Why Do Managers Use IRR in Investment Analysis?

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Abstract

Finance textbooks recommend the use of Net Present Value (NPV) as the evaluation tool for Capital Budgeting. Yet surveys of managers have consistently shown that managers prefer Internal rate of Return (IRR) to NPV. This article rigorously establishes the interpretation of IRR as the return earned on funds that remains internally invested in the project. Using this interpretation IRR can be viewed as a tool to evaluate the riskiness of the capital budgeting proposal.

I. Introduction

Capital budgeting literature tells us that the criterion for accepting or rejecting a project should be based on the Net Present Value (NPV) of the project. We should accept the project if the NPV is positive and we should reject the project if the NPV is negative. This much is drilled into every student who has taken a course in introductory finance. However, when we look at surveys of managers we find that a significantly large number of managers prefer to use Internal Rate of Return (IRR) for capital budgeting over NPV.

In order to understand this phenomenon, we need to examine the economic interpretation of IRR. In this paper we provide a rigorous justification for interpreting IRR as akin to a mortgage rate. When we take out a mortgage we prepare an amortization table. The amortization table divides every mortgage installment in two parts -interest and repayment of principal. Towards the beginning of the mortgage period, a larger fraction of the mortgage payment is going towards interest payment and towards the end of the mortgage period a larger fraction of the mortgage amount is going towards the repayment of the amount borrowed. At the end of the mortgage period, with the last payment, the account is squared off and the borrower has paid off the loan.

One possible explanation for use of IRR by managers is that managers interpret IRR as something akin to a mortgage rate and this makes it easier for them to intuitively estimate the probabilities of financial embarrassment of not meeting the return expectations of security holders. IRR thus serves as a risk measure for managers. In this article we provide a rigorous justification for interpreting IRR as akin to a mortgage rate. After establishing the validity of this particular interpretation of IRR we discuss how managers plausibly use this information to estimate the probability of the project not meeting the return expectations of security holders.

The paper is organized as follows. Section II motivates the paper and establishes the scope for the paper. Section III discusses the two economic interpretations of IRR and which of these interpretations are useful for establishing a rationale for observed managerial use of IRR, while Section IV establishes the generality of the interpretation of IRR as the rate of return on funds remaining internally invested. The next section segues into the question of managerial use of IRR and posits that managers use IRR as a risk

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measure. Section IV discusses the direction for future research. Section V summarizes and concludes. All proofs are given in the appendix.

II. Motivation and Scope for the Paper.

Finance textbooks are unanimous in their recommendation of NPV as the preferred method for evaluating capital budgets.[1] However when we look at the choices of practicing managers we find that they often prefer IRR to NPV as the choice for analyzing capital expenditure proposals.

A. Literature Review.

Surveys of corporate managers have consistently shown that managers often rank IRR ahead of NPV as their preferred means of evaluating projects. In a survey of 90 US firms and 65 Canadian firms by Payne et al. (1999), American managers ranked IRR as number one, ahead of NPV, which was judged the second most important technique. In the same survey, Canadian managers ranked IRR as number two and NPV as number one in importance. Jog and Srivastava (1995) however find that Canadian managers use IRR more than they use NPV. Kim and Ulferts (1996) summarize five surveys on capital budgeting practices of multinational companies. Each survey found that IRR is used more as a primary project evaluation technique than NPV. Sangster (1993) surveys capital budgeting practices in UK and reviews findings of other researchers and opines that
"(t)he theorist's recommendation that NPV be used rather than IRR was generally found not to have been adopted"(page 309)
Graham and Harvey (2001) find that
"74.9% of CFOs always or almost always... use net present value...; and 75.7% always or almost always use internal rate of return ..." (page 197)

The preference for IRR over NPV as a technique is ubiquitous. Block (1997) finds that small firms use IRR preferentially over NPV-a conclusion which is found to be true also for Fortune 500 companies [Burns and Walker (1997)], for subsidiaries of US multinationals [Shao and Shao (1996)] and for firms in Singapore [Kester and Chong (1998)]. The puzzle is described by Bierman (1993) as below:

"Seventy-three of the firms (99%) used IRR compared to 63 of the firms (85%) using NPV. ... Given that IRR can be improperly used (two illustrations of potentially improper use are mutually exclusive investments and multiple rates of return), the sole use of IRR without using NPV is of some concern."(page 24)

Similar concern is expressed by Brealey et al. (2006) when they write

"...for many companies DCF means IRR, not NPV."(page 99)

[1] For recent theoretical critiques of the NPV criterion see Berkovitch and Israel (2004), Magni (2002), Bert (2005), and Magni (2009). Osborne (2010) advocates use of the ratio of NPV to the absolute value of initial investment as the decision method.

