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A Mathematical Study on Dispersion of Non-buoyant Air Pollutants Emitted from Point Source Having Variable Wind Velocity

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KEYWORDS
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Vertical distance;
Concentration profile.

ABSTRACT: The objective of the present study is to discuss the dispersion of non-buoyant air pollutants emitted from point source in the atmosphere, where some kind of removal mechanism is taking place. In this paper, a partial differential equation describing unsteady state dispersion of non-buoyant pollutants is solved by considering the wind velocity in the form of wave function that varies with downwind distance. Besides wind velocity, the various other factors like downwind distance, vertical distance, and cross-wind distance are also considered. The concentration of pollutants is calculated and variations of concentration profile with various factors are shown and results are analyzed.

1. Introduction

With the rapid expansion in the global population, transportation and industrialization environmental contamination, especially air pollution has become a serious problem for the living organisms. Among various types of environment pollutions, air pollution is most hazardous and poses threat to life of human beings and other species on earth.

Air pollutants are harmful airborne substances that threaten human welfare, harm vegetation and affect visibility which gives climate change. Now-a-days, air pollution is one of the biggest problems, not only because it contributes to climate change but also because it affects public and individual health by raising morbidity and mortality rates. Air pollutants play roles indifferent kinds of human diseases (Manisalidi et al., 2020). Particulate matters (PM), are one of them which are particles with varying but extremely small diameters that enter the human respiratory system through inhalation and can lead to cancer, cardiovascular, and central nervous system disorders, as well as respiratory and other diseases. Air pollutants come from either natural source or anthropogenic (human) sources. The distinction between two sources of pollutants is not always clear. Natural sources of air-pollutants include ash and dust from volcanoes, wind-entrained dust from natural land surfaces, smoke and ash from wildfires (Whiteman, 2000).

Primary pollutants can be emitted and secondary pollutants can be produced in the atmosphere as a result of chemical or physical reactions of primary pollutants when exposed to other components of air including water vapor. Some secondary pollutants can be from photo-chemical reactions. Pollutants can (such as photo-chemical smog or ozone) also come from point source, line source, area source or volume source. The emission of pollutants may be continuous or intermittent and the source strength may be constant or variable. Dispersion of pollutants is also effected by source position.

Lead, sulfur oxides, nitrogen oxides, carbon monoxide, particle pollution, and ground-level ozone are the six main air pollutants that the World Health Organization (WHO) monitors. Air pollution can have a catastrophic impact on soil, groundwater, and other environmental elements. It also presents a significant risk to living things. There are significant ecological effects of acid rain, global warming, greenhouse effect, and climate change on air pollution (Wilson and Suh, 1997).

Moreover, dioxins, sulfur dioxide, nitrogen oxide, dioxin-containing volatile organic compounds (VOCs), and polycyclic aromatic hydrocarbons (PAHs) are all regarded as hazardous air pollutants for human health.

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High levels of carbon monoxide can even cause direct poisoning through inhalation. Depending on the extent of exposure, heavy metals like lead can cause either acute poisoning or long-term intoxication when absorbed into the human body. Air pollution affects people's health in many ways. The low levels of air pollution can also have an impact on the health of vulnerable and sensitive people. The following conditions are strongly associated with short-term exposure to air pollution: asthma, respiratory diseases, wheezing, coughing, shortness of breath, and high rates of hospitalization (Manisalidis, 2020).

The importance and the need of mathematical modelling are well known in the scientific community. There are various modeling approaches that have been used effectively in the past to deal with the dispersion of air pollutants. Also, persistent efforts are being made to improve the accuracy of predictions using latest advancements in the computing technology and improvement in the observational and modelling framework (Sharan et al., 2003).

Sharan et al., 1996 made an attempt to review the major research concerning atmospheric dispersion modelling in the last few decades. Sharan et al. have formulated a mathematical model for low wind conditions by taking into account the diffusion in the downwind direction (Sharan et al., 2003, 1996).

Srivastava et al., 2009 have presented a three dimensional atmospheric diffusion model with variable removal rate and variable wind velocity using power law profile.

Nirmaladevi et al., 2018 has presented a three dimensional analytical model for the dispersion of air pollutants emitted from elevated point source with mesoscale wind.

Verma, 2011 has given an analytical approach to the solution of the problem of dispersion of an air pollutant in steady state condition with constant wind velocity and constant removal rate taking eddy diffusivities as constant.

