A Reliable Approach to Forecast Prices of Precious and Base Metals

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Abstract
Getting reliable and trustworthy estimates of future metal prices is important. This manuscript applies a pricing model based on geometric Brownian motion simulation to test the reliability of expected price forecasts of silver, aluminum, copper, iridium, nickel, lead, palladium, platinum, rhodium, ruthenium, tin and zinc. Expected prices were estimated by totaling up the product of simulated prices and associated probabilities at the monthly, quarterly and annual frequencies, with historic mean and standard deviation based on a rolling twenty-year window as proxies for drift and diffusion. Results indicate that one-period ahead forecasts based on higher number of simulations are more reliable than those based on only one simulation. Besides monthly forecasts and quarterly forecasts may be more trustworthy than those at the annual frequency.

Keywords: geometric Brownian motion, forecasting, commodities, metals, probabilities, asset-pricing, normal distribution, expected value, mathematical methods, mathematical and simulation modeling, ensemble effect

JEL Classifications: C0, C6, G0, G12, G170

I. Introduction

Metals are an essential ingredient in the worldwide economic engine, and accordingly accurate forecasts of future metal prices is of interest to economic agents (Liu et al. 2017). Forecasting metal prices is critically important for a country's production activities and industrial policies, as metals are indispensable raw materials (Kahraman and Akay 2023; Oikonomou and Damigos 2024; Zhao et al. 2023). Numerous manuscripts have been published studying factors that influence prices, developing and testing models (Chen et al. 2014; Dooley and Lenihan 2005; Du et al. 2021; Gil-Alana and Poza 2024) that can be used to obtain accurate forecasts for investment or trading decisions. This manuscript contributes to the academic literature on modeling metal prices by applying a commonly used Econophysics (Sinha 2024e) technique called geometric Brownian motion (GBM), to obtain reliable one-period ahead forecasts of prices of six base, and six precious metals.

An exhaustive literature exists that have applied GBM to forecast asset prices. (Parungrojrat and Kidsom 2019; Reddy and Clinton 2016; Samuelson 1965; Sinha 2021, 2024a; Urama and Ezepue 2018) have applied GBM to forecast stock prices or stock indexes. (Alhagyan 2024; Lynch et al. 2021) priced energy assets using GBM. Similarly, a few of the many have applied GBM forecasting to agricultural products (Ibrahim, Misiran, and Laham 2021; Zelingher and Makowski 2024), and exchange rates (Abbas and Alhagyan 2023; Alhagyan 2022; Farzin, Moghaddam, and Shahbalaei 2024). Although only a few (Black and Scholes 1973; Germansah, Tjahjana, and Herdiana 2023; Stojkoski et al. 2020; Veestraeten 2013) are being mentioned here, some aspect of GBM modeling has been applied to derivatives securities. GBM modeling has been

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applied to metal pricing by (Hamdan, Ibrahim, and Mustafa 2020; Huang et al. 2022; Ramos et al. 2019; Roslan and Halim 2024; Soysal 2023; Wilmot 2019), as well although the approach used in this manuscript differs from these.

Typically, GBM modeling requires the application of a solution to a differential equation represented in equations *(2)* and *(6)* of this manuscript. These equations have an observed price that is used to make one period ahead forecast using a drift and diffusion terms, and a standard normal random variable that has a mean of zero and a standard deviation of one. Although, the random variable can take on numerous values, allowing for multiple one period ahead forecasts, usually an insufficient number of simulations are obtained (Kumar et al. 2024; Sinha 2021). If numerous forecasts are simulated, the issue of importance becomes as to which forecast to use. (Sinha 2021) addresses this issue by estimating an expected value from the all the simulated forecasts. As simulated values have associated probabilities, expected values from the simulated forecasts can be obtained, and compared to actual prices. (Sinha 2021, 2024f, 2024a) applied a similar methodology to forecast stock index values, and gold price forecasts (Sinha 2024d). In this manuscript, the methodology is applied to six precious metals, silver, iridium, palladium, platinum and rhodium, and six base metals, aluminum, copper, nickel, lead, tin and zinc at the monthly, quarterly, and annual frequencies, and simulation that vary from one, to one hundred thousand.

Modeling prices or return behavior of these metals is prevalent and continued interest the literature. The motivation to apply the methodology to precious metals (Askari and Askari 2011; Barczak 2014; Jalali and Heidari 2018; Oral and Unal 2019; Pierdzioch and Risse 2020; Tapia Cortez et al. 2018; Wen et al. 2017) and base metals (Dooley and Lenihan 2005; Kahraman and Akay 2023; Kriechbaumer et al. 2014; Oikonomou and Damigos 2024; Pincheira Brown and Hardy 2019; Tapia Cortez et al. 2018; Wang et al. 2021; Zhao et al. 2023) arise from the fact that these have been extensively modeled using different methodologies. Although a few manuscripts have applied GBM to obtain forecast of metal prices, this manuscript contributes to the literature by applying the methodology developed by (Sinha 2021) where expected index value of the S&P 500 were estimated using numerous GBM simulations. A similar methodology was also applied to model gold prices by (Sinha 2024d), and in this manuscript, the approach is applied model the prices of an additional six base and precious metals.

Results indicate that reliable and accurate forecasts for metal prices can be obtained especially for the monthly frequencies, at higher number of simulations. The GBM based expected metal prices estimated using one hundred thousand simulations, and the actual metal price for all twelve metals show very high correlations at the monthly frequency. Although the correlations decrease in the quarterly and annual frequencies, they are still very high at the quarterly level, and greater than 0.5 for eight of the twelve metals at the annual frequency. Regression results show good forecasting ability of the expected GBM metal price, especially at the monthly and quarterly frequencies, and for some of the metals at the annual frequency. Tests for differences in means and standard deviations between actual and expected metal prices show a similar pattern for the monthly, quarterly and annual frequencies. Results also indicate that if the expected values are estimated using only one simulation, then those values have very little if any predictable power. GBM based expected values estimated using at least one hundred simulations are found to have usefulness that may mirror those estimated using even higher number of simulations.

In this manuscript, literature review is section II. Section III is the data, while section IV discusses the methodology. The results are in section V, and section VI is the conclusion.

II. Literature Review

Developing accurate models to forecast metal prices is of prime importance as even minor enhancements in prediction accuracy can result in enormous benefits (Alameer et al. 2019; Du et al. 2021). Accordingly, a wide variety of models of different degrees of complexity have been developed and applied to forecast the price of a number of different types of metals. For example, (Schwartz 1997) applied three different stochastic differential models along with a Kalman-filter methodology to understand the behavior or crude oil, copper and gold. (Achireko and Ansong 2000) used a three-step process that included multivariate, normally distributed random variable generator, neural networks, and multiple regression analysis to model gold prices. (Chen et al. 2014) applied the PANIC method, namely the Panel Analysis of Non-stationarity in Idiosyncratic and Common Components, a procedure developed by (Bai and Ng 2004) to identify factors that influenced prices of fifty-one commodities that included metals like aluminum, copper, zinc, iron ore, lead, tin, nickel and uranium.

(Dooley and Lenihan 2005) applied an Autoregressive Integrated Moving Average (ARIMA) model to forecasts of lead and zinc prices. (Askari and Askari 2011) use the Grey model for gold price forecasting, while (Barczak 2014) compared the Grey, Holt's exponential smoothing and ARIMA model while forecasting gold prices. (Alipour, Khodaiari, and Jafari 2019) compare forecasts of ARIMA and threshold generalized autoregressive conditional heteroskedastic model (TGARCH) time-series models with models based on Stochastic Differential Equations (SDE) while predicting monthly copper prices. (García and Kristjanpoller 2019) developed a hybrid and non-hybrid approach to copper price volatility that included ARIMA, GARCH and artificial neural networks and fuzzy inference systems (FIS). (Hu, Ni, and Wen 2020) integrate Generalized Autoregressive Conditional Heteroskedasticity (GARCH) modeling and artificial neural networks to model copper price volatility predictions. (Sánchez Lasheras et al. 2015) compared the forecasts of copper prices using neural networks and ARIMA models, and found the neural networks approach to provide superior results. (Wen et al. 2017) applied different algorithms in a support vector machine (SVM) and artificial neural networks to forecast gold prices. (Wang et al. 2019) applied a price volatility network and artificial neural networks to predict copper spot prices.

(Maranon 2018) applied band-pass filters and Elliott Wave Principle to analyze and model metal price cycles. (Tapia Cortez et al. 2018) combined Chaos Theory and machine learning techniques to forecast prices of metals like, aluminum, copper, lead, nickel, silver, zinc, gold, tin, palladium, and platinum among other minerals. (Oral and Unal 2019) applied a vector autoregressive fractionally integrated moving average (VARFIMA) model to estimated local Hurst exponents, along with wavelet coherence analysis to daily prices of gold, silver and platinum. (Risse 2019) forecasts gold returns by combing wavelet decomposition and machine learning. (Kriechbaumer et al. 2014) applied wavelet-based multiresolution analysis preceding an ARIMA fitting to increase forecasting accuracy in the monthly forecasts of aluminum, copper, lead and zinc. (He et al. 2015) used a curvelet based multiscale methodology to forecast metal prices.

(Chen, He, and Zhang 2016) use a novel Grey wave forecasting approach that requires unequal-interval contour lines, and contour time filtrating to predict prices of aluminum and nickel at the monthly frequency. (Gligorić et al. 2020) applied a hybrid model based on Grey System Theory, and a stochastic differential model to monthly lead prices. (Ozdemir, Buluş, and Zor 2022) applied deep learning algorithms that required recurrent neural networks-based on long short-term memory (LSTM) and gated recurrent unit (GRU) networks to forecasted medium- and long-term horizon nickel prices. (Li et al. 2020) develop the group method of data handling (GMDH) intelligence system to predict the price of iron ore, and compared the results of the technique to ARIMA, support vector regression (SVR), artificial neural network (ANN), and classification and regression tree (CART) approaches.

(Jalali and Heidari 2018) applies the Grey GM(1,1) model to forecast palladium prices. (Pierdzioch and Risse 2020) use a multivariate random forests approach to compute out of sample forecasts of four precious metals namely gold, silver, platinum and palladium. (Du et al. 2021) apply an optimized extreme machine learning and nature inspired meta-heuristic algorithms to predict copper and gold prices. Based on a hybrid prediction framework approach, (Wang et al. 2021) looked at point and interval predictions of daily future prices of zinc, copper and lead. (Gil-Alana and Poza 2024) applied fractional integration techniques to analysis volatility persistence in the prices of gold, silver, platinum, aluminum, palladium, lead, zinc and tin. (Jermann 2024) no-arbitrage model based on minimal structural assumptions was able to capture key patterns in the gold futures prices.

Using monthly price data from the World Bank database on commodity prices, (Oikonomou and Damigos 2024) model the prices of aluminum, copper, lead, tin, nickel and zinc by applying an autoregressive Light Gradient Boosting Machine trained to only use lags of the time series to produce forecasts both as a standalone and as part of an ensemble. (Pincheira Brown and Hardy 2019) attempted to predict the returns of six base metals, namely aluminum, copper, lead, nickel, tin and zinc using the Chilean exchange rate and found evidence of predictability. (Kahraman and Akay 2023) also study the prices of base metals, specifically aluminum, copper, lead, iron, nickel, tin, and zinc, by applying numerous exponential smoothing methods. (Zhao et al. 2023) model copper and aluminum prices using a novel hybrid approach that used both interval and point forecasting. While the literature on different models used to forecast accurate metal prices, is vast and exhaustive, another approach is the application of GBM.

$$\frac{dP}{P} = \mu dt + \sigma dW_t \tag{1}$$

The biologist Robert Brown discovered Brownian motion in 1827 (Maruddani and Trimono 2018) while observing pollen grains under a microscope, and (Bachelier 1900) is widely credited for being the first for applying it to understand the price movement of stocks. (Einstein 1905) provide a stochastic differential solution, and equation (*1*), is widely used to represent GBM (Benninga 2014; Musiela and Rutkowski 2005; Navin 2007; Sengupta 2004). In this equation, dW_t is the standard Weiner variable with a mean of zero, and a standard deviation of 1, while μ and σ, are the drift and diffusion terms respectively. The change in time is represented by *dt*, while *P* is the price of the asset, and *dP* the change in price.

An implementable close form solution of equation (*1*), and widely used to price assets is represented by equation (*2*) (Benninga 2014; Benninga and Mofkadi 2021; Hull 2018; Reddy and Clinton 2016). P_t and P_0 are prices are time *t* and 0 respectively. The exponential term is represented by *e*, while Δ*t* is the change in time. The drift and diffusion terms usually measured by historical mean and standard deviation is represented by μ and σ respectively, while ε a normally distributed random variable that has an average of zero, and a standard deviation of one.

$$P_t = P_0 e^{\left(\left(\mu_t - \frac{1}{2}\sigma_t^2\right)\Delta t + \left(\sigma_t \varepsilon \sqrt{\Delta t}\right)\right)} \tag{2}$$

An exhaustive literature exists in the application of GBM to understand price movements of stocks and stock indexes (Feng et al. 2021; Kayal and Maheswaran 2018; Paluszek and Thomas 2020; Parungrojrat and Kidsom 2019; Reddy and Clinton 2016; Samuelson 1965, 1973; Sinha 2021, 2024a; Urama and Ezepue 2018). The GBM framework has also been applied to price energy assets (Al-Harthy 2007; Alhagyan 2024; Croghan, Jackman, and Min 2017; Lynch et al. 2021; Postali and Picchetti 2006), agricultural products (Ibrahim et al. 2021; Salaudeen et al. 2023; Zelingher and Makowski 2024), exchange rates (Abbas and Alhagyan 2023; Alhagyan 2022), and derivatives instruments (Black and Scholes 1973; Germansah et al. 2023), and metals and minerals prices (Hamdan et al. 2020; Ramos et al. 2019; Roslan and Halim 2024).

(Hamdan et al. 2020) modelled Malaysian gold prices using GBM. Using daily prices for the *Kijang Emas* prices obtained from the website of Bank Negara Malaysia for the period between January 4, 2016 and December 30, 2016 they model daily observations using historical means and standard deviations as proxies for the drift and diffusion terms respectively. Using data obtained from the Index Mundi database (www.indexmundi.com), (Ramos et al. 2019) modeled iron-ore prices by applying GBM and Monte Carlo simulation. (Roslan and Halim 2024) forecast world gold prices for the year 2022 using GBM. (Huang et al. 2022) compared the gold price forecasts using a mean reversion process, GBM, and a time series modeling. (Soysal 2023) applied fractional GBM while investigating super cycles in silver prices between 2005 and 2023. Applying GBM to the prices of gold, silver, copper, zinc and platinum, (Wilmot 2019) suggests that prices should follow a more complex process than GBM.

Summarizing, a few manuscripts have used GBM to study and forecast metals and minerals prices, and this manuscript contributes to the literature by applying the GBM simulations to prices of six base metals, aluminum, copper, nickel lead, tin and zinc, and six precious metal prices, silver, iridium, palladium, platinum, rhodium and ruthenium. Another contribution made in this manuscript is that the price forecasts are not based on a single GBM forecast, but rather simulates an universe of possible forecasts. Using the probabilities of each forecast, expected values are estimated by totaling up the product of the simulated price and its associated probability. This process is similar to ensemble forecasting (Feng et al. 2021), or ensemble averaging (Cherstvy et al. 2021; Semmlow 2018; Vinod et al. 2022), whereby multiple observations are averaged to obtain reliable solution. A similar process is used in (Sinha 2024d) to gold prices.

III. Data

The data in this manuscript was extracted from DataStream through Refinitiv, and comprises of prices of metals at the yearly, quarterly and monthly frequencies. The prices of the following twelve metals: silver, aluminum, copper, iridium, nickel, lead, palladium, platinum, rhodium, ruthenium, tin and zine were extracted. *Table 1* provides basic information about these metals, including their chemical symbols. The chemical symbols are not necessary for the analysis carried out in this manuscript but were used to sort the data, while running the programing codes, and accordingly have been presented in the table. The mnemonic used to extract metal prices are also presented as well as the first date of availability of the metal in DataStream. The classification of metals into precious metals and metals is based on that Refinitiv's classification. Six of theses, silver, iridium, palladium, platinum, rhodium, and ruthenium, are classified as precious metals while aluminum, copper, nickel, lead, tin and zinc are classified just as metals, or base metals. This classification into metals and precious metals is rather generic with precious metals being those that are of high economic value and comparatively rare. The prices of precious metals are reported

Table 1: Data Sources and Basic Information

Metal	Symbol	Mnemonic	Type	Availability	Source	Unit	Price Quote
Silver	Ag	SILVERH	Precious Metals	1/2/1979	Handy & Harman	Troy Ounce	USD per oz
Aluminum	Al	LAHCASH	Metals	1/31/1957	London Metal Exchange	Metric Ton	USD per ton
Copper	Cu	LCPCASH	Metals	1/31/1957	London Metal Exchange	Metric Ton	USD per ton
Iridium	Ir	JMIRIAS	Precious Metals	7/1/1992	Johnson Matthey	Troy Ounce	USD per oz
Nickel	Ni	LNI3MTH	Metals	4/23/1979	London Metal Exchange	Metric Ton	USD per ton
Lead	Pb	LEDCASH	Metals	7/5/1993	London Metal Exchange	Metric Ton	USD per ton
Palladium	Pd	PALLADM	Precious Metals	1/5/1987	London Metal Exchange	Troy Ounce	USD per oz
Platinum	Pt	PLATFRE	Precious Metals	1/2/1976	London Metal Exchange	Troy Ounce	USD per oz
Rhodium	Rh	JMRHOUS	Precious Metals	1/31/1980	Johnson Matthey	Troy Ounce	USD per oz
Ruthenium	Ru	JMRUTEU	Precious Metals	7/1/1992	Johnson Matthey	Troy Ounce	USD per oz
Tin	Sn	LTICASH	Metals	1/31/1957	London Metal Exchange	Metric Ton	USD per ton
Zinc	Zn	LZZCASH	Metals	1/31/1957	London Metal Exchange	Metric Ton	USD per ton

The availability column shows the first date from which pricing data is available in Refinitiv DataStream. The type column provides the classification as Metals or Precious Metals. The symbol is the chemical notation typically used for the corresponding metals. The DataStream mnemonic used for data extraction, the unit of weight, and price quote, as well as DataStream's source for price data are also provided.