B. Scope of the Paper

In the light of the survey evidence we should try to see if there could be a plausible rational reason why managers could be more favorably inclined towards IRR over NPV. This paper aims to supply such a rational reason. We demonstrate that IRR can be rigorously interpreted as a rate of return on funds that remain internally invested in the project. We then posit that managers use IRR as an intuitive measure of risk in a world where they face a stochastic cost of capital. It is to be emphasized here that this is a theoretical paper. The empirical validity of this theory will be the subject of future research (see Section IV).

III. Interpreting Internal Rate of Return

If managers are more favorably inclined to use IRR as a technique of capital budgeting, then IRR must be carrying some kind of meaningful economic information for managers. If we try to look at the economic interpretations of IRR we come across two interpretations.

A. The More Popular Interpretation

The more popular and standard interpretation[2] says that *internal rate of return is the return earned on the initial investment*. This interpretation follows from the structure of the equation used to determine IRR. In particular, if $-I_0, c_1, c_2, \cdots\cdots, c_n$ are the cash flows associated with the project, the *j*th term of the series being the cash flow in the *j*th period, then IRR *r* is given by

	$I_0 = \frac{c_1}{(1+r)} + \frac{c_2}{(1+r)^2} + \frac{c_3}{(1+r)^3} + \cdots\cdots + \frac{c_n}{(1+r)^n}$	(1)

After simplification Equation (1) becomes

	$I_0 (1+r)^n = \sum_{j=1}^{n} c_j (1+r)^{n-j}$	(2)

The left hand side of Equation (2) is the value of an investment I_0 after *n* periods at a compounded rate of *r* per period. However, accepting this interpretation means that we have to accept the interpretation of the right side of Equation (2) as well. This means that the intermediate cash flows are being reinvested for the remainder of the project life, at a rate equal to IRR[3]. However, there is no way to guarantee that there will be opportunities

[2] As for example Brealey et al. (2006) write:

> "Unfortunately, there is no wholly satisfactory way of defining the true rate of return of a long-lived asset. The best available concept is the so called **discounted cash flow (DCF) rate of return** or **internal rate of return (IRR)**."(page 91) (emphasis in original).

[3] For more on the assumption of reinvestment of intermediate cash flows at the rate of IRR, see Alchian (1955), Robinson (1956), Solomon (1956), Renshaw (1957) , Dudley (1972) and Levy and Sarnat (1986). A very comprehensive discussion on the reinvestment rate assumption is given in Herbst (2002).

available in future to invest intermediate cash flows at the rate of IRR. This interpretation is therefore not very useful in enhancing our understanding of the observed preference of managers for IRR.

B. An Alternative and Less Popular Interpretation.

This interpretation says that the internal rate of return is the "return earned on funds that remain internally invested in the project" Clark et al. (1989) (page 106). This interpretation of IRR does not involve consideration of reinvestment decisions for intermediate cash flows. To quote Bierman and Smidt (1993):

“The internal rate of return of a conventional investment has an interesting interpretation.... It represents the highest rate of interest that an investor could afford to pay, without losing money, if all the funds to finance the investment were borrowed and the loan (principal and accrued interest) was repaid by application of the cash proceeds from the investment as they were earned.” (page 66)

It merits reiteration that this interpretation has the advantage of bypassing the knotty problem of reinvestment rate. This interpretation treats the internal rate of return as an amortization rate such that the cash flows would be just sufficient to amortize the initial investment over the life of the project.

C. A Numerical Illustration

The alternative interpretation has the attractive feature that it bypasses the thorny issue of the investment rate completely. In this section we present a numerical illustration of this interpretation.

Consider the following cash flow (amounts in $ Millions)

Year	**0**	**1**	**2**	**3**	**4**
Cash Flow	-5000	1250	1450	1650	1918

The IRR for this cash flow is 9%. The initial investment of $ 5000 Million can be amortized as in Table 1.

D. Comments

We have given a numerical illustration of the interpretation of IRR as akin to an amortization rate. However, before we can use this interpretation of IRR to explain the widespread observation of managers using IRR, we need to establish the generality of this interpretation. We need to be able to assert that the kind of decomposition shown in Table 1 will work for all possible cash flows and for all possible finite project durations - even for non simple ones where there may be changes in sign for intermediate cash flows.