Agarwal et al., 2008 have solved an unsteady state three- dimensional atmospheric diffusion equation with a point source assuming that the wind velocity vary with downwind distance in the form of wave function and removal rate as constant.

The analytical solutions for advection–diffusion equation with different conditions and cases of the wind speed and eddy diffusivity have been studied by several researchers (Verma et al., 2011, Verma et al., 2016, Verma et al., 2015).

Particulates have the tendency to settle down on the ground of the atmosphere, and so their non-buoyant nature must be taken into consideration (Agarwal and Shukla, 2002). This is done by introducing a negative sink velocity in the vertical direction. Again, it is essential to use suitable boundary conditions for perfect observations of the dispersion process. Apart from the boundaries in the atmosphere, several removal processes (e.g., removal by rain or fog droplets, deposition on vegetative canopies, artificial removal by the introduction of some chemical species, etc.) are observed. Alam and Seinfeld 1981 have been studied the effects of removal mechanisms on the dispersion process.

In view of the above, in this paper, a study has been made to discuss the dispersion of non-buoyant air pollutants emitted from point source in an atmosphere, where some kind of removal mechanism is taking place. Here, we consider wind velocity in the form of wave function that varies with downwind distance. Further, Eddy diffusivity coefficients are taken to be constant. For solving the model, we have applied integral transform methods (i.e., Laplace transform and Fourier transform). A Dirichlet-type boundary conditions are used, which indicate total absorption at the ground and inversion layers.

2. Mathematical Model

The partial differential equation describing the unsteady state of dispersion of non-buoyant pollutants is given by

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} - w_s\frac{\partial C}{\partial z} = \frac{\partial}{\partial y}\left(K_y\frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z\frac{\partial C}{\partial z}\right) - \alpha C \quad (1)$$

where C is the concentration of the pollutants , t is time, α is the constant removal rate of pollutants, and K_y and K_z are the eddy diffusivities in the y and z directions, respectively, U is the wind velocity and w_s is the sink velocity. We consider the pollutant emitted from a point source of strength Q, which is located at stack height h_s. Here, the wind velocity U is taken in the form of a wave function as $U = U_0(1 + \varepsilon cos\frac{2\pi x}{\lambda})$, where U_o is the mean wind velocity, λ is the wave length, and ε is the amplitude ratio. The effect of buoyancy on the trajectory motion of the heavy pollutant is modeled by prescribing a negative sink velocity ($-w_s$) in the z direction, where $w_s = |w_s|$.

The initial and boundary conditions for the system (1) are taken as follows:

$C(x, y, z, t) = 0, \quad t = 0$ (2)

$C(x, y, z, t) = \frac{Q\delta(y)\delta(z-h_s)}{U(x)}, \; x = 0, t \geq 0, 0 \leq h_s \leq H$ (3)

$C(x, y, z, t) = 0, \quad y \to \pm\infty, t \geq 0$ (4)

$C(x, y, z, t) = 0, \quad z = 0, t \geq 0$ (5)

$C(x, y, z, t) = 0, \quad z = H, t \geq 0$ (6)

Condition (5) and (6) together are Dirichlet's type conditions, which imply perfectly absorbent boundaries. In other words, Dirichlet's boundary conditions indicate that contaminants are removed immediately upon contact with the boundaries, resulting in a significant concentration gradient in the vertical direction (Lin and Hildemann, 1996).

3. Method of Solution

The partial differential equation (1) describing the unsteady state of dispersion of non-buoyant pollutants and the boundary conditions are made dimensionless by using the following dimensionless quantities:

$\lambda^* = \frac{K_z\lambda}{U_0H^2}, t^* = \frac{K_zt}{H^2}, C^* = \frac{U_0H^2C}{Q}, x^* = \frac{K_zx}{U_0x^2}, \alpha^* = \frac{\alpha H^2}{K_z}, y^* = \frac{y}{H}, w^* = \frac{w_sH}{K_z}$.