Table 2: Select Days and Prices

Metal	First Day Values		Maximum Values		Minimum Values		Last Day Values	
	Date	Value	Date	Value	Date	Value	Date	Value ($)
Silver	12/30/1983	8.95	4/29/2011	48.55	2/25/1993	3.54	12/29/2023	24.25
Aluminum	12/30/1983	1,549.86	3/4/2022	3,877.50	11/29/1985	950.20	12/29/2023	2,345.50
Copper	12/30/1983	1,415.40	10/18/2021	11,299.50	10/31/1984	1,272.10	12/29/2023	8,463.92
Iridium	7/1/1992	200.00	4/28/2021	6,300.00	6/1/1995	60.00	12/29/2023	5,000.00
Nickel	12/30/1983	4,842.00	5/4/2007	51,600.00	1/12/1987	3,423.11	12/29/2023	16,603.00
Lead	7/5/1993	385.50	10/10/2007	3,989.00	10/5/1993	357.00	12/29/2023	2,034.50
Palladium	1/5/1987	118.25	3/7/2022	3,015.00	8/16/1991	78.25	12/29/2023	1,119.00
Platinum	12/30/1983	391.82	3/4/2008	2,273.00	3/11/1985	244.25	12/29/2023	1,006.00
Rhodium	12/30/1983	350.00	3/22/2021	29,800.00	1/30/1997	200.00	12/29/2023	4,425.00
Ruthenium	7/1/1992	32.00	2/9/2007	870.00	8/18/1993	18.00	12/29/2023	450.00
Tin	12/30/1983	12,341.50	3/8/2022	48,865.00	2/19/2002	3,601.00	12/29/2023	25,175.00
Zinc	12/30/1983	857.60	11/24/2006	4,603.00	11/29/1985	597.46	12/29/2023	2,640.00

The first and last day of the metal and the prices on those days are presented in this table. The last day for all the metals is the same, but the first day is the same for only eight of the metals. The maximum and minimum prices along with the dates are also presented.

Table 3: Descriptive Statistics

Metal	Annual				Quarterly				Monthly			
	Mean	Stdev	Skew	Kurt	Mean	Stdev	Skew	Kurt	Mean	Stdev	Skew	Kurt
Silver	2.5%	21.8%	0.42	0.15	0.6%	11.6%	-0.07	0.77	0.2%	7.8%	0.02	1.31
Aluminum	1.0%	24.2%	0.37	0.12	0.3%	11.3%	-0.41	2.04	0.1%	6.6%	0.05	2.53
Copper	4.5%	31.5%	0.54	1.69	1.1%	14.0%	-0.59	5.75	0.4%	7.3%	-0.34	5.21
Iridium	11.9%	44.8%	-0.95	1.65	2.7%	20.5%	1.43	5.94	0.9%	9.1%	2.38	14.80
Lead	4.9%	31.1%	-0.22	3.39	1.4%	14.0%	-0.63	3.00	0.5%	7.8%	-0.32	1.65
Nickel	3.1%	41.5%	0.38	0.03	0.8%	17.9%	0.60	0.53	0.3%	9.7%	0.35	2.53
Palladium	6.1%	38.8%	-0.27	-0.21	1.5%	16.4%	-1.21	4.84	0.5%	9.4%	-0.18	2.50
Platinum	2.4%	20.6%	-0.31	0.41	0.6%	11.2%	-1.70	10.52	0.2%	6.4%	-0.41	4.07
Rhodium	6.3%	65.9%	-0.44	-0.10	1.6%	27.0%	-0.31	3.26	0.5%	13.7%	0.17	9.84
Ruthenium	9.3%	65.3%	0.81	2.23	2.1%	26.8%	0.84	5.55	0.7%	11.8%	1.41	10.55
Tin	1.8%	26.7%	0.55	-0.10	0.4%	13.2%	-0.54	1.64	0.1%	6.8%	-0.21	2.47
Zinc	2.8%	31.5%	0.43	0.86	0.7%	13.0%	-0.05	-0.01	0.2%	7.5%	-0.37	1.77

The mean, standard deviation (stdev), skewness (skew), and kurtosis (kurt) of the log-returns of the twelve metals are presented. These are calculated for the sample period between when the pricing information becomes available to the last day of the sample period. The descriptive statistics for all three frequencies: annual, quarterly and monthly, are presented.

on troy ounces, where an ounce is approximately 31.10 grams, while those of base metals are on metric tons with a metric ton equal to one thousand kilograms. The source used by DataStream to obtain the metal prices are also presented in *Table 1.*

The data extracted for the manuscript was between December 30, 1983 and December 29, 2023 giving a forty-year sample period once log-returns are calculated. As can be seen in *Table 2,* the prices of eight metals, namely silver, aluminum, copper, nickel, platinum, rhodium, tin and zinc were available for the whole sample period. The first availability of iridium and ruthenium is from July 1, 1992, while those for lead and palladium were from July, 5, 1993 and January 5, 1987 respectively. The table also presents the observed prices of the metals on the first and last day of the sample period. The maximum and minimum prices along with the associated dates are also provided in the table. While presenting the maximum and minimum prices, only the dates for the first occurrence has been presented if those prices were observed on multiple days.

IV. Methodology

Preliminary Estimations

The returns are calculated by taking the natural log of the division of a metal price by the immediately preceding price, as shown in equation *(3)*. In this equation, $P_{t,i}$ represents the price of metal i at time t, while $P_{t-1,i}$ is the preceding price of metal i. The returns were calculated for all twelve metals and for all three frequencies, and were used to estimate the drift and diffusion terms used in the GBM simulation as represented in equations *(2)* and *(6)*.

$$r_{t,i} = ln\left(\frac{P_{t,i}}{P_{t-1,i}}\right) \tag{3}$$

The log-returns estimated using equation (*3*) were used to estimate the mean, standard deviation, skewness and kurtosis for the whole sample period and are presented as descriptive statistics in Table 3. The formulae used to estimate are standard statistical formulae, and are estimated at the annual, quarterly, and monthly frequencies. The highest average return at the annual frequency is observed for iridium at 11.9%, while the lowest is aluminum at 1.0%. The range for standard deviation is from a low of 20.6% for platinum to a high of 65.9% for rhodium, although ruthenium also has a very high standard deviation at 65.3%. Five of the twelve metals show negative skewness. For kurtosis, both positive and negative numbers are observed making some of the distribution platykurtic, and others leptokurtic. The kurtosis values for some of the metals are observed to be close to zero. Iridium is also observed to have the highest average returns, at the quarterly and monthly frequency, while aluminum has the lowest average returns at the quarterly frequency. Aluminum and tin have the lowest returns at 0.1% at the monthly frequency. The skewness numbers at the quarterly frequency show that the returns of more metals are negatively skewed than positively skewed. Besides, eleven of the twelve metals are leptokurtic, with tin being platykurtic, although the number is very close to zero at 0.1. At the monthly frequency, some metals show negative skewness, and others show positive skewness, and all metals show leptokurtosis. Numerically, the monthly frequency numbers for kurtosis of some metals are larger than those at the annual and quarterly frequencies. As the GBM modeling process is mostly dependent on the first two statistical moments, namely drift and diffusion, the impact of skewness and kurtosis on the efficacy of the simulation has not being explored in this this manuscript.

GBM Simulations and Expected Value Estimation

The drift ($\mu_{t,i}$) and diffusion ($\sigma_{t,i}$) term are proxied by the estimating average returns and standard deviation within a rolling window period, that are based on a twenty-year period as shown in equations *(4)* and *(5)* respectively. *n* has a value of twenty, eighty, and two hundred and forty at the yearly, quarterly and monthly frequencies. These formulae are standard formulae for estimating averages and standard deviations.

$$\mu_{t,i} = \frac{\sum_1^n r_i}{n} \tag{4}$$

$$\sigma_{t,i} = \sqrt{\frac{\sum_1^n (r_i - \bar{r_i})^2}{n-1}} \tag{5}$$

One thing to note here is that although the initial data sample is a forty-years, the first twenty years of the data constitutes the first estimation window for the drift and diffusion terms as shown in equations (*4*) and (*5*). However, as mentioned in section III and shown in *Table 2,* not all metals have the same starting date in the sample. Due to data availability, eight of the twelve metals, silver, aluminum, copper, nickel, platinum, rhodium, tin and zinc, have their first window based on twenty-years of data. For palladium, the first window drift and diffusion terms were estimated using sixteen years of data, while for ruthenium and iridium the first sample was based on eleven years of data. For lead, it was ten. In the rolling window, the one-periods data was dropped at the back-end, and one added at the front end. For the four metals, with less than twenty-years data for the first window, the sample principal was applied, although data was added at the front end, and the back-end data were excluded only after the window reached twenty-years of data. The twenty-year rolling window ensures twenty-years of data at the annual frequency, eighty-quarters at the quarterly frequency, and two hundred and forty months at the monthly frequency. A similar approach was used in (Sinha 2021, 2024a, 2024b, 2024d, 2024f).

The GBM formula used in this manuscript, and shown in equation *(6)*, has been used previously in a number of published articles and books (Benninga and Mofkadi 2021; Hull 2018; Maruddani and Trimono 2018; Musiela and Rutkowski 2005; Navin 2007; Reddy and Clinton 2016; Sinha 2024f, 2021, 2024a, 2024b, 2024d). In equation *(6)*, $P_{t,i,}$ is the price of metal *i* at time *t*, while $P_{sim,t+1,i}$ is the simulated price at time *t+1*, with Δt taking a value of one, for one-period ahead forecasts. $\mu_{t,i}$ and $\sigma_{t,i}$ are the drift and diffusion terms for metal *i* estimated using equations *(4)* and *(5)* respectively, at time *t*. The letter *e* represents the exponential, and *e* is a standard normal Weiner variable with a mean of zero and a standard deviation of 1.

$$P_{sim,t+1,i} = P_{t,i} e^{\left(\left(\mu_{t,i} - \frac{1}{2}\sigma_{t,i}^2\right)\Delta t + \left(\sigma_{t,i}\varepsilon\sqrt{\Delta t}\right)\right)} \tag{6}$$

As the Weiner variable, ε, is a random variable, numerous values of $P_{sim,t+1,i}$ can be simulated by change its value with each simulation. By allowing ε to change randomly but still maintaining the restriction of a mean of zero, and a standard deviation of 1, one hundred thousand simulated values were obtained for each metal, for each data frequency and each window. Using equation (7), log-returns, $r_{sim,t+1,i}$, for each of the simulated value was calculated. After sorting the simulated returns from low to high, the probabilities of the returns were estimated by taking the difference between the values of the cumulative distribution function of adjacent return values. Using the probabilities and simulated metal prices, the GBM expected metal price was estimated by multiplying them, and totaling them up, as shown in equation *(8)*. As part of the robustness check, the GBM-based expected metal price was also calculated using various numbers of simulations, specifically 1, 100, 1,000, 10,000, and 50,000 simulations.

$$r_{sim,t+1,i} = ln\left(\frac{P_{sim,t+1,i}}{P_{t,i}}\right) \quad (7)$$

$$Exp\_P_{t,i} = \sum_{1}^{100,000} P_{sim,t,i} * prob_{P_{sim,t,i}} \quad (8)$$

Testing

The expected metal price estimated using equation *(8)*, and compared to the actually values and then tested for reliability using a simple regression equation *(9)*, and also for differences in standard deviations and means using equations *(10)* and *(11)* respectively. The actual and expected metal prices were plotted against time at the annual (*Figure 1*), quarterly (*Figure 2*) and monthly (*Figure 3*) frequencies for silver. The plots of actual prices against the GBM based expected prices for silver along with a trendline were also generate. These plots are in *Figure 4, Figure 5* and *Figure 6* at the annual quarterly and monthly frequencies respectively. While producing the figures 1 to 3, the dollar prices were used, and the natural log of the actual and expected prices were used for producing figures 4 to 6. For these figures, the GBM based expected prices were based on one hundred thousand simulations. In the manuscript though, the plots for only silver is presented.

$$ln\left(P_{t,i}\right) = \alpha_i + \beta_i\, ln\left(Exp\_P_{t,i}\right) + \varepsilon_i \quad (9)$$

In the simple regression equation, the actual metal price, $P_{t,i}$, is the dependent variable while, $Exp\_P_{t,i}$, the GBM based expected metal price as obtained by equation (8) is the independent variable. If the GBM based expected metal price is an accurate and reliable predictor of the actual metal price, the coefficient β_i, will have a value of one, and the intercept, α_i, will have a value equal to zero. A similar logic was used in (Sinha 2021, 2024a, 2024f) while testing the accuracy of expected stock index values and (Sinha 2024d) while testing the accuracy of gold prices. (Grinblatt and Titman 1992) also used a similar approach while testing for persistence in mutual fund performance. As equation (*9*) is a simple regression, the correlation coefficients between the actual metal prices and the GBM based expected price, are obtained by computing the square root of R-square of the regression equation. These results are presented at the annual frequency are presented in *Table 4*, while those that at quarterly frequency are in *Table 5*, and the monthly frequency in *Table 6*.

The differences in standard deviation and means are tested using a standard statistical procedure discussed in many textbooks on statistics like (Berenson, Levine, and Krehbiel 2015). The hypothesis being tested here is whether the time series of the GBM based expected prices of metals have standard deviation and means similar to those of the actual metal prices. (Sinha 2021, 2024a, 2024d, 2024f) also used a similar logic to test the differences between the actual values and the GBM based expected values. The F-stat, as estimated using equation *(10)* is used to test for differences in standard deviation, with the hypothesis being that the difference between the standard deviation between the expected prices and actual prices are equal to zero.

Table 4: Annual Frequency Simple Regression Results

Metal	Intercept		GBM Expected Price			R^2	Adj R^2	ρ	F-stat
	Coefficient	t-stat ($H_0 = 0$)	Coefficient	t-stat ($H_0 = 0$)	t-stat ($H_0 = 1$)				
Silver	0.92	**3.00**	0.68	**6.41**	**3.08**	69.5%	67.8%	0.83	**41.03**
Aluminum	5.51	**3.59**	0.28	1.38	**3.60**	9.5%	4.5%	0.31	1.90
Copper	4.90	**3.65**	0.44	**2.89**	**3.69**	31.6%	27.8%	0.56	**8.33**
Iridium	0.92	1.65	0.87	**10.52**	1.64	86.0%	85.2%	0.93	**110.77**
Nickel	6.06	**2.99**	0.37	1.80[a]	**3.06**	15.3%	10.6%	0.39	3.24[b]
Lead	5.06	**4.27**	0.33	2.11[a]	**4.33**	19.8%	15.4%	0.45	4.45[a]
Palladium	1.09	1.68	0.82	**8.49**	1.80[b]	80.0%	78.9%	0.89	**72.03**
Platinum	3.62	2.75[a]	0.48	2.58[a]	2.80[a]	27.0%	22.9%	0.52	**6.66**
Rhodium	2.31	2.05[b]	0.69	**4.97**	2.22[a]	57.8%	55.5%	0.76	**24.66**
Ruthenium	1.88	2.14 [a]	0.61	**3.68**	2.39[a]	43.0%	39.8%	0.66	**13.57**
Tin	3.84	**3.00**	0.61	**4.66**	**3.02**	54.7%	52.2%	0.74	**21.72**
Zinc	5.58	**4.03**	0.28	1.56	**4.08**	12.0%	7.1%	0.35	2.44

The results of the simple regression equation with natural log of the actual price as the dependent variable, and the natural log of the GBM based expected price are presented in this table. The GBM based expected metal prices are based on one-hundred thousand simulations. The t-stat for the intercept and the slope coefficient being equal to zero are presented. The t-stat for the coefficient equal to one are also provided. Equation statistics like R-square, adjusted R-square and F-stats are also provided. ρ is the correlation coefficient between the actual and expected values, and were obtained by taking square root of R-square. The highlighted values indicate significance at 1%, while the superscripts *a* and *b* indicate significance at 5% and 10% respectively. The regression was based on twenty-years of data for the period between 2004 and 2023.