In order to answer this question we need to establish the validity of the interpretation formally.[4] This is done in the next section.

Table 1: **Amortization of the Project Cash Flows Using IRR. (Amounts in $ Millions Rounded)**

Row Number	**Item**	**How Calculated**	**Year 1**	**Year 2**	**Year 3**	**Year 4**
1	Cash Flow	From the Cash Flow Data	1250	1450	1650	1918
2	Outstanding Investment at the beginning of the year	For Year 1, it is the initial investment. For other years it is from Row 5 of the previous year.	5000	4200	3128	1760
3	Interest Paid	Row 2 Multiplied by IRR, in this case 9%.	450	378	282	158
4	Principal Repaid (Amortized)	Row 1 minus Row 3	800	1072	1368	1760
5	Outstanding Investment at the end of the year	Row 2 minus Row 4	4200	3128	1760	0

IV. Interpreting IRR as a Rate of Return on Funds Remaining Internally Invested-A Generalized Result.

In this section we formally and rigorously prove that IRR is the return on funds remaining internally invested and that it is both a necessary and a sufficient condition. We set out the notation in this section and state the theorems. The formal proofs are given in the Appendix. We state the theorems for the case where the initial cash flow is negative. The proofs and the theorems can be trivially extended to the cases where the initial cash flow is positive. We also assume that the cash flows are such that the IRR exists.

Let $-I_0, c_1, c_{2,} \cdots\cdots, c_n, I_0 > 0$ be the cash flows associated with the project, the *j*th term of the series being the cash flow in the *j*th period. It is to be noted that $c_j,\ j \neq 0$ are unrestricted in sign. Each cash flow $c_j,\ j \neq 0$ can be notionally divided into the following two parts:

[4] The following perspicacious quote from Russell (1986) may be recalled in this context.

> "What I do wish to maintain - and it is here that the scientific attitude becomes imperative - is that insight, untested and unsupported, is an insufficient guarantee of truth, in spite of the fact that much of the most important truth is first suggested by its means." (page 30)

- A part of $c_j, j \neq 0$ goes toward providing one period return (say *i*) **on** the amount remaining internally invested in the project at the end of the previous period.

- The balance part of $c_j, j \neq 0$ goes toward repayment (in full or in part) **of** the amount remaining internally invested in the project at the end of the previous period.

Note that if a particular $c_j < 0, j \neq 0,$ then there will be negative repayment or in other words additional investment. Without loss of generality we assume that the cash flows take place at the end of each period. It is to be noted that this assumption in no way affects the validity of the formal proofs.

Let

i be the return per period on the funds remaining internally invested.

r be the discount rate, at which the cash flows are discounted to arrive at the net present value.

r^* be the internal rate of return.

R_j be the repayment of investment (either in whole or in part) in period $j, j \geq 1$.

n be the total number of periods in the cash flow stream, i.e. $c_j = 0 \forall j \geq n+1$.

The decomposition of the cash flows is shown in Table 2

Table 2: Decomposition of Cash Flows

Column 1	*Column 2*	*Column 3*	*Column 4*	*Column 5*
Period	**Funds Remaining Internally Invested at the Beginning of the Period**	**Return *On* Funds Remaining Internally Invested During the Period**	**Repayment *of* Investment During the Period**	**Total Cash Flow =** c_j **= Column 3 + Column 4**
1	I_0	iI_0	R_1	$c_1 = R_1 + iI_0$
2	$I_0 - R_1$	$i(I_0 - R_1)$	R_2	$c_2 = R_2 + i(I_0 - R_1)$
3	$I_0 - R_1 - R_2$	$i(I_0 - R_1 - R_2)$	R_3	$c_3 = R_3 + i(I_0 - R_1 - R_2)$
…	…	…	…	…
…	…	…	…	…
j	$I_0 - R_1 - R_2 - \cdots - R_{j-1}$	$i(I_0 - R_1 - R_2 - \cdots - R_{j-1})$	R_j	$c_j = R_j + i(I_0 - R_1 - R_2 - \cdots - R_{j-1})$
…	…	…	…	…
…	…	…	…	…
n	$I_0 - R_1 - R_2 - \cdots - R_{n-1}$	$i(I_0 - R_1 - R_2 - \cdots - R_{n-1})$	R_n	$c_n = R_n + i(I_0 - R_1 - R_2 - \cdots - R_{n-1})$

Since *i* is defined as the return on funds remaining internally invested and R_j is defined as the repayment of investment balance (in part or in full) in period *j*, therefore, it follows that, the total repayment of investment balances must equal the initial investment amount I_0. This means that we can write

	$\sum_{j=1}^{n} R_j = I_0$	(3)

Please note that this does not preclude any of the R_j's from being negative. We want to establish the following results

- If *i* is any return on the funds remaining internally invested for the given cash flow, then *i* is an IRR for this given cash flow.