On dropping astricks (*) and using $U = U_0[1 + \varepsilon\cos(\frac{2\pi x}{\lambda})]$, the equation (1) becomes

$$\frac{\partial C}{\partial t} + [1 + \varepsilon\cos(\frac{2\pi x}{\lambda})]\frac{\partial C}{\partial x} - w\frac{\partial C}{\partial z} = \beta\frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2} - \alpha C \quad (7)$$

where $\beta = \frac{K_y}{K_z}$ and boundary conditions become

$C = 0, \quad t = 0$ (8)

$C = \frac{Q\delta(y)\delta(z-h_s)}{\left(1+\varepsilon\cos(\frac{2\pi x}{\lambda})\right)}, \; x = 0, t \geq 0,$ (9)

$C = 0, \quad y \to \pm\infty, t \geq 0$ (10)

$C = 0, \quad z = 0, t \geq 0$ (11)

$C = 0, \; z = 1, \; t \geq 0$ (12)

Taking the Laplace transform of equation (7), w. r. t. 't' , we get

$$S\bar{C} + [1 + \varepsilon\cos(\frac{2\pi x}{\lambda})]\frac{\partial \bar{C}}{\partial x} - w\frac{\partial \bar{C}}{\partial z} = \beta\frac{\partial^2 \bar{C}}{\partial y^2} + \frac{\partial^2 \bar{C}}{\partial z^2} - \alpha\bar{C} \quad (13)$$

where bar (—) denotes the Laplace transform of the function and 'S' is the corresponding Laplace transform parameter.

Again, taking the Fourier transform of equation (13) w. r. t. 'y', we get

$$[1 + \varepsilon\cos(\frac{2\pi x}{\lambda})]\frac{\partial \hat{\bar{C}}}{\partial x} + (\alpha + S + \beta p^2)\hat{\bar{C}} = \frac{\partial^2 \hat{\bar{C}}}{\partial z^2} + w\frac{\partial \hat{\bar{C}}}{\partial z} \quad (14)$$

where cap (∧) denotes the Fourier transform of the function and p is the corresponding Fourier transform parameter.

Using the above-mentioned integral transforms (Laplace and Fourier transforms), the boundary conditions (8-12) become

$\hat{\bar{C}} = \frac{Q\delta(z-h_s)}{S[1+\varepsilon\cos(\frac{2\pi x}{\lambda})]}, \quad x = 0$ (15)

$\hat{\bar{C}} = 0, \quad z = 0$ (16)

$\hat{\bar{C}} = 0, \quad z = 1$ (17)

To use the method of separation of variables for solving equation (14), we take

$\hat{\bar{C}} = M(x)\,N(z)$ (18)

where $M(x)$ and N(z) are the functions of x and z, respectively.

Using (18) in (14), we get the following two ordinary differential equations:

$$[1 + \varepsilon\cos(\frac{2\pi x}{\lambda})]\frac{dM}{dx} + (\alpha + S + \beta p^2 + k^2)M = 0 \quad (19)$$

$$\frac{d^2N}{dz^2} + w\frac{dN}{dz} + k^2N = 0 \quad (20)$$

where k^2 is a separation constant.

For the solution of equation (19), we write it in the form

$$[1 + \varepsilon\cos(\frac{2\pi x}{\lambda})]\frac{dM}{dx} = -(\alpha + S + \beta p^2 + k^2)M$$

or $\frac{dM}{M} + \frac{(\alpha+S+\beta p^2+k^2)}{\left(1+\varepsilon\cos(\frac{2\pi x}{\lambda})\right)}dx = 0$

which on integration gives

$$\log M = -(\alpha + S + \beta p^2 + k^2)\int\frac{dx}{[1+\varepsilon\cos(\frac{2\pi x}{\lambda})]} = -(\alpha + S + \beta p^2 + k^2)\int\frac{dx}{(1-\varepsilon)+2\varepsilon\cos^2\left(\frac{\pi x}{\lambda}\right)}$$

which can also be written as

$$\log M = -(\alpha + S + \beta p^2 + k^2)\int\frac{\sec^2\left(\frac{\pi x}{\lambda}\right)}{(1-\varepsilon)\sec^2\left(\frac{\pi x}{\lambda}\right)+2\varepsilon}dx$$

or $\log M = -\frac{\lambda(\alpha+S+\beta p^2+k^2)}{\pi\sqrt{1-\varepsilon^2}}\tan^{-1}\left[\tan\left(\frac{\pi x}{\lambda}\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\right)\right] + \log C_1$

where C_1 is the arbitrary constant of integration.

or $M = C_1\exp[-(\alpha + S + \beta p^2 + k^2)]g(x)$ (21)

where $g(x) = \frac{\lambda}{\pi\sqrt{1-\varepsilon^2}}\tan^{-1}\left[\tan\left(\frac{\pi x}{\lambda}\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\right)\right]$ (22)

Now, the solution of equation (20) is given by

$$N(z) = C_2e^{\left[\left\{\frac{-w+\sqrt{w^2-4k^2}}{2}\right\}\right]z} + C_3e^{\left[\left\{\frac{-w-\sqrt{w^2-4k^2}}{2}\right\}\right]z} \quad (23)$$

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where C_2 and C_3 are another arbitrary constants of integration.