Table 5: Quarterly Frequency Simple Regression Results

Metal	Intercept		GBM Expected Price			R^2	Adj R^2	ρ	F-stat
	Coefficient	t-stat ($H_0 = 0$)	Coefficient	t-stat ($H_0 = 0$)	t-stat ($H_0 = 1$)				
Silver	0.30	**2.95**	0.89	**24.98**	**3.02**	88.89%	88.74%	0.94	**623.79**
Aluminum	1.52	**3.11**	0.80	**12.47**	**3.13**	66.60%	66.17%	0.82	**155.54**
Copper	1.59	**3.63**	0.82	**16.43**	**3.66**	77.59%	77.30%	0.88	**270.09**
Iridium	0.30	1.84[b]	0.96	**40.38**	1.81[b]	95.43%	95.38%	0.98	**1630.30**
Nickel	1.35	2.53[a]	0.86	**15.73**	2.58[a]	76.03%	75.72%	0.87	**247.43**
Lead	1.40	**3.64**	0.81	**15.90**	**3.68**	76.41%	76.11%	0.87	**252.70**
Palladium	0.30	1.78[b]	0.95	**37.47**	1.86[b]	94.74%	94.67%	0.97	**1,404.18**
Platinum	1.02	**2.64**	0.85	**15.48**	**2.69**	75.45%	75.13%	0.87	**239.70**
Rhodium	0.48	1.95[a]	0.94	**30.25**	2.05[a]	92.15%	92.05%	0.96	**915.17**
Ruthenium	0.33	1.76[b]	0.93	**25.23**	1.95[a]	89.08%	88.94%	0.94	**636.59**
Tin	1.16	**2.99**	0.88	**22.17**	**3.00**	86.30%	86.13%	0.93	**491.55**
Zinc	1.12	**3.11**	0.85	**18.21**	**3.13**	80.96%	80.72%	0.90	**331.69**

The results of the simple regression equation with natural log of the actual price as the dependent variable, and the natural log of the GBM based expected price are presented in this table. The GBM based expected metal prices are based on one-hundred thousand simulations. The t-stat for the intercept and the slope coefficient being equal to zero are presented. The t-stat for the coefficient equal to one are also provided. Equation statistics like R-square, adjusted R-square and F-stats are also provided. ρ is the correlation coefficient between the actual and expected values, and were obtained by taking square root of R-square. The highlighted values indicate significance at 1%, while the superscripts *a* and *b* indicate significance at 5% and 10% respectively. The regression was based on eighty-quarters of data for the period between 1st quarter of 2004 and the last quarter of 2023.

Table 6: Monthly Frequency Simple Regression Results

Metal	Intercept		GBM Expected Price			R^2	Adj R^2	ρ	F-stat
	Coefficient	t-stat ($H_0 = 0$)	Coefficient	t-stat ($H_0 = 0$)	t-stat ($H_0 = 1$)				
Silver	0.11	**2.92**	0.96	**70.26**	**3.02**	95.40%	95.38%	0.98	**4,936.58**
Aluminum	0.42	**2.79**	0.94	**47.59**	**2.80**	90.49%	90.45%	0.95	**2,265.13**
Copper	0.44	**3.39**	0.95	**63.95**	**3.42**	94.50%	94.48%	0.97	**4,090.11**
Iridium	0.09	2.13[a]	0.99	**167.58**	1.98[a]	99.16%	99.16%	1.00	**28,082.30**
Nickel	0.42	2.40[a]	0.96	**53.47**	2.45[a]	92.32%	92.28%	0.96	**2,859.14**
Lead	0.47	**3.52**	0.94	**53.52**	**3.56**	92.33%	92.30%	0.96	**2,864.73**
Palladium	0.09	1.77[b]	0.99	**121.26**	1.85[b]	98.41%	98.40%	0.99	**14,704.90**
Platinum	0.30	**2.44**	0.96	**55.49**	**2.48**	92.83%	92.80%	0.96	**3,079.46**
Rhodium	0.14	1.93[a]	0.98	**108.27**	2.00[a]	98.01%	98.00%	0.99	**11,723.20**
Ruthenium	0.08	1.60	0.98	**102.39**	1.69[b]	97.78%	97.77%	0.99	**10,484.20**
Tin	0.34	**2.91**	0.97	**80.61**	**2.92**	96.47%	96.45%	0.98	**6,497.50**
Zinc	0.38	**3.04**	0.95	**58.54**	**3.06**	93.51%	93.48%	0.97	**3,426.68**

The results of the simple regression equation with natural log of the actual price as the dependent variable, and the natural log of the GBM based expected price are presented in this table. The GBM based expected metal prices are based on one-hundred thousand simulations. The t-stat for the intercept and the slope coefficient being equal to zero are presented. The t-stat for the coefficient equal to one are also provided. Equation statistics like R-square, adjusted R-square and F-stats are also provided. ρ is the correlation coefficient between the actual and expected values, and were obtained by taking square root of R-square. The highlighted values indicate significance at 1%, while the superscripts *a* and *b* indicate significance at 5% and 10% respectively. The regression was based on two-hundred and forty months of data for the period between January 2004 and December 2023.

$$F - stat_i = \frac{\left(ln\left(Exp\_P_{t,i}\right)\right)^2}{\left(ln\left(P_{t,i}\right)\right)^2} \quad (10)$$

The differences between the means of the GBM based expected metal prices, and the actual metal prices is tested using a t-stat calculated using equation *(12)*, where the pooled variance was calculated equation *(11)*. The hypothesis tested here is $\left(\bar{\mu}_{Exp\_P_{t,i}} - \bar{\mu}_{P_{t,i}}\right) = 0$, with $n_1 = n_2$. The results for the tests for differences in standard deviation and means are presented in *Table 7, Table 8* and *Table 9* for the annual, quarterly and monthly frequencies respectively when the expected metal prices are based on one hundred thousand simulations.

$$\sigma^2_{pooled,i} = \frac{(n_1-1)\sigma^2_{P_{t,i}} + (n_2-1)\sigma^2_{Exp\_P_{t,i}}}{(n_{1,i}-1)+(n_{2,i}-1)} \quad (11)$$

$$t_{stat,i} = \frac{\left(\bar{\mu}_{Exp\_P_{t,i}} - \bar{\mu}_{P_{t,i}}\right) - (\mu_{1,i} - \mu_{2,i})}{\sqrt{\sigma^2_{pooled,i}\left(\frac{1}{n_{1,i}} + \frac{1}{n_{2,i}}\right)}} \quad (12)$$

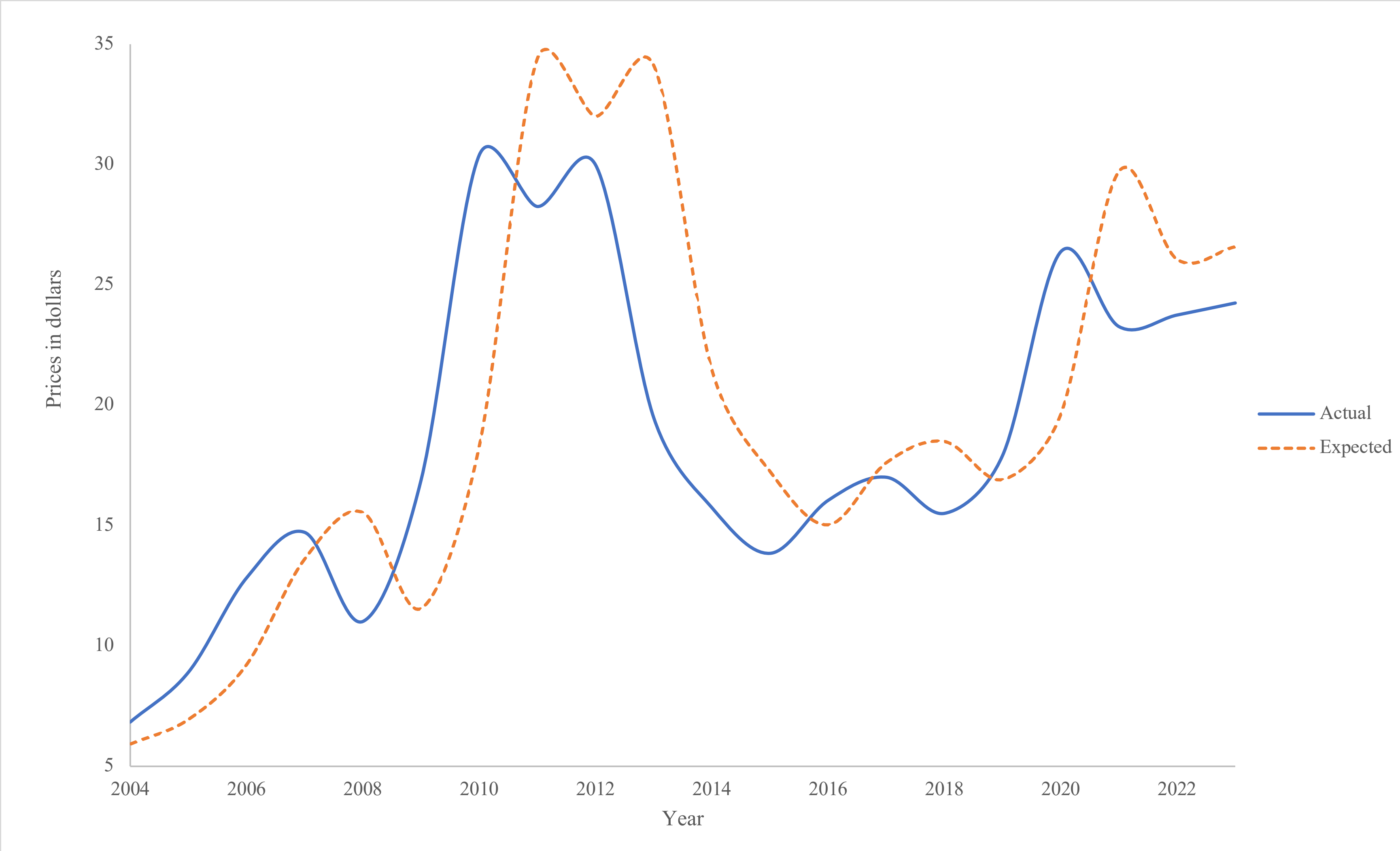
As mentioned earlier and as part of robustness check, the expected prices were estimated using different number of simulations, specifically one, one hundred, one thousand, ten thousand, and fifty thousand. These expected prices were also tested using the regression, and difference in standard deviation and means. The regression results are presented in *Table 10, Table 11*, and *Table 12* at the annual, quarterly and monthly frequencies. The results of the differences in standard deviation and means at the annual, quarterly and monthly frequencies are presented *Table 13, Table 14* and *Table 16* respectively.

V. Results

The plots of the actual metal price and the GBM based expected prices for silver are presented in *Figure 1* for the annual frequency, *Figure 2* for the quarterly frequency, and *Figure 3* for the monthly frequency. In these plots the actual year-end dollar price, and the GBM based expected metal price in dollars are plotted. In these plots, we can observe that the expected and actual prices follow the similar pattern, but the gap between the actual and expected decreases from the annual to the monthly frequency.

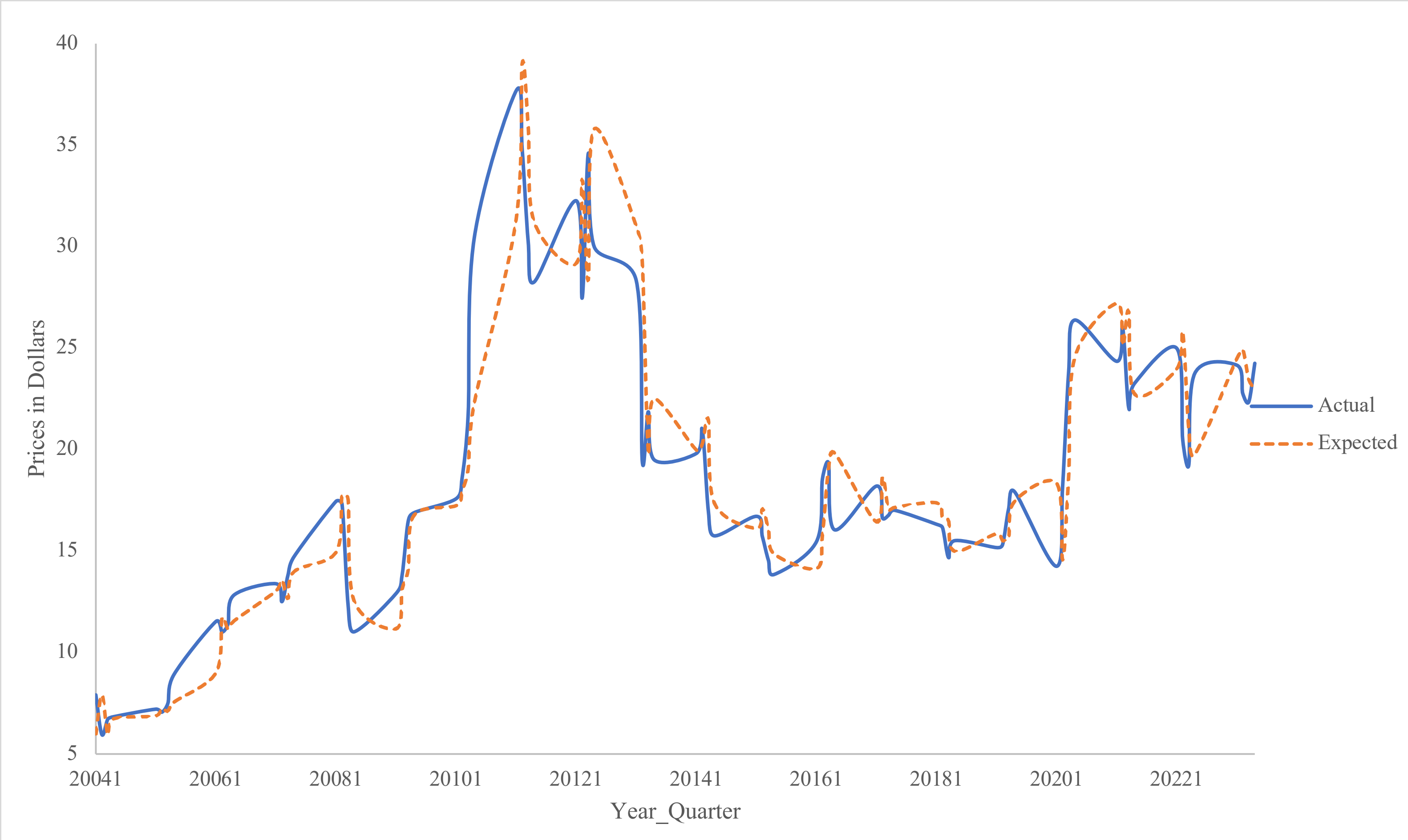
The plots of natural log of the year-end actual price for silver against the natural log of the GBM based expected silver price are in *Figure 4, Figure 5* and *Figure 6* at the annual, quarterly and monthly frequencies respectively. The plot for the annual frequency seems to be too scattered for a trend to not be very pronounced to the naked eye, although a linear trendline can be fitted to the plot. A trend in the quarterly and monthly frequency seem to be more pronounced to the naked eye, and the trendline in both these plots seems to be starting from the intersection of the x-axis and y-axis, and rising upwards to perhaps suggest a linear relationship. The trend seems to be more

Figure 1: Annual Actual and Expected Silver Prices



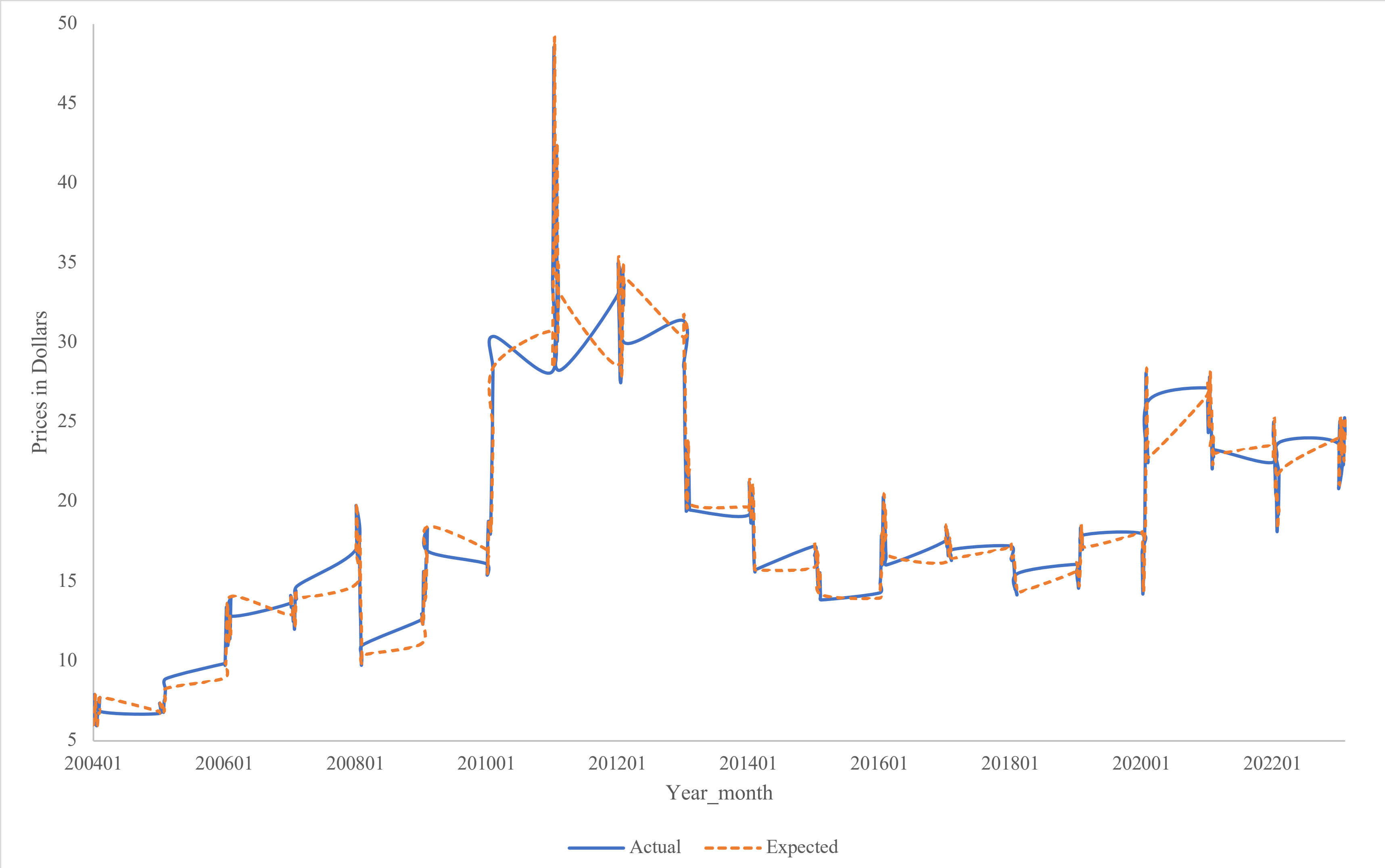
The actual and GBM based expected silver prices are plotted in this figure. The prices are actual year end dollar prices for the period between 2004 and 2023. The expected prices are based on one hundred thousand simulations.