- If r^* is any IRR for the given cash flow, then r^* is a return on the funds remaining internally invested for this given cash flow.

We state the above results in the form of formal theorems.

Theorem 1: *Let* $-I_0, c_1, c_2, \cdots\cdots, c_n, I_0 > 0, c_n \neq 0$ *and* $c_j = 0 \forall j \geq n+1$ *be the cash flows associated with the project, the jth term of the series being the cash flow in the jth period. Let i be the return on funds remaining internally invested,* $i \neq -1$, *such that* $c_j = R_j + i\left(I_0 - R_1 - R_2 - \cdots - R_{j-1}\right), j = 1, 2, 3, \cdots n.$ $R_j s$ *can be generated by the recursive relationship* $R_j = c_j - i\left(I_0 - R_1 - R_2 - \cdots - R_{j-1}\right)$. *Since i is by definition a return on the funds remaining internally invested, therefore* $R_j s$ *also satisfy the additional condition* $\sum_{j=1}^{n} R_j = I_0$. *Then i is an IRR for the cash flow so defined.*

Proof: See Appendix.

Theorem 2: *Let* $-I_0, c_1, c_2, \cdots\cdots, c_n, I_0 > 0, c_n \neq 0$ *and* $c_j = 0 \forall j \geq n+1$ *be the cash flows associated with the project, the jth term of the series being the cash flow in the jth period. Let the IRR* r^* *exist for this cash flow,* $r^* \neq -1$. *Let* $R_j s$ *can be generated by the recursive relationship* $R_j = c_j - r^*\left(I_0 - R_1 - R_2 - \cdots - R_{j-1}\right)$. *Then* $\sum_{j=1}^{n} R_j = I_0$ *i.e* r^* *is the return on the funds remaining internally invested.*

Proof: See Appendix.

Together these two theorems establish that IRR as the return on funds remaining internally invested is both a necessary and a sufficient condition.

V. Why Managers Use Internal Rate of Return.

Internal rate of return is derived without any reference to the conditions in the capital market. In order to derive the IRR we just need to have the cash flows. According to the IRR criterion of Capital Budgeting Decision, in deciding upon the acceptability of the project, managers will have to compare the internal rate of return against the cost of capital. *However, the cost of capital can change randomly depending on the capital market conditions. Indeed Chatrath and Seiler (1997) find that managers face a stochastic cost of capital.* A rational manager will therefore treat cost of capital as a random variable following some probability distribution and knowing the IRR of the project can determine the tail end probability of the project being financially untenable. We should emphasize here that we are not suggesting any particular form for the probability distribution. The only thing we are assuming here is that managers know or, at the least have some intuitive idea about the probability distribution for their cost of capital.