Now, using (16) and (17) in (23), we get the following eigen value equation:

$2\, k_n = w$, where n = 1, 2, 3,…

Putting the values of M(x) and N(z) in equation (18), we get the solution as

$$\hat{\bar{C}} = \sum_{n=1}^{\infty} C_n e^{\{-(\alpha+S+\beta p^2+k_n{}^2)g(x)\}} = \left[e^{\left(\frac{-w+\sqrt{w^2-4k_n{}^2}}{2}\right)z} + e^{\left(\frac{-w-\sqrt{w^2-4k_n{}^2}}{2}\right)z}\right] \qquad (24)$$

where $C_n = C_1C_2C_3$

From boundary condition (15), we get

$$\frac{Q\delta(z-h_s)}{S(1+\varepsilon)} = \sum_{n=1}^{\infty} C_n f_n(z) \qquad (25)$$

where $f_n(z)$ is given by equation (23).

Multiplying throughout by f_m (z) and integrating w.r.t. z from 0 to 1, we get

$$\int_0^1 \frac{Q\delta(z-h_s)f_m(z)}{S(1+\varepsilon)} = \sum_{n=1}^{\infty} C_n \int_0^1 f_n(z) f_m(z)dz$$

Using the results $\int_0^1 \delta(z-h_s)f_m(z)dz = f_m(h_s)$ and $\int_0^1 f_m(z)f_n(z)dz = 0 \; if \; m \neq n$, we get

$$C_n = \sum_{n=1}^{\infty} \frac{Qf_n(h_s)}{S(1+\varepsilon)\int_0^1 \{f_n(z)\}^2 dz} \qquad (26)$$

Therefore, using the value of C_n from (26), the value of $\hat{\bar{C}}$ becomes

$$\hat{\bar{C}} = \sum_{n=1}^{\infty} \frac{Qe^{-(\alpha+S+\beta p^2+k_n{}^2)}g(x)f_n(h_s)f_m(z)}{S(1+\varepsilon)\int_0^1 \{f_n(z)\}^2 dz} \qquad (27)$$

Now, taking inverse transforms of equation (27), we get

$$C = \frac{1.414QH(t-g(x))}{\sqrt{\beta g(x)}} \, exp\left\{-\frac{y^2}{4}\beta g(x)\right\}. \sum_{n=1}^{\infty} \frac{f_n(h_s)f_n(z)}{(1+\varepsilon)\int_0^1 \{f_n(z)\}^2 dz} e^{-(\alpha+k_n{}^2)g(x)} \qquad (28)$$

where $H(t-g(x))$ is the Heavy side function and $g(x)$ is given by (22).

4. Results and Discussion

Here, the case of dispersion of non- buoyant air pollutants by a continuous point source, where the wind velocity is taken in the form of wave function varying with downwind distance is studied. The concentration of pollutant is calculated by using equation (26). The dimensionless parametric values used in the analysis are taken as follows:

$\alpha = 2$, $h_s = 0.2$, $H = 1$, $\beta = 10$, $Q = 1$, $\lambda = 10$, $\varepsilon = 0.005$.

In order to illustrate the behavior of concentration profile, the dimensionless concentration when Q = 1 is displayed graphically for the variety of conditions as shown below:

In figure (1), the concentration profile of non- buoyant pollutant is plotted against the vertical distance ($0 \leq z \leq 1$) for different values of downwind distance (x= 0.3, 0.5, 0.7). The value of y is taken to be zero. It is observed that for a particular downwind distance (x = 0.3), the concentration profile of non buoyant pollutant attains its peak at (z = 0.2). But with increasing downwind distance, the concentration profile decreases and approaches uniform distribution. As a consequence, the concentration profile of pollutants near the ground becomes high, which may prove dangerous.

