setminus \mathit{Diff}\,(w', r)$ and $\mathit{Diff}\,(w, r') = \mathit{Diff}\,(w, w') \setminus \mathit{Diff}\,(w', r') = \mathcal{P} \setminus \mathit{Diff}\,(w', r')$. Consequently, we derive that $\mathcal{P} \setminus \mathit{Diff}\,(w', r) \subset \mathcal{P} \setminus \mathit{Diff}\,(w', r')$. This again entails that $\mathit{Diff}\,(w', r') \subset \mathit{Diff}\,(w', r)$. Therefore, we can conclude that $\mathit{Diff}\,(K, r) \subset \mathit{Diff}\,(K, r')$ and $\mathit{Diff}\,(H, r') \subset \mathit{Diff}\,(H, r)$. Now, let $\preceq_K$, $\preceq_H$ be the faithful preorders that $*$ assigns (via (F$*$)) at K, H, respectively. Then, we derive from condition (SRR) that $r \prec_K r'$ and $r' \prec_H r$. This, however, contradicts condition (UR), which imposes that $r \preceq_K r'$ iff $r \preceq_H r'$. Consequently, we have shown that there is no uniform revision operator that satisfies postulate (RR), as desired.

Thereafter, let $\diamond$ be an arbitrary uniform update operator, which by definition satisfies condition (UU). Suppose, towards contradiction, that $\diamond$ satisfies postulate (RU). Let $\preceq_w$, $\preceq_{w'}$ be the faithful preorders that $\diamond$ assigns (via (F$\diamond$)) at w, w', respectively. Then, we derive from condition (SRU) that $r \prec_w r'$ and $r' \prec_{w'} r$. This, however, contradicts condition (UU), which imposes that $r \preceq_w r'$ iff $r \preceq_{w'} r'$. Consequently, we have shown that there is no uniform update operator that satisfies postulate (RU), as desired. ■

As far as parametrized-difference belief change is concerned, already in Theorem 4 of Peppas and Williams (2016), it was shown that PD revision operators satisfy postulate (RR). Theorem 12 of Aravanis (2021) also proved that PD update operators satisfy postulate (RU). Therefore, contrary to uniform belief change, parametrized-difference belief change *respects* Parikh's notion of relevance, as stated in Theorem 25 below.

Theorem 25 (Peppas & Williams, 2016; Aravanis, 2021)**.** *PD revision operators satisfy postulate (RR), and PD update operators satisfy postulate (RU).*

9. Interestingly, uniform belief revision respects a variant of the strong version of Parikh's axiom for belief revision, as it has been proven by Aravanis (2020, Theorem 6.7).

Finally, we investigate the relation between the introduced consistency principle, encoded into postulates (PC1) and (PC2), and relevance, encoded into postulates (RR) and (RU). First, note that since parametrized-difference belief change respects all (PC1) & (PC2) and (RR) & (RU), it follows that consistency and relevance are *compatible*. However, as it is pointed out in Example 26, there exist AGM revision operators and KM update operators that respect consistency and violate relevance, and conversely, there exist AGM revision operators and KM update operators that respect relevance and violate consistency.

Example 26 (Consistency & Relevance)**.** *Let $\mathcal{P} = \{a, b, c\}$, let w_1, w_2, r_1, r_2, r_3 be possible worlds of $\mathbb{M}$ such that $w_1 = \{a, b, c\}$, $w_2 = \{\neg a, \neg b, \neg c\}$, $r_1 = \{\neg a, b, c\}$, $r_2 = \{\neg a, \neg b, c\}$ and $r_3 = \{a, b, \neg c\}$, and let K, H be two theories such that $[K] = \{w_1, w_2\}$ and $[H] = \{w_1\}$. Consider an AGM revision operator $*$ and a KM update operator $\diamond$ that satisfy postulates (PC1) and (PC2). Clearly then, $*$ and $\diamond$ induce (via (F$*$) and (F$\diamond$)) two families $\{\preceq_K\}_{K\in\mathbb{T}}$ and $\{\preceq_w\}_{w\in\mathbb{M}}$ of faithful preorders over worlds that satisfy conditions (C1) and (C2). Assume that $r_1 \prec_{w_1} r_2$, $r_1 \prec_{w_2} r_2$, and $r_1 \prec_K r_2$, a scenario fully compatible with conditions (C1) and (C2). In that case however, the faithful preorder $\preceq_{w_2}$ violates condition (SRU), which requires that $r_2 \prec_{w_2} r_1$. Consequently, $\diamond$ violates postulate (RU).*