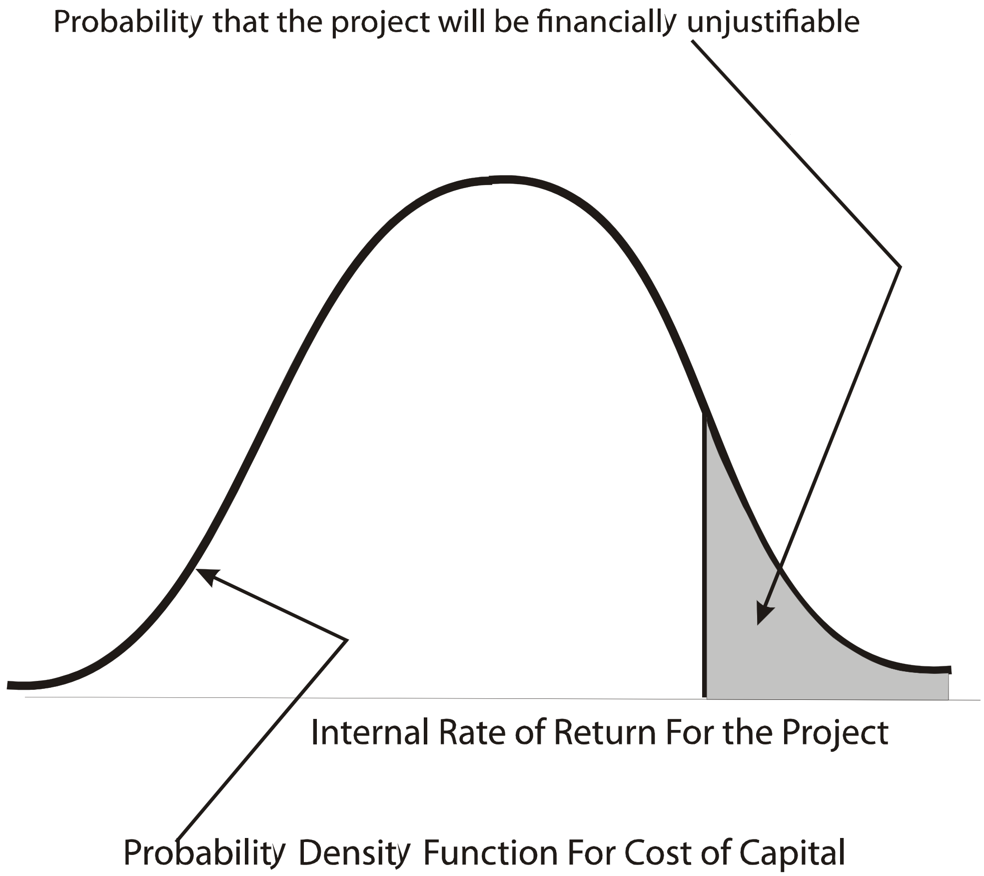

Figure 1: IRR as a Risk Measure

We can illustrate this by referring to Figure 1. Figure 1 shows the probability density function for the cost of capital. We reiterate that the probability distribution sought to be depicted in Figure 1 is a general probability distribution and is not necessarily a normal distribution. For a particular project the IRR can be determined with the knowledge of the expected cash flows alone. From the probability density function we can then determine the probability of the project being financially unviable. In other words IRR conveys the risk information pertinent to the project. Note that for this rendition to be valid IRR needs to be interpreted as the return earned on funds that remain internally invested-an interpretation which has been shown to be rigorously valid.