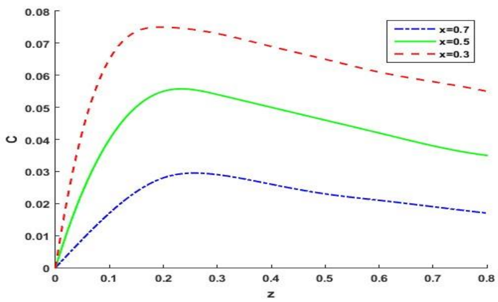

Fig 1: Dimensionless concentration **C(*x*, 0, *z*)** plotted against vertical height.

In figure (2), the concentration profile is plotted against the downwind distance ($0 \leq x \leq 1$) for different values of vertical distance (z=0.2, 0.5, 0.8), keeping the value of cross-wind distance fixed at y=0. It is seen that for a particular downwind, i.e, x=0.2, the profile is very high, but on increasing the downwind distance, we observe that the concentration profile of non- buoyant pollutants decreases regularly.

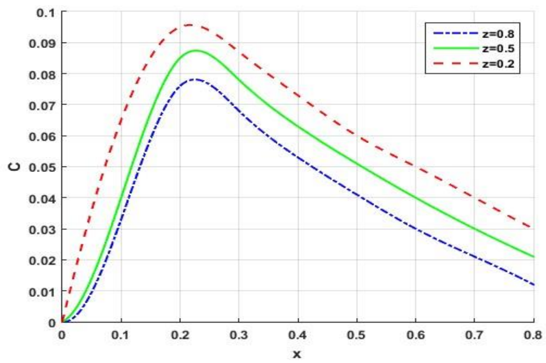

Fig 2: Dimensionless concentration **C(*x*, 0, *z*)** plotted against downwind distance ***x***.

In figure 3, the concentration profile is plotted with respect to downwind distance ($0 \le x \le 1$) for different values of cross-wind distance, i.e. (y = 0.2, 0.6, 0.9), keeping the value of vertical distance fixed on the value (z = 0.2). It is observed from the figure that concentration profile attains its peak at low distances, while for higher crosswind distances, the concentration profile decreases regularly.

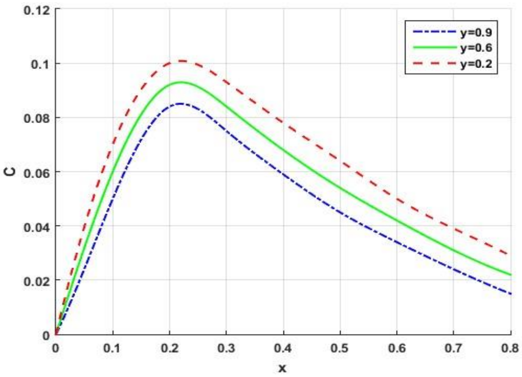

Fig 3: Dimensionless concentration **C(*x*, *y*, 0.2)** plotted against downwind distance ***x***.

In figure 4, the concentration profile is plotted against the vertical distance ($0 \le z \le 1$) with respect to different values (y = 0.2, 0.6, 1.0) for the constant value of the downwind distance, i.e., (x=0.2). It is seen that in the vertical direction, the concentration profile along the centerline of the plume reaches a point of maximum concentration, followed by extended spreading. It is also seen that the concentration level of non- buoyant pollutant decreases, and there is lateral spreading with increasing cross-wind distance ($y \ge 1$) from the source.

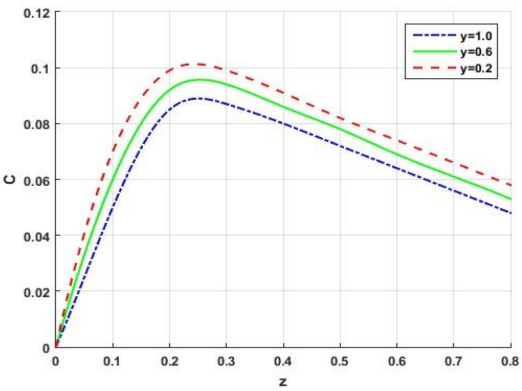

Fig 4: Dimensionless concentration **C (0.2, *y*, *z*)** plotted against vertical height ***z***.

5. Conclusion

In this study, a mathematical model for dispersion of non-buoyant air pollutants is constructed and is solved incorporating different parametric values with the wind velocity in the form of wave function that vary with down-wind distance. The concentration profile of pollutants against the vertical and downwind distances for different parametric values is investigated.

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