Thereafter, consider an AGM revision operator $*$ and a KM update operator $\diamond$ that satisfy postulates (RR) and (RU), respectively. Clearly then, $*$ and $\diamond$ induce (via (F$*$) and (F$\diamond$)) two families $\{\preceq_K\}_{K\in\mathbb{T}}$ and $\{\preceq_w\}_{w\in\mathbb{M}}$ of faithful preorders over worlds that satisfy conditions (SRR) and (SRU), respectively. Assume that $r_1 \prec_{w_1} r_3$ and $r_3 \prec_H r_1$, a scenario fully compatible with conditions (SRR) and (SRU). This compatibility is due to the fact that $\preceq_H$ and $\preceq_{w_1}$ trivially satisfy (SRR) and (SRU), respectively, since the antecedent of both conditions is falsified, as the sets Diff (w_1, r_1), Diff (w_1, r_3) are incomparable with respect to set inclusion, as are the sets Diff (H, r_1), Diff (H, r_3). However, under these circumstances, $\{\preceq_K\}_{K\in\mathbb{T}}$ and $\{\preceq_w\}_{w\in\mathbb{M}}$ violate condition (C2) at theory H, implying that $*$ and $\diamond$ violate postulate (PC2).

It is noteworthy that the proposed consistency principle and the notion of relevance share certain conceptual and computational aspects. Conceptually, both principles, each in its own way, impose a form of "well-behavedness" in the belief-change process. Computationally, both offer significant practical advantages: relevance ensures that belief change remains confined to a local portion of a knowledge base, while consistency potentially reduces the agent's need to provide excessive information, as discussed in Subsection 6.3.

The results of the present study are summarized in Figure 5, which provides an abstract overview of the types of belief change examined. In the context of Figure 5, a particular belief-change type should be regarded as a pair $\langle *, \diamond \rangle$ of an AGM revision operator $*$ and a KM update operator $\diamond$; hence, each closed curve in the figure circumscribes pairs of rational revision and update operators.

10. Conclusion

In this article, we further illuminated the interrelation between two fundamental processes of belief change, namely, belief revision and belief update. At first, we introduced axiomatically and semantically a principle of consistency between belief revision and belief update, which relates in a coherent manner the revision and update policies employed by a single rational

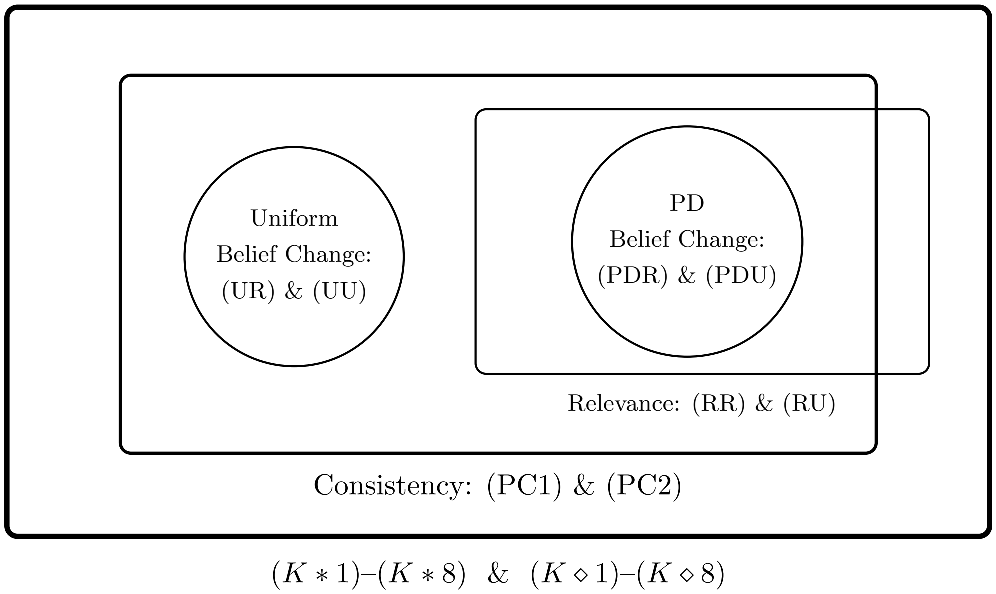

Figure 5: The types of belief change discussed in the present study. Each closed curve circumscribes pairs of rational revision and update operators.

agent. It turned out that, in view of this consistency principle, revision and update are indistinguishable in the realm of complete states of belief, as both processes lead to identical outcomes. Additionally, we showed that uniform belief change and parametrized-difference belief change —prominent belief-change types with applicable properties— serve as two well-behaved proof of concepts for the introduced consistency principle. Furthermore, by defining uniform belief update, we proved an interesting property of uniform belief change; that is, uniform belief revision and uniform belief update by epistemic input that contradicts the initial state of belief are equivalent. Lastly, we established that uniform belief change is incompatible with Parikh's concept of relevance. Consequently, in view of previous results, we demonstrated that parametrized-difference belief change is relevance-sensitive (indicating that the proposed notion of consistency is compatible with relevance), while uniform belief change is not.

As belief revision and belief update are typically carried out by a single rational agent, there may be other plausible and interesting ways —beyond consistency— that link these two fundamental operations of belief change, either conceptually or technically. We are currently investigating such potential interrelations.

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