Figure 2: Quarterly Actual and Expected Silver Prices



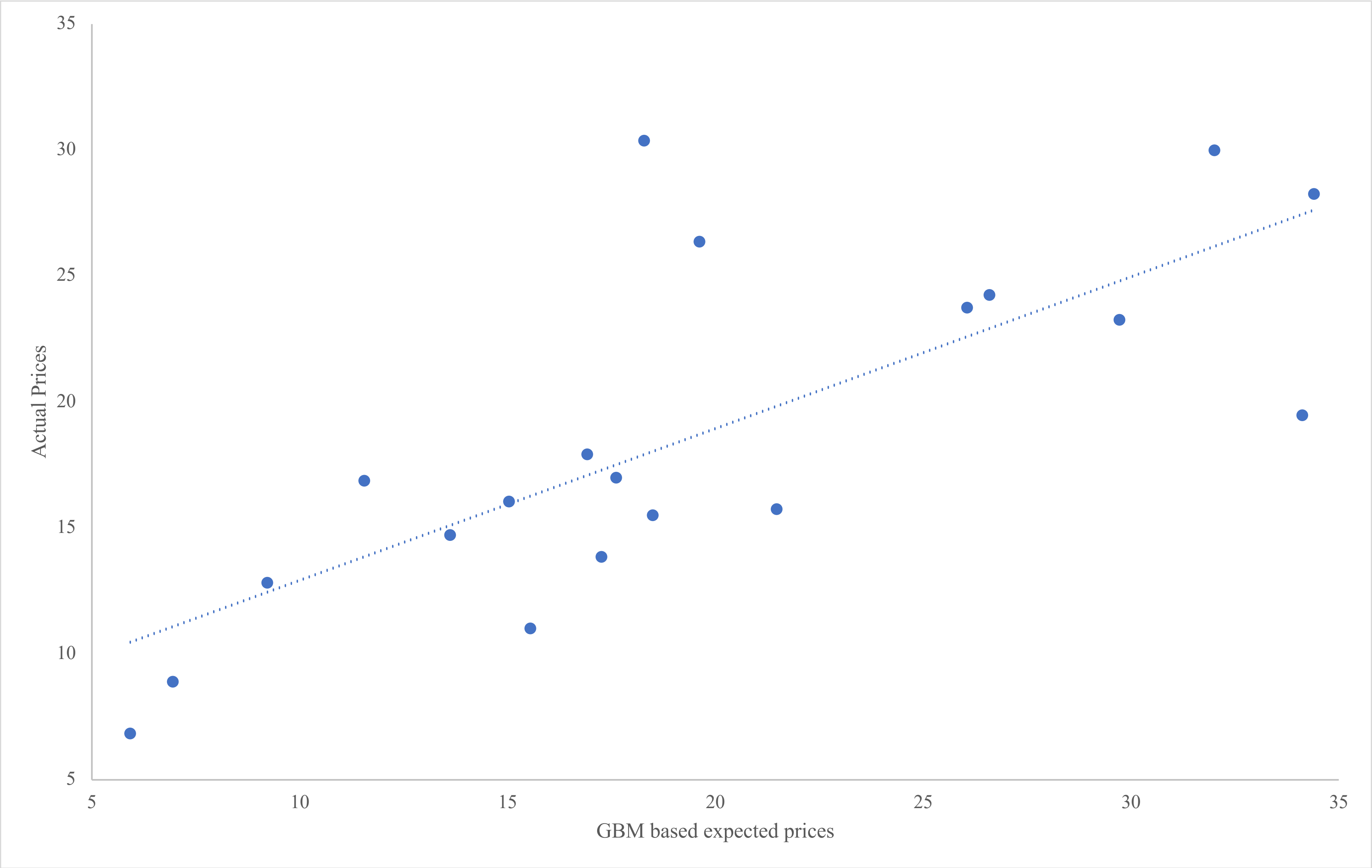
The actual and GBM based expected silver prices are plotted in this figure. The prices are natural logs of the quarter-end prices for the period between 2004 and 2023. The expected prices are based on one hundred thousand simulations.

Figure 3: Monthly Actual and Expected Silver Prices



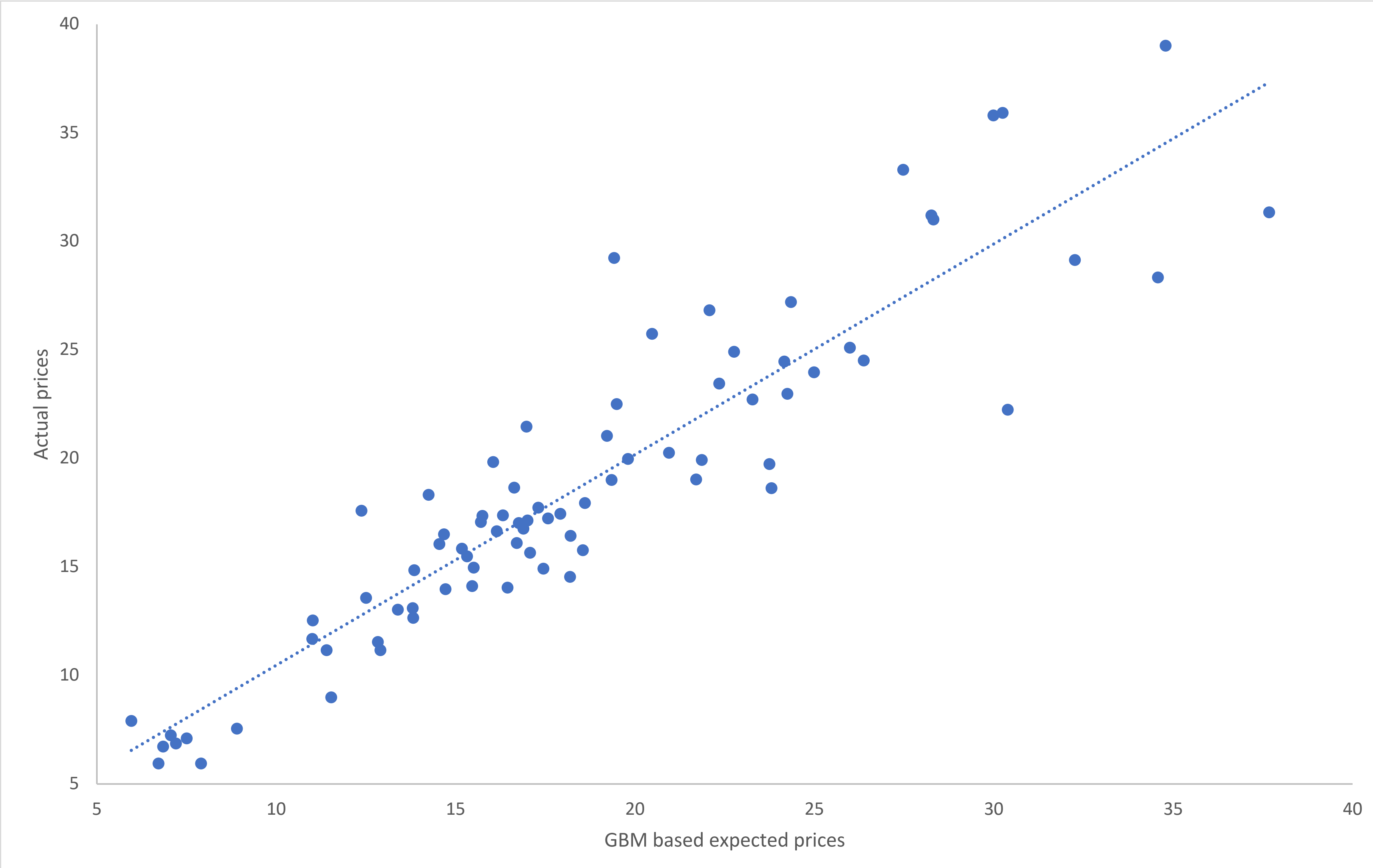
The actual and GBM based expected silver prices are plotted in this figure. The prices are natural logs of the month-end prices for the period between 2004 and 2023. The expected prices are based on one hundred thousand simulations.

Figure 4: Annual Actual Vs Expected Silver Prices



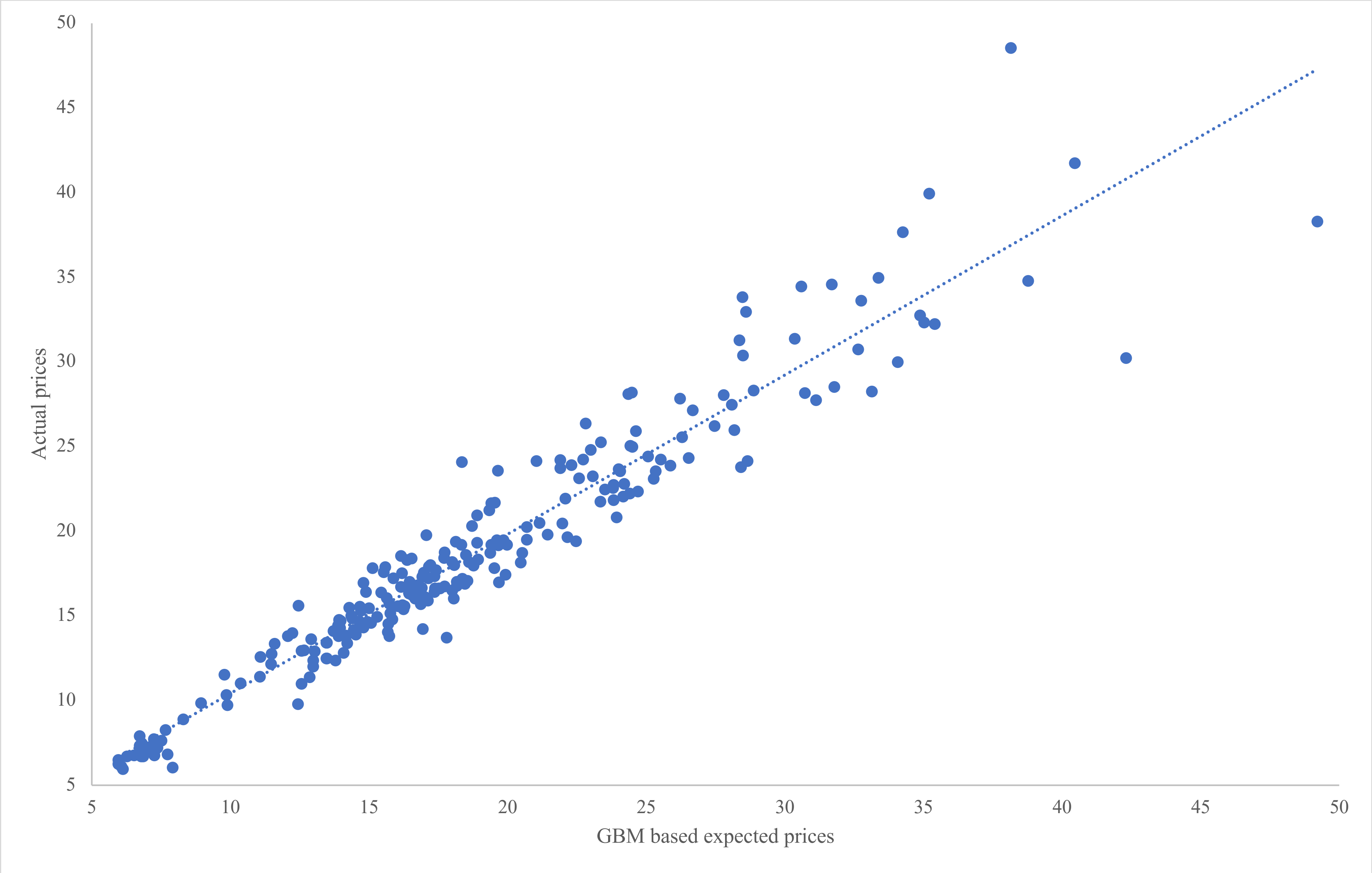
In this figure the actual prices are plotted against the GBM based expected Silver year-end prices for the period between 2003 and 2004. The expected prices are based on one hundred thousand simulations.

Figure 5: Quarterly Actual Vs Expected Silver Prices



In this figure the actual prices are plotted against the GBM based expected Silver quarter-end prices for the period between 2003 and 2004. The expected prices are based on one hundred thousand simulations.

Figure 6: Monthly Actual and Expected Silver Prices



In this figure the actual prices are plotted against the GBM based expected Silver monthly prices for the period between 2003 and 2004. The expected prices are based on one hundred thousand simulations.

Table 7: Annual Frequency Differences in Means and Standard Deviation Results

Metal	Standard Deviation					Mean				
	Actual	Expected	Pooled	F-stat	p-value	Actual	Expected	Difference	t-stat	p-value
Silver	0.40	0.49	0.45	1.52	0.37	2.86	2.87	-0.01	-0.08	0.94
Aluminum	0.18	0.20	0.19	1.25	0.64	7.63	7.65	-0.02	-0.40	0.69
Copper	0.32	0.41	0.36	1.64	0.29	8.77	8.81	-0.05	-0.42	0.68
Iridium	0.98	1.05	1.02	1.15	0.76	6.75	6.73	0.02	0.05	0.96
Nickel	0.36	0.38	0.37	1.11	0.82	9.70	9.84	-0.14	-1.22	0.23
Lead	0.29	0.39	0.34	1.85	0.19	7.56	7.62	-0.07	-0.62	0.54
Palladium	0.77	0.83	0.80	1.18	0.73	6.55	6.62	-0.07	-0.28	0.78
Platinum	0.23	0.25	0.24	1.18	0.73	7.00	7.06	-0.06	-0.82	0.42
Rhodium	0.99	1.09	1.04	1.21	0.68	7.88	8.05	-0.17	-0.52	0.61
Ruthenium	0.88	0.96	0.92	1.17	0.74	5.06	5.25	-0.18	-0.63	0.53
Tin	0.43	0.52	0.47	1.49	0.40	9.78	9.80	-0.02	-0.12	0.91
Zinc	0.32	0.40	0.36	1.56	0.34	7.74	7.80	-0.05	-0.48	0.64

The table provides the results for the differences in standard deviation and means of the natural log of the actual and GBM based expected metal prices. In this table, 'Actual' is the natural log the actual metal price, while 'Expected' is the natural log of the GBM based expected metal price estimated using one hundred thousand simulations, while 'Pooled' is the pooled standard deviation. F-stat is used to test for the differences in standard deviations, while the pooled standard deviation is used to estimate the t-stat for differences in means. The table reports the p-values for the F-stats and t-stats. This table is for the annual frequency and was based on twenty-years of data for the period between 2004 and 2023.