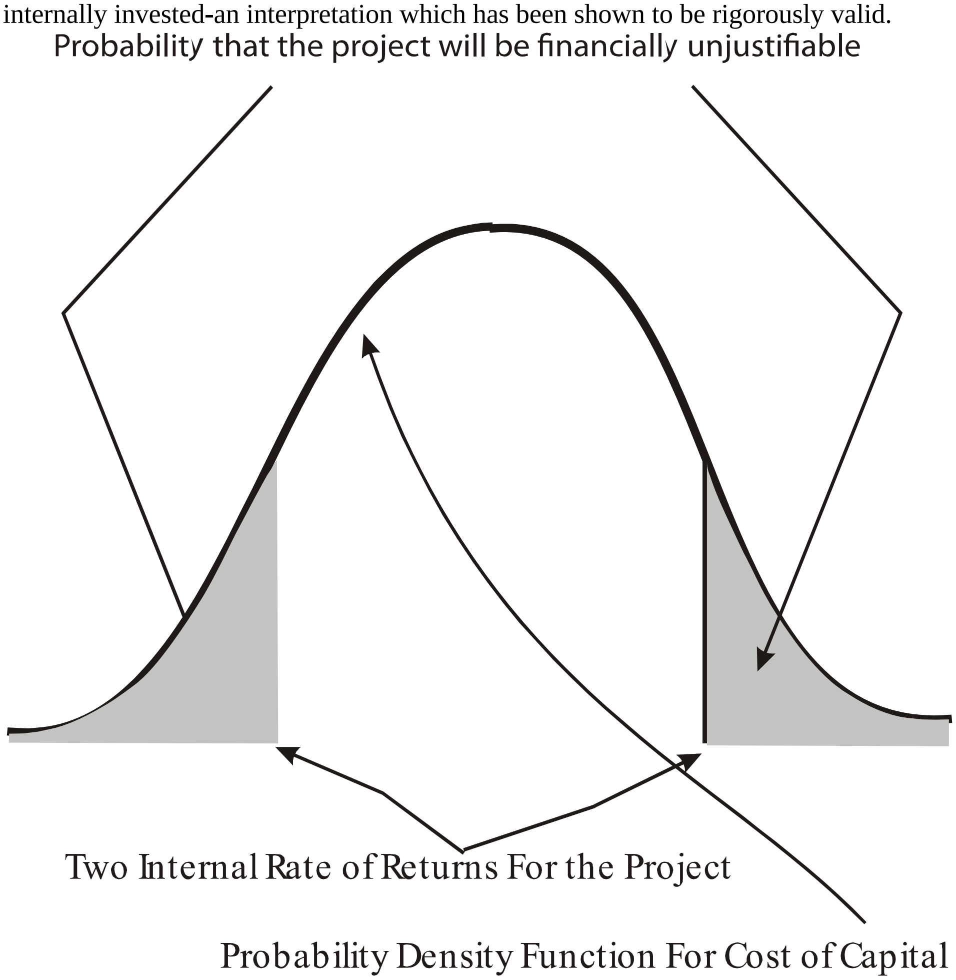

Figure 2: IRR as a Risk Measure-The Case for Double IRRs

One of the criticisms frequently advanced against the IRR method is that there could be cases when a particular set of cash flows could generate multiple IRRs.[5] Once

[5] The issue of multiple IRR started with the famous pump problem of Lorie and Savage (1955). In this problem there is a cash outflow when the company purchases a pump of higher capacity, pumps out additional oil which gives the company positive cash flows in the period of operation of the pump. In the terminal period there is a cash outflow because the company has pumped out the additional oil

managers know the probability distribution of the cost of capital, they can estimate the probability of the project causing financial embarrassment using the multiple IRRs. Figure 2 shows graphically how the probability of financial embarrassment can be estimated for a set of cash flows with two IRRs.

IV. Directions for Future Research

This paper has advanced a theoretical rationale to explain the puzzling observation that while the theories of finance recommend use of NPV in investment appraisal, in practice managers often use IRR in preference to NPV. A theory however needs to be validated by empirical tests. This will the subject of future research.

There are two ways to test the rationale posited here. One is the survey method where we can design a questionnaire to test the hypothesis that managers use IRR as a rough measure of the riskiness of the project. The second method is to use the approaches of experimental finance. We can set up an experiment where we present subjects with choices of projects and then observe their choices. The choices will have to be so designed that once we observe the choice we can infer whether they are using IRR as a rough measure of riskiness or not. Designing such surveys and such experiments are full-fledged research projects by themselves and will be subjects of future research.

V. Summary and Conclusion

In this paper we have tried to understand why managers use IRR in preference to NPV much to the chagrin of finance academics. For example Brealey et al. (2006) lament: “... for many companies DCF means IRR, not NPV.” (page 99)

We have provided a rigorous justification for the interpretation of IRR as the rate of return on funds that remain internally invested in the project. Based on this interpretation it can be argued that IRR may be used by managers as a risk assessment tool.[6] Future research efforts would involve understanding this aspect in more depth either through survey methodology or through experimental finance.

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and exhausted the supply and thus it no longer has the incremental cash flow. The cash flow of the pump problem is characterized by a negative cash flow in the initial period , followed by periods of positive cash flows and then followed by periods of negative cash flows. This kind of situation can also arise in mining where after extracting the minerals the company might be required to fill the mine with sand in order to prevent cave ins. A series of negative cash flows towards the end of project life might also occur when companies might be required to rectify environmental degradation. In all such cases the pattern of cash flows will be an initial negative cash flow, followed by a set of positive cash flows, followed by a set of negative cash flows. Using Descartes' Rule of signs [Herbst (2002)] we can see that in such cases there will be at best two IRRs.

[6] We may note here that payback period is another popular technique of capital budgeting which finds ever lesser approval among academics than IRR. Weingartner (1969) suggests that payback period may be a measure of the rate of resolution of uncertainty. Readers may find it interesting to look up Booth (1996) for some relations betweenpayback period and IRR.

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Appendix: Proof of Theorems

Theorem 1: *Let* $-I_0, c_1, c_2, \cdots\cdots, c_n, I_0 > 0, c_n \neq 0$ *and* $c_j = 0 \forall j \geq n+1$ *be the cash flows associated with the project, the jth term of the series being the cash flow in the jth period. Let i be the return on funds remaining internally invested,* $i \neq -1$, *such that* $c_j = R_j + i\left(I_0 - R_1 - R_2 - \cdots - R_{j-1}\right), j = 1, 2, 3, \cdots n.$ $R_j s$ *can be generated by the recursive relationship* $R_j = c_j - i\left(I_0 - R_1 - R_2 - \cdots - R_{j-1}\right)$. *Since i is by definition a return on the funds remaining internally invested, therefore* $R_j s$ *also satisfy the additional condition* $\sum_{j=1}^{n} R_j = I_0$. *Then i is an IRR for the cash flow so defined.*

Proof:

Case 1: *i=0*.