Table 8: Quarterly Frequency Differences in Means and Standard Deviation Results

Metal	Standard Deviation					Mean				
	Actual	Expected	Pooled	F-stat	p-value	Actual	Expected	Difference	t-stat	p-value
Silver	0.40	0.43	0.42	1.12	0.62	2.84	2.84	0.00	-0.06	0.95
Aluminum	0.19	0.20	0.20	1.04	0.86	7.62	7.63	0.00	-0.16	0.87
Copper	0.30	0.33	0.32	1.16	0.51	8.76	8.77	-0.01	-0.20	0.84
Iridium	0.95	0.97	0.96	1.04	0.86	6.75	6.74	0.01	0.05	0.96
Nickel	0.35	0.36	0.36	1.03	0.89	9.72	9.74	-0.03	-0.48	0.63
Lead	0.30	0.32	0.31	1.16	0.51	7.53	7.55	-0.02	-0.31	0.76
Palladium	0.75	0.77	0.76	1.04	0.85	6.53	6.54	-0.01	-0.10	0.92
Platinum	0.25	0.26	0.26	1.04	0.86	7.02	7.03	-0.02	-0.40	0.69
Rhodium	1.00	1.03	1.01	1.05	0.83	7.87	7.89	-0.02	-0.13	0.90
Ruthenium	0.87	0.89	0.88	1.03	0.88	5.06	5.09	-0.03	-0.22	0.82
Tin	0.41	0.43	0.42	1.11	0.64	9.78	9.78	0.00	-0.07	0.95
Zinc	0.31	0.33	0.32	1.11	0.64	7.71	7.72	-0.01	-0.12	0.90

The table provides the results for the differences in standard deviation and means of the natural log of the actual and GBM based expected metal prices. In this table, 'Actual' is the natural log the actual metal price, while 'Expected' is the natural log of the GBM based expected metal price estimated using one hundred thousand simulations, while 'Pooled' is the pooled standard deviation. F-stat is used to test for the differences in standard deviations, while the pooled standard deviation is used to estimate the t-stat for differences in means. The table reports the p-values for the F-stats and t-stats. This table is for the quarterly frequency and was based on eighty-quarters of data for the period between 1st quarter of 2004 and the last quarter of 2023.

Table 9: Monthly Frequency Differences in Means and Standard Deviation Results

Metal	Standard Deviation					Mean				
	Actual	Expected	Pooled	F-stat	p-value	Actual	Expected	Difference	t-stat	p-value
Silver	0.42	0.43	0.42	1.04	0.78	2.84	2.84	0.00	-0.06	0.95
Aluminum	0.19	0.20	0.20	1.01	0.91	7.63	7.63	0.00	-0.09	0.93
Copper	0.31	0.32	0.32	1.05	0.71	8.76	8.77	0.00	-0.08	0.93
Iridium	0.95	0.96	0.95	1.02	0.91	6.72	6.72	0.01	0.08	0.94
Nickel	0.36	0.36	0.36	1.01	0.94	9.72	9.73	-0.01	-0.27	0.79
Lead	0.31	0.31	0.31	1.05	0.71	7.54	7.54	0.00	-0.16	0.87
Palladium	0.75	0.76	0.75	1.01	0.91	6.53	6.53	0.00	-0.05	0.96
Platinum	0.25	0.26	0.25	1.01	0.92	7.03	7.03	0.00	-0.22	0.83
Rhodium	1.02	1.02	1.02	1.02	0.90	7.87	7.87	0.00	-0.05	0.96
Ruthenium	0.87	0.87	0.87	1.01	0.94	5.05	5.06	0.00	-0.04	0.97
Tin	0.41	0.42	0.42	1.04	0.79	9.78	9.78	0.00	-0.02	0.98
Zinc	0.33	0.33	0.33	1.04	0.79	7.71	7.71	0.00	-0.07	0.94

The table provides the results for the differences in standard deviation and means of the natural log of the actual and GBM based expected metal prices. In this table, 'Actual' is the natural log the actual metal price, while 'Expected' is the natural log of the GBM based expected metal price estimated using one hundred thousand simulations, while 'Pooled' is the pooled standard deviation. F-stat is used to test for the differences in standard deviations, while the pooled standard deviation is used to estimate the t-stat for differences in means. The table reports the p-values for the F-stats and t-stats. This table is for the monthly frequency and was based on two-hundred and forty months of data for the period between January 2004 and December 2023.

pronounced in the monthly than in the quarterly figure. Perhaps a stronger relationship between expected price and the actual price are to be expected at the monthly and quarterly frequency, than at the annual frequency.

$$ln(P_{t,i}) = \widehat{\alpha_\iota} + \widehat{\beta_\iota}\, ln(Exp\_P_{t,i}) \quad (13)$$

Equation (*13*) represents the output of the simple regression equation represented in equation *(9)*, with $\widehat{\alpha_\iota}$ being the intercept, and $\widehat{\beta_\iota}$ the coefficient of the dependent variable $Exp\_P_{t,i}$. As mentioned earlier, if the GBM based expected metal price is an absolute reliable representation of the actual price, the intercept would have a value of zero, and the slope coefficient would have a value of one. Nevertheless, the regression model with a non-zero intercept, and slope different from one, could still be useful in making predictions about the future prices.

Table 4 presents the results of the regression for all twelve metals at the annual frequency, with GBM based expected metal price estimated from one hundred thousand simulations. The value of the intercept ranges from 0.92 for silver to 6.06 for nickel. The hypothesis that the intercept is statistically different from zero is rejected for silver, aluminum, copper, nickel, lead, tin, and zinc at the 1% confidence level, and the 5% for platinum and ruthenium. The hypothesis is rejected at the 10% level for rhodium, with iridium being the only metal for which the hypothesis is not rejected. The value of the slope coefficient ranges from a minimum of 0.28 for aluminum and silver to a maximum of 0.87 for iridium. These coefficients are tested for being different from zero, and one. For aluminum and silver, the hypothesis that the slope coefficient is equal to zero is not rejected, but rejected for silver, copper, iridium, palladium, rhodium, ruthenium, and tin at the 1% confidence level. For nickel, lead, platinum the rejection is at the 5% level. The statistics show the slope to be being statistically different from one, for eleven of the twelve metals at either the one, five or ten percent confidence levels. Only for one metal, iridium, the hypothesis cannot be rejected. Except for aluminum and zinc, the regressions have double digit R-squares and adjusted R-squares, with maximum value observed for iridium. The regression F-stat is significant for ten of the twelve metals, indicating that except for aluminum and zinc the regression equations may be useful to forecast one-period ahead forecasts. Only for iridium, the GBM based expected price could be used as a proxy for a forecast. This may also be surmised from the correlation coefficient between the actual and expected price, which is also the highest for iridium at 0.93. Palladium has a high correlation coefficient at 0.89, while the lowest is found to be for aluminum at 0.31.

The regression results at the quarterly frequency is in *Table 5*. The hypothesis that the intercept is equal to zero is rejected for all the metals at the one, five or ten percent confidence levels. For silver, aluminum, copper, lead, platinum, tin and zinc, it is at the one percent level, while it is at the five percent level for nickel and rhodium. For iridium and palladium, the rejection is at the ten percent level. The value of the intercept ranges from 0.30 for silver to 1.59 for copper. The value of the slope coefficients, although not equal to one, are numerically close to one ranging from a low of 0.81 for lead, to a high of 0.96 for iridium. The t-stat for the hypothesis that the slope coefficient is equal to zero is rejected at the one percent level for all the metals. The hypothesis that they are statistically equal to one is also rejected for all metals as well, at one confidence level or another. The rejection is at the one percent level for silver, aluminum, copper, lead, platinum, tin and zinc. It is rejected at the five percent level for nickel, rhodium, ruthenium, and at the ten percent level for iridium. The equations have very high and significant F-stat, and all R-squares and adjusted R-squares are greater than 66%. The actual metal price and GBM based expected prices show very high correlation coefficients, with the lowest being 0.82 for aluminum. The highest is for iridium at 0.98. The results at the quarterly frequency may be interpreted as that although the GBM based expected metal price is not an exact proxy of the actual price, it may be used to obtain reliable one period ahead forecasts of the actual metal prices.

Table 6 presents the results of equation *(13)* at the monthly frequency. The regressions for all metals have very high R-squares and adjusted R-squares, with the minimum being 90.49% for aluminum. All metals have very high regression F-stats that are significant at one percent, indicating appropriateness of the simple regression model. The correlation coefficients between the expected and actual prices are also very high at this frequency, with the lowest being 0.95 for aluminum and the highest being 1.00 for iridium. The numerical values of the intercepts are low, with the lowest being for ruthenium at 0.08, and the highest being 0.47 for lead. The hypothesis that the intercept is statistical equal to zero, are rejected for all metals, except ruthenium. This rejection is at the one percent confidence level for silver, aluminum, copper, lead, platinum, tin and zinc. It is at the five percent level for iridium, nickel and rhodium, while it is at the ten percent level for palladium. The numerical value of the slope coefficient, $\widehat{\beta_i}$, is close to, but not quite one for all metals with the lowest being 0.94 for aluminum and lead, and the highest being 0.99 for iridium and palladium. The hypotheses that these coefficients are statistically equal to zero are rejected for all metals at the one percent significance level. Although, the values of the slope coefficients approach one, they are not equal to one, at least statistically. The hypothesis that they are equal to one are rejected at the one percent level for sliver, aluminum, copper, lead, platinum, tin and zinc, and at the five percent level for iridium, nickel and rhodium. The rejection is at the ten percent level for palladium and ruthenium. Given the regression statistics at the monthly frequency, one may conclude that while the GBM based expected metal price may not be an exact proxy for the actual metal price, the simple regression model may be used to produce reliable forecast of the actual while using it as the independent variable.

The results for the test for differences in standard deviation and means are presented in *Table 7, Table 8* and *Table 9* at the annual, quarterly and monthly frequency respectively. In these tables, the GBM based expected metal prices are estimated using one hundred thousand simulations. The hypotheses tested is that the differences in standard deviation and means between the natural logs of the actual and GBM based expected prices, are equal to zero. F-stats as estimated using equation (*10*) were used to test the hypotheses related to standard deviation, while the t-stat estimated using equation (*12*) was for the differences in means. The results from all three tables indicate that the hypotheses related to standard deviation cannot be rejected at all three frequencies, implying that the standard deviation of the GBM based expected prices are not statistically different from those of the actual prices. Similarly, the hypotheses related to differences in means for all the metals cannot be rejected at all three frequencies, indicating that the means of the GBM based expected prices are not statistically different from those of the actual prices.

As part of robustness check, the GBM based expected metal prices were estimate at different simulation runs, specifically one, one hundred, one thousand, ten thousand and fifty thousand. The prices were then compared to actual prices using simple regressions, and also tested for differences in standard deviation and means. The simple regression results are presented in *Table 10, Table 11* and *Table 12* at the annual, quarterly and monthly frequencies respectively. The results of the tests for the differences in standard deviation and means at the annual, quarterly and monthly frequencies are provided in *Table 13*, *Table 14*, and *Table 15* respectively. These tables are organized into panels, with one panel for each metal.

Table 10: Annual Simple Regression Tests for Different Simulation Runs

No. of Simulations	Intercept Coef.	Intercept t-stat ($H_0 = 0$)	GBM Expected Price Coef.	t-stat (H_0 = 0)	t-stat ($H_0 = 1$)	R^2	Adj R^2	ρ	F-stat
Panel A: Silver									
1	2.53	**8.78**	0.13	1.20	**7.74**	7.41%	2.26%	0.27	1.44
100	0.94	**3.22**	0.67	**6.67**	**3.25**	71.20%	69.60%	0.84	**44.50**
1,000	0.92	**3.00**	0.68	**6.40**	**3.08**	69.45%	67.75%	0.83	**40.92**
10,000	0.92	**3.00**	0.68	**6.40**	**3.08**	69.50%	67.80%	0.83	**41.01**
50,000	0.92	**3.00**	0.68	**6.41**	**3.08**	69.51%	67.82%	0.83	**41.04**
Panel B: Aluminum									
1	7.47	**46.94**	0.03	1.05	**40.04**	5.74%	0.51%	0.24	1.10
100	5.54	**3.58**	0.27	1.35	**3.58**	9.23%	4.19%	0.30	1.83
1,000	5.52	**3.58**	0.28	1.37	**3.60**	9.45%	4.42%	0.31	1.88
10,000	5.51	**3.59**	0.28	1.38	**3.61**	9.55%	4.53%	0.31	1.90
50,000	5.51	**3.58**	0.28	1.38	**3.60**	9.54%	4.52%	0.31	1.90
Panel C:Copper									
1	8.24	**22.59**	0.07	1.46	**19.62**	10.53%	5.56%	0.32	2.12
100	5.04	**3.71**	0.42	**2.75**	**3.74**	29.58%	25.67%	0.54	**7.56**
1,000	4.88	**3.63**	0.44	**2.89**	**3.67**	31.65%	27.85%	0.56	**8.33**
10,000	4.90	**3.65**	0.44	**2.89**	**3.69**	31.63%	27.83%	0.56	**8.33**
50,000	4.90	**3.65**	0.44	**2.89**	**3.69**	31.64%	27.84%	0.56	**8.33**
Panel D: Iridium									
1	5.61	**11.58**	0.25	2.55[a]	**7.74**	26.53%	22.45%	0.52	6.50[a]
100	0.96	1.70	0.86	**10.39**	1.66	85.70%	84.91%	0.93	**107.88**
1,000	0.92	1.65	0.87	**10.57**	1.63	86.12%	85.35%	0.93	**111.66**
10,000	0.92	1.65	0.87	**10.53**	1.64	86.02%	85.25%	0.93	**110.79**
50,000	0.92	1.65	0.87	**10.52**	1.64	86.02%	85.24%	0.93	**110.76**
Panel E: Nickel									
1	9.31	**21.68**	0.04	0.93	**20.22**	4.63%	-0.67%	0.22	0.87
100	6.23	**3.20**	0.35	1.79[b]	**3.27**	15.08%	10.36%	0.39	3.20[b]
1,000	5.97	**2.92**	0.38	1.83[b]	**2.99**	15.66%	10.98%	0.40	3.34[b]
10,000	6.06	**2.99**	0.37	1.80[b]	**3.06**	15.21%	10.50%	0.39	3.23[b]
50,000	6.06	**2.99**	0.37	1.80[b]	**3.06**	15.25%	10.54%	0.39	3.24[b]
Panel F: Lead									
1	7.16	**18.07**	0.06	1.02	**16.20**	5.50%	0.25%	0.23	1.05
100	5.12	**4.35**	0.32	2.08[a]	**4.41**	19.33%	14.85%	0.44	4.31[a]
1,000	5.07	**4.28**	0.33	2.09[a]	**4.34**	19.59%	15.12%	0.44	4.39[a]
10,000	5.06	**4.27**	0.33	2.11[a]	**4.33**	19.79%	15.33%	0.44	4.44[a]
50,000	5.06	**4.27**	0.33	2.11[a]	**4.33**	19.81%	15.35%	0.45	4.45[a]
Panel G: Palladium									
1	6.13	**16.41**	0.09	1.26	**13.02**	8.09%	2.98%	0.28	1.58
100	1.23	1.86[b]	0.81	**8.11**	1.94[b]	78.52%	77.33%	0.89	**65.79**
1,000	1.10	1.69	0.82	**8.41**	1.80[b]	79.72%	78.59%	0.89	**70.75**
10,000	1.09	1.68	0.82	**8.49**	1.80[b]	80.02%	78.91%	0.89	**72.09**
50,000	1.09	1.68	0.82	**8.49**	1.80[b]	80.01%	78.90%	0.89	**72.06**
Panel H: Platinum									
1	6.89	**25.91**	0.02	0.44	**21.54**	1.05%	-4.45%	0.10	0.19

100	3.73	**2.89**	0.46	2.53[a]	**2.93**	26.30%	22.20%	0.51	6.42[a]
1,000	3.61	2.75[a]	0.48	2.59[a]	2.80[a]	27.09%	23.04%	0.52	6.69[a]
10,000	3.62	2.75[a]	0.48	2.58[a]	2.80[a]	27.00%	22.95%	0.52	6.66[a]
50,000	3.62	2.75[a]	0.48	2.58[a]	2.80[a]	26.99%	22.94%	0.52	6.66[a]
Panel I: Rhodium									
1	5.55	**6.54**	0.37	**2.81**	**4.81**	30.52%	26.66%	0.55	**7.91**
100	2.24	2.01[b]	0.70	**5.10**	2.15[a]	59.10%	56.82%	0.77	**26.01**
1,000	2.31	2.05[b]	0.69	**4.98**	2.21[a]	57.99%	55.66%	0.76	**24.85**
10,000	2.31	2.05[b]	0.69	**4.96**	2.21[a]	57.76%	55.41%	0.76	**24.61**
50,000	2.32	2.05[b]	0.69	**4.96**	2.22[a]	57.77%	55.43%	0.76	**24.63**
Panel J: Ruthenium									
1	4.33	**7.97**	0.21	1.44	**5.54**	10.32%	5.34%	0.32	2.07
100	1.81	2.12[a]	0.63	**3.86**	2.31[a]	45.26%	42.22%	0.67	**14.89**
1,000	1.89	2.15[a]	0.61	**3.67**	2.38[a]	42.76%	39.58%	0.65	**13.45**
10,000	1.88	2.14[a]	0.61	**3.68**	2.39[a]	43.00%	39.83%	0.66	**13.58**
50,000	1.88	2.14[a]	0.61	**3.68**	2.38[a]	42.98%	39.81%	0.66	**13.57**
Panel K: Tin									
1	9.36	**20.41**	0.05	0.96	**17.75**	4.82%	-0.46%	0.22	0.91
100	3.78	**2.95**	0.61	**4.70**	**2.96**	55.12%	52.63%	0.74	**22.11**
1,000	3.83	**3.00**	0.61	**4.67**	**3.02**	54.76%	52.25%	0.74	**21.79**
10,000	3.83	**3.00**	0.61	**4.66**	**3.02**	54.71%	52.20%	0.74	**21.75**
50,000	3.84	**3.00**	0.61	**4.66**	**3.02**	54.69%	52.17%	0.74	**21.72**
Panel L: Zinc									
1	7.58	**21.26**	0.03	0.47	**18.09**	1.21%	-4.28%	0.11	0.22
100	5.75	**4.21**	0.26	1.47	**4.25**	10.66%	5.70%	0.33	2.15
1,000	5.57	**4.02**	0.28	1.57	**4.06**	12.02%	7.13%	0.35	2.46
10,000	5.58	**4.03**	0.28	1.56	**4.08**	11.91%	7.02%	0.35	2.43
50,000	5.58	**4.03**	0.28	1.56	**4.08**	11.95%	7.05%	0.35	2.44

The results of the simple regression equation with natural log of the actual price as the dependent variable, and the natural log of the GBM based expected price arepresented in this table. The GBM based expected metal prices are based on simulations, ranging from one to fifty-thousand. The t-stat for the intercept and the slope coefficient being equal to zero are presented. The t-stat for the coefficient equal to one are also provided. Equation statistics like R-square, adjusted R-square and F-stats are also provided. ρ is the correlation coefficient between the actual and expected values, and were obtained by taking square root of R-square. The highlighted values indicate significance at 1%, while the superscripts a and b indicate significance at 5% and 10% respectively. The regression was based on twenty-years of data for the period between 2004 and 2023.

Table 11: Quarterly Simple Regression Tests for Different Simulation Runs

No. of	Intercept		GBM Expected Price			R^2	Adj R^2	ρ	F-stat
Simulations	Coef.	$H_0 = 0$	Coef.	$H_0 = 0$	$H_0 = 1$				
Panel A: Silver									
1	2.84	**33.31**	0.00	-0.05	**24.60**	0.00%	-1.28%	0.00	0.00
100	0.32	**3.17**	0.89	**24.99**	**3.13**	88.89%	88.75%	0.94	**624.35**
1,000	0.30	**2.96**	0.89	**24.98**	**3.03**	88.89%	88.74%	0.94	**623.79**
10,000	0.30	**2.95**	0.89	**24.97**	**3.02**	88.88%	88.74%	0.94	**623.55**
50,000	0.30	**2.95**	0.89	**24.98**	**3.03**	88.89%	88.74%	0.94	**623.77**
Panel B: Aluminum									
1	7.24	**70.46**	0.06	**3.82**	**61.58**	15.78%	14.70%	0.40	**14.61**
100	1.61	**3.29**	0.79	**12.26**	**3.28**	65.82%	65.38%	0.81	**150.21**
1,000	1.52	**3.11**	0.80	**12.48**	**3.12**	66.63%	66.20%	0.82	**155.74**
10,000	1.52	**3.11**	0.80	**12.47**	**3.13**	66.60%	66.17%	0.82	**155.51**
50,000	1.52	**3.11**	0.80	**12.47**	**3.13**	66.60%	66.17%	0.82	**155.54**
Panel C:Copper									
1	8.29	**34.77**	0.06	2.01[a]	**30.53**	4.92%	3.71%	0.22	4.04[b]
100	1.62	**3.68**	0.82	**16.30**	**3.68**	77.30%	77.01%	0.88	**265.57**
1,000	1.59	**3.62**	0.82	**16.42**	**3.65**	77.56%	77.28%	0.88	**269.66**
10,000	1.59	**3.63**	0.82	**16.43**	**3.66**	77.58%	77.30%	0.88	**269.96**
50,000	1.59	**3.63**	0.82	**16.43**	**3.66**	77.59%	77.30%	0.88	**270.07**
Panel D: Iridium									
1	4.83	**12.87**	0.34	**5.27**	**10.12**	26.29%	25.35%	0.51	**27.82**
100	0.34	2.05[a]	0.95	**38.89**	1.95[b]	95.09%	95.03%	0.98	**1,512.15**
1,000	0.30	1.85[b]	0.96	**40.39**	1.81[b]	95.44%	95.38%	0.98	**1,631.36**
10,000	0.30	1.84[b]	0.96	**40.38**	1.81[b]	95.43%	95.38%	0.98	**1,630.47**
50,000	0.30	1.84[b]	0.96	**40.38**	1.81[b]	95.43%	95.38%	0.98	**1,630.54**
Panel E: Nickel									
1	9.31	**42.09**	0.05	1.89[b]	**37.67**	4.38%	3.15%	0.21	3.57[b]
100	1.30	2.46[a]	0.86	**15.95**	2.51[a]	76.52%	76.22%	0.87	**254.25**
1,000	1.34	2.51[a]	0.86	**15.75**	2.56[a]	76.09%	75.78%	0.87	**248.17**
10,000	1.35	2.53[a]	0.86	**15.73**	2.58[a]	76.04%	75.73%	0.87	**247.52**
50,000	1.35	2.53[a]	0.86	**15.73**	2.58[a]	76.03%	75.72%	0.87	**247.44**
Panel F: Lead									
1	6.84	**37.67**	0.11	**3.90**	**33.07**	16.32%	15.25%	0.40	**15.22**
100	1.42	**3.63**	0.81	**15.62**	**3.65**	75.77%	75.46%	0.87	**243.98**
1,000	1.41	**3.64**	0.81	**15.86**	**3.68**	76.34%	76.04%	0.87	**251.65**
10,000	1.40	**3.64**	0.81	**15.90**	**3.68**	76.41%	76.11%	0.87	**252.66**
50,000	1.40	**3.64**	0.81	**15.90**	**3.68**	76.42%	76.11%	0.87	**252.72**
Panel G: Palladium									
1	4.73	**14.94**	0.33	**5.84**	**11.75**	30.43%	29.54%	0.55	**34.12**
100	0.32	1.89[b]	0.95	**37.39**	1.91[b]	94.72%	94.65%	0.97	**1,398.15**
1,000	0.30	1.77[b]	0.95	**37.57**	1.84[b]	94.76%	94.70%	0.97	**1,411.34**
10,000	0.30	1.78[b]	0.95	**37.48**	1.86[b]	94.74%	94.67%	0.97	**1,405.01**
50,000	0.30	1.78[b]	0.95	**37.48**	1.86[b]	94.74%	94.67%	0.97	**1,404.44**
Panel H: Platinum									
1	6.69	**51.10**	0.06	**2.53**	**42.89**	7.56%	6.37%	0.27	**6.38**
100	1.01	2.52[a]	0.86	**15.06**	2.54[a]	74.40%	74.08%	0.86	**226.73**

1,000	1.02	**2.63**	0.85	**15.52**	**2.67**	75.53%	75.21%	0.87	**240.72**
10,000	1.02	**2.64**	0.85	**15.48**	**2.69**	75.45%	75.14%	0.87	**239.75**
50,000	1.02	**2.64**	0.85	**15.48**	**2.69**	75.45%	75.14%	0.87	**239.73**
Panel I: Rhodium									
1	5.38	**16.57**	0.38	**7.92**	**13.13**	44.57%	43.86%	0.67	**62.73**
100	0.46	1.88[b]	0.94	**30.50**	1.95[b]	92.26%	92.16%	0.96	**930.22**
1,000	0.48	1.94[b]	0.94	**30.33**	2.03[a]	92.18%	92.08%	0.96	**919.76**
10,000	0.48	1.95[b]	0.94	**30.26**	2.05[a]	92.15%	92.05%	0.96	**915.59**
50,000	0.48	1.95[b]	0.94	**30.25**	2.05[a]	92.15%	92.04%	0.96	**915.04**
Panel J: Ruthenium									
1	3.27	**15.11**	0.43	**8.74**	**11.52**	49.50%	48.85%	0.70	**76.45**
100	0.37	1.92[b]	0.92	**25.03**	2.03[a]	88.93%	88.79%	0.94	**626.70**
1,000	0.33	1.76[b]	0.93	**25.23**	1.94[b]	89.09%	88.95%	0.94	**636.64**
10,000	0.33	1.76[b]	0.93	**25.23**	1.95[b]	89.09%	88.95%	0.94	**636.74**
50,000	0.33	1.76[b]	0.93	**25.23**	1.95[b]	89.08%	88.94%	0.94	**636.56**
Panel K: Tin									
1	9.19	**27.35**	0.07	1.76[b]	**24.25**	3.84%	2.61%	0.20	3.11[b]
100	1.24	**3.16**	0.87	**21.79**	**3.15**	85.89%	85.71%	0.93	**474.74**
1,000	1.15	**2.96**	0.88	**22.20**	**2.97**	86.34%	86.16%	0.93	**492.81**
10,000	1.16	**2.99**	0.88	**22.17**	**3.00**	86.31%	86.13%	0.93	**491.65**
50,000	1.16	**2.99**	0.88	**22.17**	**3.00**	86.30%	86.13%	0.93	**491.52**
Panel L: Zinc									
1	7.24	**36.04**	0.07	2.37[a]	**31.23**	6.73%	5.54%	0.26	5.63[a]
100	1.17	**3.26**	0.85	**18.16**	**3.26**	80.87%	80.62%	0.90	**329.73**
1,000	1.13	**3.11**	0.85	**18.20**	**3.12**	80.93%	80.69%	0.90	**331.13**
10,000	1.13	**3.11**	0.85	**18.20**	**3.13**	80.95%	80.70%	0.90	**331.37**
50,000	1.12	**3.11**	0.85	**18.21**	**3.13**	80.96%	80.72%	0.90	**331.68**

The results of the simple regression equation with natural log of the actual price as the dependent variable, and the natural log of the GBM based expected price are presented in this table. The GBM based expected metal prices are based on simulations, ranging from one to fifty-thousand. The t-stat for the intercept and the slope coefficient being equal to zero are presented. The t-stat for the coefficient equal to one are also provided. Equation statistics like R-square, adjusted R-square and F-stats are also provided. ρ is the correlation coefficient between the actual and expected values, and were obtained by taking square root of R-square. The highlighted values indicate significance at 1%, while the superscripts a and b indicate significance at 5% and 10% respectively. The regression was based on equity-quarters of data for the period between 1st quarter of 2004 and the last quarter of 2023.

Table 12: Monthly Simple Regression Tests for Different Simulation Runs

No. of	Intercept		GBM Expected Price			R^2	Adj R^2	ρ	F-stat
Simulations	Coef.	$H_0 = 0$	Coef.	$H_0 = 0$	$H_0 = 1$				
Panel A: Silver									
1	2.65	**56.16**	0.11	**4.88**	**40.93**	9.08%	8.70%	0.30	**23.78**
100	0.14	**3.46**	0.95	**68.96**	**3.31**	95.23%	95.21%	0.98	**4,755.13**
1,000	0.11	**2.93**	0.96	**70.35**	**3.00**	95.41%	95.39%	0.98	**4,948.74**
10,000	0.11	**2.93**	0.96	**70.27**	**3.02**	95.40%	95.38%	0.98	**4,937.19**
50,000	0.11	**2.93**	0.96	**70.26**	**3.02**	95.40%	95.38%	0.98	**4,936.31**
Panel B: Aluminum									
1	7.43	**99.97**	0.03	**2.72**	**87.86**	3.02%	2.61%	0.17	**7.41**
100	0.44	**2.85**	0.94	**47.12**	**2.81**	90.32%	90.28%	0.95	**2,219.85**
1,000	0.42	**2.77**	0.95	**47.77**	**2.77**	90.55%	90.52%	0.95	**2,281.84**
10,000	0.42	**2.79**	0.94	**47.58**	**2.80**	90.49%	90.45%	0.95	**2,264.02**
50,000	0.42	**2.79**	0.94	**47.59**	**2.80**	90.49%	90.45%	0.95	**2,265.08**
Panel C: Copper									
1	8.19	**57.88**	0.08	**4.10**	**50.57**	6.60%	6.21%	0.26	**16.82**
100	0.49	**3.78**	0.94	**63.15**	**3.73**	94.37%	94.34%	0.97	**3,988.12**
1,000	0.44	**3.40**	0.95	**63.95**	**3.42**	94.50%	94.48%	0.97	**4,090.00**
10,000	0.44	**3.39**	0.95	**63.96**	**3.42**	94.50%	94.48%	0.97	**4,091.47**
50,000	0.44	**3.39**	0.95	**63.96**	**3.42**	94.50%	94.48%	0.97	**4,090.60**
Panel D: Iridium									
1	4.06	19.48	0.46	**13.12**	**15.14**	41.99%	41.74%	0.65	**172.24**
100	0.10	2.50[a]	0.99	**164.53**	2.11[a]	99.13%	99.12%	1.00	**27,070.50**
1,000	0.09	2.15[a]	0.99	**167.32**	1.98[a]	99.16%	99.15%	1.00	**27,996.80**
10,000	0.09	2.14[a]	0.99	**167.56**	1.99[a]	99.16%	99.16%	1.00	**28,077.70**
50,000	0.09	2.13[a]	0.99	**167.58**	1.98[a]	99.16%	99.16%	1.00	**28,082.00**
Panel E: Nickel									
1	8.87	**54.88**	0.10	**5.35**	**49.05**	10.74%	10.37%	0.33	**28.64**
100	0.43	2.42[a]	0.96	**52.59**	2.42[a]	92.08%	92.04%	0.96	**2,765.85**
1,000	0.42	2.40[a]	0.96	**53.47**	2.45[a]	92.32%	92.28%	0.96	**2,859.08**
10,000	0.42	2.40[a]	0.96	**53.47**	2.45[a]	92.32%	92.28%	0.96	**2,858.99**
50,000	0.42	2.40[a]	0.96	**53.47**	2.45[a]	92.32%	92.28%	0.96	**2,859.33**
Panel F: Lead									
1	7.08	**63.36**	0.07	**4.14**	**55.00**	6.71%	6.32%	0.26	**17.13**
100	0.47	**3.53**	0.94	**52.73**	**3.50**	92.11%	92.08%	0.96	**2,780.27**
1,000	0.47	**3.55**	0.94	**53.42**	**3.58**	92.30%	92.27%	0.96	**2,853.74**
10,000	0.46	**3.52**	0.94	**53.55**	**3.56**	92.34%	92.30%	0.96	**2,867.16**
50,000	0.47	**3.52**	0.94	**53.52**	**3.56**	92.33%	92.30%	0.96	**2,864.86**
Panel G: Palladium									
1	4.59	**26.69**	0.35	**11.60**	**21.28**	36.10%	35.83%	0.60	**134.47**
100	0.10	1.92[b]	0.99	**119.23**	1.81[b]	98.35%	98.35%	0.99	**14,215.90**
1,000	0.10	1.79[b]	0.98	**121.16**	1.85[b]	98.40%	98.40%	0.99	**14,680.20**
10,000	0.09	1.77[b]	0.99	**121.28**	1.85[b]	98.41%	98.40%	0.99	**14,708.90**
50,000	0.09	1.77[b]	0.99	**121.27**	1.85[b]	98.41%	98.40%	0.99	**14,705.30**
Panel H: Platinum									
1	6.63	**77.16**	0.07	4.73	**65.79**	8.60%	8.22%	0.29	**22.40**
100	0.31	2.47[a]	0.96	53.89	2.44[a]	92.42%	92.39%	0.96	**2,903.64**

1,000	0.30	2.45[a]	0.96	55.50	2.48[a]	92.83%	92.80%	0.96	**3,080.77**
10,000	0.30	2.44[a]	0.96	55.50	2.48[a]	92.83%	92.80%	0.96	**3,080.38**
50,000	0.30	2.44[a]	0.96	55.49	2.48[a]	92.83%	92.80%	0.96	**3,079.47**
Panel I: Rhodium									
1	5.24	**24.02**	0.39	**12.36**	**19.60**	39.10%	38.84%	0.63	**152.80**
100	0.16	2.22[a]	0.98	**107.27**	2.17[a]	97.97%	97.97%	0.99	**11,506.60**
1,000	0.14	1.95[b]	0.98	**108.36**	2.01[a]	98.01%	98.01%	0.99	**11,742.00**
10,000	0.14	1.93[b]	0.98	**108.27**	2.00[a]	98.01%	98.00%	0.99	**11,721.90**
50,000	0.14	1.93[b]	0.98	**108.27**	2.00[a]	98.01%	98.00%	0.99	**11,723.40**
Panel J: Ruthenium									
1	3.72	**31.24**	0.34	**12.04**	**23.60**	37.87%	37.61%	0.62	**145.04**
100	0.10	2.06[a]	0.98	**101.56**	1.96[a]	97.74%	97.74%	0.99	**10,314.90**
1,000	0.08	1.61	0.98	**102.35**	1.68[b]	97.78%	97.77%	0.99	**10,476.40**
10,000	0.08	1.60	0.98	**102.38**	1.69[b]	97.78%	97.77%	0.99	**10,482.30**
50,000	0.08	1.60	0.98	**102.39**	1.69[b]	97.78%	97.77%	0.99	**10,483.50**
Panel K: Tin									
1	8.79	**47.82**	0.11	**5.46**	**42.78**	11.11%	10.74%	0.33	**29.76**
100	0.39	**3.39**	0.96	**80.78**	**3.31**	96.48%	96.47%	0.98	**6,525.69**
1,000	0.34	**2.93**	0.96	**80.62**	**2.93**	96.47%	96.45%	0.98	**6,499.20**
10,000	0.34	**2.91**	0.97	**80.59**	**2.92**	96.47%	96.45%	0.98	**6,495.32**
50,000	0.34	**2.91**	0.97	**80.60**	**2.92**	96.47%	96.45%	0.98	**6,496.86**
Panel L: Zinc									
1	7.04	**63.60**	0.10	**6.12**	**54.41**	13.61%	13.24%	0.37	**37.48**
100	0.40	**3.16**	0.95	**58.54**	**3.11**	93.51%	93.48%	0.97	**3,426.97**
1,000	0.38	**3.05**	0.95	**58.65**	**3.06**	93.53%	93.50%	0.97	**3,440.06**
10,000	0.38	**3.04**	0.95	**58.55**	**3.06**	93.51%	93.48%	0.97	**3,427.93**
50,000	0.38	**3.04**	0.95	**58.54**	**3.06**	93.51%	93.48%	0.97	**3,427.14**

The results of the simple regression equation with natural log of the actual price as the dependent variable, and the natural log of the GBM based expected price are presented in this table. The GBM based expected metal prices are based on simulations, ranging from one to fifty-thousand. The t-stat for the intercept and the slope coefficient being equal to zero are presented. The t-stat for the coefficient equal to one are also provided. Equation statistics like R-square, adjusted R-square and F-stats are also provided. ρ is the correlation coefficient between the actual and expected values, and were obtained by taking square root of R-square. The highlighted values indicate significance at 1%, while the superscripts a and b indicate significance at 5% and 10% respectively. The regression was based on two-hundred and forty months of data for the period between January 2004 and December 2023.