In this case $c_j = R_j \forall n \geq j \geq 1$.
From the given condition

$$\sum_{j=1}^{n} R_j = I_0$$

$$\Rightarrow \quad -I_0 + \sum_{j=1}^{n} R_j = 0$$

$\therefore i = 0$ is an Internal Rate of Return (IRR).

Case 2: $i \neq 0, i \neq -1$.

We know from the definition of $R_j s$.

$$\begin{aligned}
R_1 &= c_1 - iI_0 \\
R_2 &= c_2 - i\left(I_0 - R_1\right) \\
&= c_2 - i\left(I_0 - c_1 + iI_0\right) \\
&= c_2 + ic_1 - i\left(1+i\right)I_0 \\
R_2 &= c_3 - i\left(I_0 - R_1 - R_2\right) \\
&= c_3 - i\left(I_0 - c_1 + iI_0 - c_2 - ic_1 + i\left(1+i\right)I\right) \\
&= c_3 + ic_2 + i\left(1+i\right)c_1 - i\left(1+i\right)^2 I_0
\end{aligned}$$

Suppose we assume that

$$R_j = c_j + ic_{j-1} + i(1+i)c_{j-2} + i(1+i)^2 c_{j-3} + \cdots + i(1+i)^{j-2} c_1 - i(1+i)^{j-1} I_0$$
$$j = 1, 2, 3\cdots, k, i \neq 0, i \neq -1$$

i.e. $R_\tau = c_\tau + i\sum_{j=1}^{\tau-1}(1+i)^\tau (1+i)^{-(j+1)} c_j - i(1+i)^{(\tau-1)} I_0 \ \ \forall k \geq \tau \geq 1.$

$$\therefore \quad \sum_{\tau=1}^{k} R_\tau = \sum_{\tau=1}^{k} c_\tau + i\sum_{\tau=1}^{k}\sum_{j=1}^{\tau-1}(1+i)^\tau (1+i)^{-(j+1)} c_j - i\sum_{\tau=1}^{k}(1+i)^{(\tau-1)} I_0$$

$$\Rightarrow \quad \sum_{\tau=1}^{k} R_\tau = \sum_{\tau=1}^{k} c_\tau + i\sum_{j=1}^{k-1}\sum_{\tau=j+1}^{k}(1+i)^\tau (1+i)^{-(j+1)} c_j - I_0\left[(1+i)^k - 1\right]$$

$$\Rightarrow \quad \sum_{\tau=1}^{k} R_\tau = \sum_{\tau=1}^{k} c_\tau + \sum_{j=1}^{k-1}\left[(1+i)^{k-j} - 1\right]c_j - I_0\left[(1+i)^k - 1\right]$$

$$\Rightarrow \quad \sum_{\tau=1}^{k} R_\tau = c_k + \sum_{j=1}^{k-1}(1+i)^{k-j} c_j - I_0\left[(1+i)^k - 1\right]$$

$$\Rightarrow \quad R_{k+1} = c_{k+1} + ic_k + i\sum_{j=1}^{k-1}(1+i)^{k-j} c_j - i(1+i)^k I_0 \ \because R_{k+1} = c_{k+1} - i\left[I_0 - \sum_{\tau=1}^{k} R_\tau\right]$$

$\therefore$By Mathematical Induction,

$$R_j = c_j + ic_{j-1} + i(1+i)c_{j-2} + i(1+i)^2 c_{j-3} + \cdots + i(1+i)^{j-2} c_1 - i(1+i)^{j-1} I_0$$
$$\forall j = 1, 2, 3\cdots, n, i \neq 0, i \neq -1$$

Now $R_j = c_j + i\sum_{\tau=1}^{j-1}(1+i)^{-\tau}(1+i)^{j-1} c_\tau - i(1+i)^{(j-1)} I_0 \ \ \forall n \geq j \geq 1.$

$$\therefore \quad \sum_{j=1}^{n} R_j = \sum_{j=1}^{n} c_j + i\sum_{j=1}^{n}\sum_{\tau=1}^{j-1}(1+i)^{-\tau}(1+i)^{j-1} c_\tau - i\sum_{j=1}^{n}(1+i)^{(j-1)} I_0$$

$$\Rightarrow \quad \sum_{j=1}^{n} R_j = \sum_{j=1}^{n} c_j + i\sum_{\tau=1}^{n-1}\sum_{j=\tau+1}^{n}(1+i)^{-\tau}(1+i)^{j-1} c_\tau - I_0\left[(1+i)^n - 1\right]$$