Table 13: Annual Standard Deviation and Means Tests for Different Simulation

No. of	Standard Deviation					Mean				
Simulations	Act.	Exp.	P/S	F-stat	p-value	Act.	Exp.	Diff.	t-stat	p-value
Panel A: Silver										
1	0.40	0.81	0.64	4.11	0.00	2.86	2.45	0.41	2.01	0.05
100	0.40	0.50	0.45	1.57	0.33	2.86	2.85	0.00	0.03	0.98
1,000	0.40	0.49	0.45	1.52	0.37	2.86	2.87	-0.01	-0.07	0.94
10,000	0.40	0.49	0.45	1.52	0.37	2.86	2.87	-0.01	-0.08	0.94
50,000	0.40	0.49	0.45	1.52	0.37	2.86	2.87	-0.01	-0.08	0.94
Panel B: Aluminum										
1	0.18	1.69	1.20	88.41	0.00	7.63	6.33	1.30	3.43	0.00
100	0.18	0.20	0.19	1.23	0.66	7.63	7.64	-0.01	-0.10	0.92
1,000	0.18	0.20	0.19	1.24	0.64	7.63	7.65	-0.02	-0.39	0.70
10,000	0.18	0.20	0.19	1.25	0.63	7.63	7.65	-0.02	-0.39	0.70
50,000	0.18	0.20	0.19	1.25	0.64	7.63	7.65	-0.02	-0.40	0.69
Panel C:Copper										
1	0.32	1.49	1.08	22.08	0.00	8.77	7.55	1.21	3.56	0.00
100	0.32	0.41	0.36	1.65	0.28	8.77	8.80	-0.04	-0.32	0.75
1,000	0.32	0.40	0.36	1.63	0.30	8.77	8.81	-0.05	-0.42	0.68
10,000	0.32	0.41	0.36	1.64	0.29	8.77	8.81	-0.05	-0.42	0.68
50,000	0.32	0.41	0.36	1.64	0.29	8.77	8.81	-0.05	-0.42	0.68
Panel D: Iridium										
1	0.98	2.04	1.60	4.32	0.00	6.75	4.58	2.17	4.30	0.00
100	0.98	1.05	1.02	1.15	0.76	6.75	6.71	0.03	0.11	0.92
1,000	0.98	1.05	1.01	1.15	0.77	6.75	6.73	0.02	0.06	0.95
10,000	0.98	1.05	1.01	1.15	0.77	6.75	6.73	0.02	0.05	0.96
50,000	0.98	1.05	1.02	1.15	0.76	6.75	6.73	0.02	0.05	0.96
Panel E: Nickel										
1	0.36	1.75	1.26	23.70	0.00	9.70	8.92	0.78	1.96	0.06
100	0.36	0.39	0.38	1.21	0.69	9.70	9.83	-0.13	-1.10	0.28
1,000	0.36	0.37	0.37	1.09	0.85	9.70	9.84	-0.14	-1.20	0.24
10,000	0.36	0.38	0.37	1.11	0.82	9.70	9.84	-0.14	-1.22	0.23
50,000	0.36	0.38	0.37	1.11	0.82	9.70	9.84	-0.14	-1.22	0.23
Panel F: Lead										
1	0.29	1.13	0.82	15.57	0.00	7.56	6.73	0.82	3.17	0.00
100	0.29	0.39	0.34	1.89	0.18	7.56	7.62	-0.06	-0.56	0.58
1,000	0.29	0.39	0.34	1.85	0.19	7.56	7.62	-0.07	-0.60	0.55
10,000	0.29	0.39	0.34	1.85	0.19	7.56	7.62	-0.07	-0.61	0.54
50,000	0.29	0.39	0.34	1.85	0.19	7.56	7.62	-0.07	-0.62	0.54
Panel G: Palladium										
1	0.77	2.48	1.84	10.41	0.00	6.55	4.75	1.79	3.09	0.00
100	0.77	0.84	0.81	1.21	0.69	6.55	6.59	-0.04	-0.18	0.86
1,000	0.77	0.83	0.80	1.18	0.73	6.55	6.61	-0.07	-0.26	0.79
10,000	0.77	0.83	0.80	1.18	0.73	6.55	6.62	-0.07	-0.28	0.78
50,000	0.77	0.83	0.80	1.18	0.73	6.55	6.62	-0.07	-0.28	0.78
Panel H: Platinum										
1	0.23	1.19	0.86	26.55	0.00	7.00	5.73	1.28	4.71	0.00
100	0.23	0.25	0.24	1.22	0.67	7.00	7.06	-0.05	-0.70	0.49

1,000	0.23	0.25	0.24	1.17	0.73	7.00	7.06	-0.06	-0.80	0.43
10,000	0.23	0.25	0.24	1.18	0.73	7.00	7.06	-0.06	-0.82	0.42
50,000	0.23	0.25	0.24	1.17	0.73	7.00	7.06	-0.06	-0.82	0.42
Panel I: Rhodium										
1	0.99	1.49	1.27	2.24	0.09	7.88	6.30	1.58	3.94	0.00
100	0.99	1.09	1.04	1.20	0.70	7.88	8.02	-0.14	-0.42	0.68
1,000	0.99	1.09	1.04	1.21	0.69	7.88	8.04	-0.16	-0.48	0.63
10,000	0.99	1.09	1.04	1.21	0.68	7.88	8.05	-0.17	-0.51	0.61
50,000	0.99	1.09	1.04	1.21	0.68	7.88	8.05	-0.17	-0.51	0.61
Panel J: Ruthenium										
1	0.88	1.38	1.16	2.42	0.06	5.06	3.55	1.52	4.15	0.00
100	0.88	0.95	0.92	1.16	0.75	5.06	5.20	-0.13	-0.46	0.65
1,000	0.88	0.95	0.92	1.16	0.75	5.06	5.24	-0.17	-0.60	0.55
10,000	0.88	0.96	0.92	1.17	0.74	5.06	5.25	-0.18	-0.63	0.54
50,000	0.88	0.96	0.92	1.17	0.74	5.06	5.25	-0.18	-0.63	0.54
Panel K: Tin										
1	0.43	1.83	1.33	18.50	0.00	9.78	8.39	1.40	3.33	0.00
100	0.43	0.51	0.47	1.46	0.41	9.78	9.79	0.00	-0.02	0.98
1,000	0.43	0.52	0.47	1.48	0.40	9.78	9.80	-0.02	-0.11	0.91
10,000	0.43	0.52	0.47	1.49	0.40	9.78	9.80	-0.02	-0.12	0.91
50,000	0.43	0.52	0.47	1.49	0.40	9.78	9.80	-0.02	-0.12	0.91
Panel L: Zinc										
1	0.32	1.39	1.01	18.91	0.00	7.74	6.48	1.27	3.97	0.00
100	0.32	0.41	0.37	1.62	0.30	7.74	7.78	-0.04	-0.31	0.76
1,000	0.32	0.40	0.36	1.55	0.35	7.74	7.79	-0.05	-0.45	0.66
10,000	0.32	0.40	0.36	1.56	0.34	7.74	7.80	-0.05	-0.48	0.64
50,000	0.32	0.40	0.36	1.56	0.34	7.74	7.80	-0.05	-0.48	0.64

The table provides the results for the differences in standard deviation and means of the natural log of the actual and GBM based expected metal prices. In this table, ‘Actual’ is the natural log the actual metal price, while ‘Expected’ is the natural log of the GBM based expected metal price, while ‘Pooled’ is the pooled standard deviation. F-stat is used to test for the differences in standard deviations, while the pooled standard deviation is used to estimate the t-stat for differences in means. The table reports the p-values for the F-stats and t-stats. This table is for the annual frequency and was based on twenty-years of data for the period between 2004 and 2023. The results in this table are based on different number of simulations.

Table 14: Quarterly Standard Deviation and Means Test Results for Different Simulation

No. of	Standard Deviation					Mean				
Simulations	Act.	Exp.	Pooled	F-stat	p-value	Act.	Exp.	Diff.	t-stat	p-value
Panel A: Silver										
1	0.40	1.12	0.84	7.72	0.00	2.84	1.77	1.07	7.99	0.00
100	0.40	0.43	0.42	1.13	0.60	2.84	2.83	0.01	0.11	0.92
1,000	0.40	0.43	0.42	1.12	0.62	2.84	2.84	0.00	-0.05	0.96
10,000	0.40	0.43	0.42	1.12	0.62	2.84	2.84	0.00	-0.06	0.95
50,000	0.40	0.43	0.42	1.12	0.62	2.84	2.84	0.00	-0.06	0.95
Panel B: Aluminum										
1	0.19	1.31	0.94	46.16	0.00	7.62	6.59	1.03	6.98	0.00
100	0.19	0.20	0.20	1.06	0.80	7.62	7.62	0.00	0.11	0.91
1,000	0.19	0.20	0.20	1.04	0.86	7.62	7.63	0.00	-0.13	0.90
10,000	0.19	0.20	0.20	1.04	0.86	7.62	7.63	0.00	-0.16	0.87
50,000	0.19	0.20	0.20	1.04	0.86	7.62	7.63	0.00	-0.16	0.87
Panel C:Copper										
1	0.30	1.09	0.80	12.91	0.00	8.76	7.68	1.08	8.55	0.00
100	0.30	0.33	0.32	1.16	0.51	8.76	8.76	0.00	0.01	0.99
1,000	0.30	0.33	0.31	1.16	0.52	8.76	8.77	-0.01	-0.17	0.86
10,000	0.30	0.33	0.32	1.16	0.51	8.76	8.77	-0.01	-0.20	0.85
50,000	0.30	0.33	0.32	1.16	0.51	8.76	8.77	-0.01	-0.20	0.84
Panel D: Iridium										
1	0.95	1.42	1.21	2.24	0.00	6.75	5.60	1.15	6.01	0.00
100	0.95	0.97	0.96	1.05	0.83	6.75	6.73	0.02	0.13	0.90
1,000	0.95	0.97	0.96	1.04	0.86	6.75	6.74	0.01	0.06	0.95
10,000	0.95	0.97	0.96	1.04	0.86	6.75	6.74	0.01	0.05	0.96
50,000	0.95	0.97	0.96	1.04	0.86	6.75	6.74	0.01	0.05	0.96
Panel E: Nickel										
1	0.35	1.54	1.12	19.18	0.00	9.72	8.61	1.11	6.25	0.00
100	0.35	0.36	0.35	1.02	0.91	9.72	9.74	-0.02	-0.40	0.69
1,000	0.35	0.36	0.36	1.03	0.90	9.72	9.74	-0.03	-0.46	0.64
10,000	0.35	0.36	0.36	1.03	0.89	9.72	9.74	-0.03	-0.48	0.63
50,000	0.35	0.36	0.36	1.03	0.89	9.72	9.74	-0.03	-0.48	0.63
Panel F: Lead										
1	0.30	1.15	0.84	14.66	0.00	7.53	6.61	0.92	6.93	0.00
100	0.30	0.32	0.31	1.15	0.53	7.53	7.54	-0.01	-0.17	0.87
1,000	0.30	0.32	0.31	1.16	0.51	7.53	7.55	-0.01	-0.28	0.78
10,000	0.30	0.32	0.31	1.16	0.51	7.53	7.55	-0.01	-0.31	0.76
50,000	0.30	0.32	0.31	1.16	0.51	7.53	7.55	-0.02	-0.31	0.76
Panel G: Palladium										
1	0.75	1.25	1.03	2.76	0.00	6.53	5.43	1.10	6.76	0.00
100	0.75	0.77	0.76	1.05	0.84	6.53	6.53	0.00	-0.02	0.98
1,000	0.75	0.77	0.76	1.04	0.85	6.53	6.54	-0.01	-0.08	0.93
10,000	0.75	0.77	0.76	1.04	0.85	6.53	6.54	-0.01	-0.09	0.92
50,000	0.75	0.77	0.76	1.04	0.85	6.53	6.54	-0.01	-0.10	0.92
Panel H: Platinum										
1	0.25	1.26	0.91	24.45	0.00	7.02	5.82	1.20	8.37	0.00
100	0.25	0.26	0.25	1.02	0.94	7.02	7.02	-0.01	-0.13	0.89
1,000	0.25	0.26	0.26	1.04	0.87	7.02	7.03	-0.02	-0.37	0.71

10,000	0.25	0.26	0.26	1.04	0.87	7.02	7.03	-0.02	-0.40	0.69
50,000	0.25	0.26	0.26	1.04	0.86	7.02	7.03	-0.02	-0.40	0.69
Panel I: Rhodium										
1	1.00	1.78	1.44	3.15	0.00	7.87	6.61	1.26	5.53	0.00
100	1.00	1.02	1.01	1.04	0.85	7.87	7.88	-0.01	-0.08	0.94
1,000	1.00	1.03	1.01	1.05	0.83	7.87	7.89	-0.02	-0.11	0.91
10,000	1.00	1.03	1.01	1.05	0.83	7.87	7.89	-0.02	-0.13	0.90
50,000	1.00	1.03	1.01	1.05	0.83	7.87	7.89	-0.02	-0.13	0.90
Panel J: Ruthenium										
1	0.87	1.42	1.18	2.66	0.00	5.06	4.15	0.91	4.88	0.00
100	0.87	0.89	0.88	1.04	0.86	5.06	5.07	-0.02	-0.11	0.91
1,000	0.87	0.88	0.88	1.03	0.89	5.06	5.09	-0.03	-0.21	0.83
10,000	0.87	0.89	0.88	1.03	0.88	5.06	5.09	-0.03	-0.22	0.82
50,000	0.87	0.89	0.88	1.03	0.88	5.06	5.09	-0.03	-0.22	0.82
Panel K: Tin										
1	0.41	1.19	0.89	8.34	0.00	9.78	8.67	1.11	7.94	0.00
100	0.41	0.44	0.42	1.12	0.60	9.78	9.77	0.01	0.08	0.93
1,000	0.41	0.43	0.42	1.11	0.64	9.78	9.78	0.00	-0.05	0.96
10,000	0.41	0.43	0.42	1.11	0.64	9.78	9.78	0.00	-0.06	0.95
50,000	0.41	0.43	0.42	1.11	0.64	9.78	9.78	0.00	-0.07	0.95
Panel L: Zinc										
1	0.31	1.16	0.85	13.50	0.00	7.71	6.65	1.06	7.90	0.00
100	0.31	0.33	0.32	1.12	0.60	7.71	7.71	0.00	0.05	0.96
1,000	0.31	0.33	0.32	1.11	0.64	7.71	7.71	-0.01	-0.11	0.92
10,000	0.31	0.33	0.32	1.11	0.64	7.71	7.72	-0.01	-0.12	0.90
50,000	0.31	0.33	0.32	1.11	0.64	7.71	7.72	-0.01	-0.12	0.90

The table provides the results for the differences in standard deviation and means of the natural log of the actual and GBM based expected metal prices. In this table, 'Actual' is the natural log the actual metal price, while 'Expected' is the natural log of the GBM based expected metal price, while 'Pooled' is the pooled standard deviation. F-stat is used to test for the differences in standard deviations, while the pooled standard deviation is used to estimate the t-stat for differences in means. The table reports the p-values for the F-stats and t-stats. This table is for the quarterly frequency and was based on eighty-quarters of data for the period between 1st quarter of 2004 and the last quarter of 2023. The results in this table are based on different number of simulations.

Table 15: Monthly Standard Deviation and Means Test Results for Different Simulation

No. of	Standard Deviation					Mean				
Simulations	Act.	Exp.	POOLED	F-stat	p-value	Act.	Exp.	Diff.	t-stat	p-value
				Panel A: Silver						
1	0.42	1.18	0.89	8.02	0.00	2.84	1.81	1.03	12.74	0.00
100	0.42	0.43	0.42	1.05	0.73	2.84	2.83	0.01	0.19	0.85
1,000	0.42	0.43	0.42	1.04	0.78	2.84	2.84	0.00	-0.04	0.97
10,000	0.42	0.43	0.42	1.04	0.77	2.84	2.84	0.00	-0.06	0.95
50,000	0.42	0.43	0.42	1.04	0.78	2.84	2.84	0.00	-0.06	0.95
				Panel B: Aluminum						
1	0.19	1.12	0.81	33.41	0.00	7.63	6.63	0.99	13.49	0.00
100	0.19	0.20	0.20	1.01	0.91	7.63	7.62	0.01	0.36	0.72
1,000	0.19	0.20	0.20	1.01	0.92	7.63	7.63	0.00	-0.03	0.97
10,000	0.19	0.20	0.20	1.01	0.91	7.63	7.63	0.00	-0.09	0.93
50,000	0.19	0.20	0.20	1.01	0.91	7.63	7.63	0.00	-0.09	0.93
				Panel C: Copper						
1	0.31	1.07	0.79	11.73	0.00	8.76	7.66	1.10	15.31	0.00
100	0.31	0.32	0.32	1.06	0.66	8.76	8.76	0.01	0.23	0.82
1,000	0.31	0.32	0.32	1.05	0.71	8.76	8.76	0.00	-0.05	0.96
10,000	0.31	0.32	0.32	1.05	0.71	8.76	8.76	0.00	-0.08	0.94
50,000	0.31	0.32	0.32	1.05	0.71	8.76	8.76	0.00	-0.08	0.93
				Panel D: Iridium						
1	0.95	1.33	1.15	1.95	0.00	6.72	5.74	0.98	9.35	0.00
100	0.95	0.96	0.95	1.02	0.90	6.72	6.71	0.02	0.19	0.85
1,000	0.95	0.96	0.95	1.02	0.91	6.72	6.72	0.01	0.09	0.93
10,000	0.95	0.96	0.95	1.02	0.91	6.72	6.72	0.01	0.08	0.94
50,000	0.95	0.96	0.95	1.02	0.91	6.72	6.72	0.01	0.08	0.94
				Panel E: Nickel						
1	0.36	1.19	0.88	11.10	0.00	9.72	8.71	1.02	12.68	0.00
100	0.36	0.36	0.36	1.01	0.95	9.72	9.72	0.00	0.01	0.99
1,000	0.36	0.36	0.36	1.01	0.94	9.72	9.73	-0.01	-0.24	0.81
10,000	0.36	0.36	0.36	1.01	0.94	9.72	9.73	-0.01	-0.26	0.79
50,000	0.36	0.36	0.36	1.01	0.94	9.72	9.73	-0.01	-0.27	0.79
				Panel F: Lead						
1	0.31	1.13	0.83	13.70	0.00	7.54	6.51	1.03	13.53	0.00
100	0.31	0.31	0.31	1.05	0.72	7.54	7.53	0.00	0.16	0.87
1,000	0.31	0.31	0.31	1.05	0.70	7.54	7.54	0.00	-0.13	0.90
10,000	0.31	0.31	0.31	1.05	0.71	7.54	7.54	0.00	-0.16	0.87
50,000	0.31	0.31	0.31	1.05	0.71	7.54	7.54	0.00	-0.16	0.87
				Panel G: Palladium						
1	0.75	1.28	1.05	2.90	0.00	6.53	5.51	1.03	10.70	0.00
100	0.75	0.76	0.75	1.01	0.92	6.53	6.52	0.01	0.10	0.92
1,000	0.75	0.76	0.75	1.01	0.91	6.53	6.53	0.00	-0.03	0.97
10,000	0.75	0.76	0.75	1.01	0.91	6.53	6.53	0.00	-0.05	0.96
50,000	0.75	0.76	0.75	1.01	0.91	6.53	6.53	0.00	-0.05	0.96
				Panel H: Platinum						
1	0.25	1.11	0.81	19.10	0.00	7.03	5.96	1.07	14.59	0.00
100	0.25	0.26	0.25	1.01	0.94	7.03	7.02	0.00	0.13	0.90
1,000	0.25	0.26	0.25	1.01	0.92	7.03	7.03	0.00	-0.18	0.86

10,000	0.25	0.26	0.25	1.01	0.92	7.03	7.03	0.00	-0.21	0.83
50,000	0.25	0.26	0.25	1.01	0.92	7.03	7.03	0.00	-0.22	0.83
Panel I: Rhodium										
1	1.02	1.64	1.36	2.61	0.00	7.87	6.78	1.09	8.71	0.00
100	1.02	1.03	1.02	1.02	0.88	7.87	7.86	0.01	0.06	0.95
1,000	1.02	1.02	1.02	1.02	0.90	7.87	7.87	0.00	-0.04	0.97
10,000	1.02	1.02	1.02	1.02	0.90	7.87	7.87	0.00	-0.05	0.96
50,000	1.02	1.02	1.02	1.02	0.90	7.87	7.87	0.00	-0.05	0.96
Panel J: Ruthenium										
1	0.87	1.58	1.27	3.32	0.00	5.05	3.94	1.11	9.54	0.00
100	0.87	0.87	0.87	1.02	0.90	5.05	5.05	0.01	0.08	0.94
1,000	0.87	0.87	0.87	1.01	0.94	5.05	5.05	0.00	-0.03	0.98
10,000	0.87	0.87	0.87	1.01	0.94	5.05	5.06	0.00	-0.04	0.97
50,000	0.87	0.87	0.87	1.01	0.94	5.05	5.06	0.00	-0.04	0.97
Panel K: Tin										
1	0.41	1.22	0.91	8.69	0.00	9.78	8.78	1.00	12.03	0.00
100	0.41	0.42	0.42	1.05	0.73	9.78	9.77	0.01	0.27	0.79
1,000	0.41	0.42	0.42	1.04	0.78	9.78	9.78	0.00	0.00	1.00
10,000	0.41	0.42	0.42	1.04	0.79	9.78	9.78	0.00	-0.02	0.98
50,000	0.41	0.42	0.42	1.04	0.79	9.78	9.78	0.00	-0.02	0.98
Panel L: Zinc										
1	0.33	1.19	0.87	13.31	0.00	7.71	6.60	1.11	13.97	0.00
100	0.33	0.33	0.33	1.04	0.78	7.71	7.70	0.01	0.23	0.82
1,000	0.33	0.33	0.33	1.04	0.79	7.71	7.71	0.00	-0.04	0.97
10,000	0.33	0.33	0.33	1.04	0.79	7.71	7.71	0.00	-0.07	0.95
50,000	0.33	0.33	0.33	1.04	0.79	7.71	7.71	0.00	-0.07	0.94

The table provides the results for the differences in standard deviation and means of the natural log of the actual and GBM based expected metal prices. F-stat is used to test for the differences in standard deviations, while the pooled standard deviation is used to estimate the t-stat for differences in means. The table reports the p-values for the F-stats and t-stats. This table is for the monthly frequency and was based on two-hundred and forty months of data for the period between January 2004 and December 2023. The results in this table are based on different number of simulations.

The simple regression results for equation *(13)* at the annual frequency in *Table 10*, show that when the expected value is based on only one simulation, the hypothesis that the intercept of the equation is equal to zero, is rejected for all twelve metals. Numerically, the intercept ranges from a minimum of 2.53 for silver to a maximum of 9.36 for tin. The slope coefficients range from a minimum of 0.02 for platinum, to a maximum of 0.37 for rhodium. Statistically, however, these coefficients are not different from zero for eleven of twelve metals. For iridium, it takes the value of 0.25 and the hypothesis is rejected at the five percent confidence level. The correlation coefficient between the actual and expected metal prices are observed to be rather low, with only iridium and rhodium with a value greater than 0.50 but still less than 0.60. These regressions also have low R-squares and adjusted R-squares, and equation F-stats do not reject the inappropriateness of the model for ten out of twelve metals. Only for iridium and rhodium the hypotheses are rejected at five and one percent confidence level respectively.

It is observed that as the number of simulations increases towards fifty thousand, regression R-squares and adjusted R-squares increases, especially when the number of simulations increases from one to one hundred, although relatively speaking the improvement is small for aluminum, nickel and zinc. The increase in R-squares and adjusted R-squares for simulations higher than one hundred are not as dramatic than the increase from one to one hundred simulations. A similar pattern is observed for equation F-stats. As the number of simulations increases from one to one hundred, the F-stat value increases from insignificance to significance at one percent confidence level for silver, copper, palladium, rhodium, ruthenium and tin. For nickel, the improved is from insignificance to significance at the ten percent level, while for platinum the improvement is to a five percent significance level. For iridium the improvement is from a five to one percent significance level. The F-stat, however, does not improve to have a significance levels for aluminum and zinc.

The value of the regression equation intercepts decreases as the number of simulations increases, although the decreases is not uniform for all the metals. In most of the cases, the value of the intercept does not change much after one hundred or one thousand simulations. For iridium and palladium, the hypothesis that the value is not different from zero can be rejected. The value of the slope coefficient decreases for all metals with increases in simulation numbers, and the hypotheses that they are equal to zero are rejected for all metals except for aluminum and zinc. The rejection is at the one percent level for silver copper, iridium, palladium, rhodium, ruthenium and tin, if expected values are estimate with one hundred or higher simulations. For lead and platinum, the rejection is at one percent, while it is at the five percent level for nickel. The numerical value of the intercept limits towards one, although it is not quite one for all metals. The hypothesis that they are equal to one is rejected for all metals with at least ten percent confidence except iridium.

The results of the equation *(13)* simple regression at the quarterly frequency in *Table 11* is similar to that of the annual frequency in *Table 10* but with some exceptions. When expected values are based on only one simulation, F-stats are insignificant, and the regressions have poor R-squares and adjusted R-squares. The values of the regression intercepts are not statistically equal to zero for all metals as the hypotheses are reject at the one percent confidence level. The intercept values decrease and stabilizes at either one hundred or one thousand simulations. The hypotheses that the values are equal to zero are also rejected at one hundred or higher simulations at the ten percent, five percent or one percent significance levels. The values of the slope coefficients are low (or equal to zero in case of silver), but increases as the number of simulations increases. The hypothesis that they are equal to zero are also rejected for all metals except for silver when expected value is based on only one simulation. The slope coefficient values are also not statistically equal to one, but numerically they are greater than 0.8 for simulation runs greater than one hundred for all twelve metals. The correlation coefficient between expected and actual values are also greater than 0.81 for all twelve metals for simulations runs one hundred or greater.

The expected and actual metal prices demonstrate very high correlation at the monthly frequency, as can be inferred from the correlation coefficients in *Table 12*, especially when the expected prices were based on at least one hundred simulations. The lowest correlation coefficient for simulation runs one hundred or greater is 0.95 for aluminum, and for iridium it is 1.00. Even when the expected price was based on only one simulation run, some of the metals have very high correlation coefficient between the actual and expected prices, though not as good as those for greater number of simulations. The F-stats for regression equation (*13*) are significant for all equations at all simulation runs including one. The R-squares and adjusted R-squares are greater than 90% for all metals at simulation runs one hundred or greater, although they are comparatively lower when expected prices are estimated using only one simulation run. The hypothesis that the intercept value is statistically equal to zero is rejected for all metals at all simulation runs, expect for ruthenium for simulations runs one thousand or greater. Numerically, though, the numbers are in decimals for all metals when expected prices were estimated using at least one hundred simulations. The value of the slope coefficient is not statistically equal to zero for all metals for all simulation runs as the hypothesis that they are equal to zero can be rejected at the one percent confidence level. The hypothesis that the slope coefficient is statistically equal to one, can also be rejected for all metals, with the hypotheses rejected at the one percent level for silver, aluminum, copper, lead, ruthenium, tin and zinc at the one percent confidence level. The hypotheses can be rejected at the five percent level for iridium, nickel and rhodium, though for iridium the rejection for expected price based on one simulation run is at the one percent level. The hypotheses are rejected with ten percent confidence for palladium. Numerically, however, the slope coefficients are greater than 0.94 for all metals when the expected prices are estimated using at least one hundred simulations.

The results for the test for differences in standard deviations and means between the actual and expected metal prices when the expected prices were estimated at different simulations runs are in *Table 13*, *Table 14* and *Table 15* at the annual, quarterly and monthly frequencies respectively. Results show that the hypotheses of no difference between the standard deviations of the natural log of the actual and expected metal prices cannot not be rejected for all metals and all simulation runs, expect when the expected prices were estimated using only one simulation. The rejection is with one percent confidence. Similarly, the hypothesis that the differences between the means of the natural logs of actual and expected metal prices are equal to zero can be reject for all metals and all frequencies when the expected prices were estimated using at least one hundred simulations. The hypotheses that the differences are equal to zero cannot be rejected for all metals for all frequencies with at least ten percent confidence level, when the expected prices were estimated using only one simulation run.

Summarizing, this study evaluates the reliability of GBM-based forecasts for twelve metal prices, six base and six precious, across annual, quarterly, and monthly frequencies. Graphical analysis shows that the GBM-expected and actual prices align more closely at higher frequencies. Regression analyses based on a log-linear model reveal that while the GBM forecasts are not perfect proxies, they display strong statistical relationships with actual prices, particularly at the monthly level. The strength of the relationship improves with the number of simulations, with at least 100 simulations yielding significantly better predictive power than lower number of simulation runs. Tests for differences in means and standard deviations of the natural log of actual and expected prices further affirm the statistical similarity, especially at higher simulation counts. Robustness checks confirm that forecasts based on a single simulation are unreliable, while those based on 100 or more, could be useful. The findings underscore that GBM-based models, when properly specified and sufficiently simulated, can produce useful one-period-ahead forecasts, especially for shorter time horizons. These results consistent with earlier studies on gold (Sinha 2024d) and equity index forecasts (Sinha 2021, 2024a, 2024f).

VI. Conclusion

Empirical research in metal prices has mushroomed (Pierdzioch and Risse 2020), as accurate metal price forecasts are of great importance to economic agents (Liu et al. 2017). Researchers have studied and established links between vital economic variables like exchange rates (Balcilar, Gupta, and Pierdzioch 2017; Pierdzioch, Risse, and Rohloff 2016a, 2016b; Pukthuanthong and Roll 2011; Reboredo 2013b), interest rates (Agbeyegbe 1989; Baffes and Savescu 2014), inflation rates (Beckmann and Czudaj 2013a), oil prices (Beckmann and Czudaj 2013b; Reboredo 2013a), business cycle (Fama and French 1988), and mineral activity (Huang et al. 2022) to mention a few. Accordingly, a number of different methodologies have been developed to study factors influencing and forecast metal prices (Chen et al. 2014; Dooley and Lenihan 2005; Du et al. 2021; Gil-Alana and Poza 2024). Among the various forecasting methodological approach is GBM, a commonly applied Econophysics (Sinha 2024e, 2024c) technique.

GBM is a stochastic differential equation-based modeling approach that requires drift and diffusion terms along with a standard normal random variable that has an average value of zero, and a standard deviation of one. This approach has been applied to forecast stock prices and indexes (Parungrojrat and Kidsom 2019; Reddy and Clinton 2016; Samuelson 1965; Urama and Ezepue 2018), energy assets (Alhagyan 2024; Lynch et al. 2021), agricultural products (Ibrahim et al. 2021; Zelingher and Makowski 2024), and exchange rates are (Abbas and Alhagyan 2023; Alhagyan 2022), and metals (Hamdan et al. 2020; Huang et al. 2022; Ramos et al. 2019; Roslan and Halim 2024; Soysal 2023; Wilmot 2019). Although each iteration of the random term, generates a forecast, most manuscripts use only a limited number of simulations (Kumar et al. 2024; Sinha 2021). If more than one simulation is used, and each simulation producing a forecast, the issue becomes as to which forecast to use. This was resolved in (Sinha 2021) by estimating expected forecasted values from the forecasts and their associate probabilities. (Sinha 2024a, 2024f) applied this approach to forecast stock index values, while (Sinha 2024d) applied this approach to forecast gold prices.

This study estimates expected metal prices using the (Sinha 2021) framework for six base metals—aluminum, copper, nickel, lead, tin, and zinc—and six precious metals—silver, iridium, palladium, platinum, and rhodium. Forecasts are generated at monthly, quarterly, and annual frequencies using an initial set of 100,000 simulations. The results reveal strong correlations between actual market prices and GBM-derived expected prices at the monthly and quarterly levels across all metals, and at the annual level for most. Regression analyses and statistical tests for differences in means and standard deviations further support the reliability of the GBM-based forecasts. Robustness assessments demonstrate that forecasts based on a single simulation are unreliable, whereas forecasts derived from a minimum of 100 simulations provide reliable estimates.

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