$$\Rightarrow \quad \sum_{j=1}^{n} R_j = \sum_{j=1}^{n} c_j + \sum_{\tau=1}^{n-1}\left[(1+i)^{n-\tau} - 1\right]c_\tau - I_0\left[(1+i)^n - 1\right]$$

$$\Rightarrow \quad \sum_{j=1}^{n} R_j = c_n + \sum_{j=1}^{n-1}(1+i)^{n-j} c_j - I_0\left[(1+i)^n - 1\right]$$

$$\Rightarrow \quad -(1+i)^n I_0 + \sum_{j=1}^{n-1}(1+i)^{n-j} c_j + c_n = 0 \because \sum_{j=1}^{n} R_j = I_0$$

$$\Rightarrow \quad -I_0 + \sum_{j=1}^{n}\frac{c_j}{(1+i)^j} = 0$$

$\therefore$ i is an IRR for the cash flow.

Theorem 2: *Let* $-I_0, c_1, c_2, \cdots\cdots, c_n, I_0 > 0, c_n \neq 0$ *and* $c_j = 0 \forall j \geq n+1$ *be the cash flows associated with the project, the jth term of the series being the cash flow in the jth period. Let the IRR* r^* *exist for this cash flow,* $r^* \neq -1$. *Let* $R_j s$ *can be generated by the recursive relationship* $R_j = c_j - r^*\left(I_0 - R_1 - R_2 - \cdots - R_{j-1}\right)$. *Then* $\sum_{j=1}^{n} R_j = I_0$ *i.e* r^* *is the return on the funds remaining internally invested.*

Proof:

Since r^* is the IRR for the given cash flow,

	$\therefore -I_0 + \sum_{j=1}^{n} \frac{c_j}{\left(1+r^*\right)^j} = 0$	(4)

Case 1: $r^* = 0$

From Equation (4) we get

	$\sum_{j=1}^{n} c_j = I_0$	(5)

From the definition of R_j we get,

	$\sum_{j=1}^{n} R_j = \sum_{j=1}^{n} c_j$	(6)

Equations (5) and (6) show that $\sum_{j=1}^{n} R_j = I_0$.

Case 2: $r^* \neq 0, r^* \neq -1$.

Put $k = \frac{1}{1+r^*}, \therefore k \neq 0, k \neq 1$.

From Equation (4) we get,

	$\therefore -I_0 + \sum_{j=1}^{n} k^j c_j = 0$	(7)

Also from the definition of R_j

	$c_j = R_j + r^*\left[I_0 - \sum_{u=1}^{j-1} R_u\right] = R_j + \frac{1-k}{k}\left[I_0 - \sum_{u=1}^{j-1} R_u\right]$	(8)

From Equations (7) and (8), we get

$$-I_0 + \sum_{j=1}^{n} k^j \left[R_j + \frac{1-k}{k} \left[I_0 - \sum_{u=1}^{j-1} R_u \right] \right] = 0$$

$$\Rightarrow\ -I_0 + \sum_{j=1}^{n} k^j R_j + \frac{1-k}{k} I_0 \sum_{j=1}^{n} k^j - \frac{1-k}{k} \sum_{j=1}^{n} k^j \sum_{u=1}^{j-1} R_u = 0$$

$$\Rightarrow\ -I_0 + \sum_{j=1}^{n} k^j R_j + \frac{1-k}{k} \frac{k\left(k^n - 1\right)}{k-1} I_0 - \frac{1-k}{k} \sum_{u=1}^{n-1} R_u \sum_{j=u+1}^{n} k^j = 0$$

$$\Rightarrow\ -I_0 + \sum_{j=1}^{n} k^j R_j - \left(k^n - 1\right) I_0 + \frac{k-1}{k} \sum_{u=1}^{n-1} R_u \frac{k^{u+1}\left[k^{n-u} - 1\right]}{k-1} = 0$$

$$\Rightarrow\ \sum_{j=1}^{n} k^j R_j - k^n I_0 + \sum_{j=1}^{n-1} k^n R_j - \sum_{j=1}^{n-1} k^j R_j = 0$$

$$\Rightarrow\ k^n R_n - k^n I_0 + k^n \sum_{j=1}^{n-1} R_j = 0$$

$$\Rightarrow\ k^n \left[\sum_{j=1}^{n} R_j - I_0 \right] = 0$$

$$\Rightarrow\ \sum_{j=1}^{n} R_j = I_0 \because k \neq 0.$$