An Option Valuation Analysis of the Value of Tax Shields for Risky Debt

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Abstract

Several different models for the appropriate discount rate for the tax shields and the present value of the tax shields resulting from the use of debt have appeared in the finance literature. This research uses an option valuation framework to develop models for the appropriate discount rate for the risky tax shields, the present value of the tax shields, and the probability of default resulting from the use of debt. Relationships between cash flows and their present values for firms with constant, perpetual growth are developed. The reinvestment required to support the constant, perpetual growth is incorporated in the expressions for the relationships between cash flows and present values. This analysis using option valuation theory and relationships between cash flows and their present values provides a framework for unifying the major apparently dissimilar results for the appropriate discount rate for the tax shields and the present value of the tax shields that have appeared in the finance literature.

I. Introduction

Different models for the appropriate discount rate for the tax shields and the present value of the tax shields resulting from the use of debt have appeared in the finance literature. This research uses an option valuation framework to develop models for the appropriate discount rate for the tax shields, the present value of the tax shields, and the probability of default resulting from the use of debt. Analysis using option valuation theory provides a framework that unifies the major apparently dissimilar results that have appeared in the finance literature. Section 2 of the paper presents the major models for the appropriate discount rate for the tax shields and the present value of the tax shields resulting from the use of debt that have appeared in the finance literature. Models showing the relationships between cash flows and present values are developed in Section 3. Section 4 applies option valuation analysis to securities that are claims or options on the value of the firm. In this formulation, the underlying asset for the contingent claim is the value of the firm's operating assets. Debt and equity are valued as contingent claims on the value of the firm's operating assets. In Section 4, the option valuation model is developed when there are no taxes. Section 5 extends the models of Section 4 to a business environment with taxes. This section provides a framework for unifying the major results for the appropriate discount rate for the tax shields and the present value of the tax shields that have appeared in the finance literature. A numerical illustration of the relationships between the relative amount of debt and the probability of default, the beta coefficient for risky debt, the appropriate discount rate for risky debt and risky tax shields, and the present value of the tax shields is presented in Section 6 and the accompanying tables and figures. Section 7 provides a summary of the results of this study and conclusions with respect to the major models for the appropriate discount rate for the tax shields and the present value of the tax shields resulting from the use of debt.

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II. Previous Models for the Appropriate Discount Rate and the Present Value for the Tax Shields Resulting from the Use of Debt

Some of the different models for the present value of the tax shields ($V_{TS,0}$) resulting from the use of debt that have appeared in the finance literature are presented in this section of the paper. For companies with constant perpetual cash flows, the discounted value of the tax shields is given by (Modigliani and Miller 1963)

$$V_{TS,0} = \frac{D_0 \cdot T_C \cdot R_F}{R_F} = D_0 \cdot T_C \tag{1}$$

where D_0 is the present value of debt, T_C is the corporate income tax rate, and R_F is the risk-free rate.

Technically, $V_{TS,0}$ is not a cash flow's present value. It is the difference between the present value of the tax stream for the company without debt which should be discounted at the unlevered cost of equity, R_U, and the present value of the tax stream for the company with debt which should be discounted at the levered cost of equity, R_L. The difference between the present value of the tax stream for the company without debt and the present value of the tax stream for the company with debt provides the following expression for the present value of the tax shields when there is constant, perpetual growth.

$$V_{TS,0} = \frac{TAX_{U,1}}{R_U - g} - \frac{TAX_{L,1}}{R_L - g} \tag{2}$$

where $TAX_{U,1}$ is the tax for the unlevered firm for time 1, $TAX_{L,1}$ is the tax for the levered firm for time 1, and g is the constant growth rate. For companies with constant perpetual growth, even though the present value of the tax shields is the difference between the present values of two different tax streams, its value is given by (Lewellen and Emery 1986; Fernandez 2002 and 2004)

$$V_{TS,0} = \frac{D_0 \cdot T_C \cdot R_U}{R_U - g}. \tag{3}$$

In this formulation, this result does not mean that the unlevered cost of equity is the appropriate discount rate for the tax shields. The cash flow stream being discounted here is larger than the stream of tax savings resulting from the interest expenses.

Another position holds that since the difference in the taxes is caused by the use of debt financing, the risk of the tax saving from the use of debt is the same as the risk of debt and the discounted value of the tax shields due to interest payments should be calculated using (Inselbag and Kaufold 1997; Luehrman 1997; Myers 1974; Ruback 2002)

$$V_{TS,0} = \frac{D_0 \cdot T_C \cdot R_D}{R_D - g} \tag{4}$$

where R_D is the risk-adjusted required rate of return for debt.

A different point-of-view is that the risk of the tax savings from the use of debt is the same as the risk of the taxes of the unlevered firm and the discounted value of the tax shield due to interest payments should be calculated using (Harris and Pringle 1985; Ruback 1995; Taggart 1991)

$$V_{TS,0} = \frac{D_0 \cdot T_C \cdot R_D}{R_U - g}. \tag{5}$$

A variant of this model that treats the tax savings from the use of debt as having the same risk as the taxes of the unlevered firm is (Miles and Ezzell 1980)

$$V_{TS,0} = \left(\frac{D_0 \cdot T_C \cdot R_D}{R_U - g}\right) \cdot \left(\frac{1 + R_U}{1 + R_D}\right). \tag{6}$$

In these models, there is disagreement about whether the appropriate discount rate for the tax shields is the risk-free rate, the risk-adjusted discount rate for debt, or the discount rate for unlevered equity. There is also disagreement about how the numerator of the model for the present value of the tax shields should be formulated.

III. Relationships between Cash Flows and Present Values

This section of the paper examines relationships between cash flows and their present values for firms with constant, perpetual growth. Growth requires reinvestment. The reinvestment required to support the constant, perpetual growth is incorporated in the expressions developed for the relationships between cash flows and present values. The relationships for free cash flows, required capital reinvestment, the present value of the unlevered firm, and the present value of the taxes for the unlevered firm are presented. The relationships for debt cash flows, required debt reinvestment, the present value of the debt, and the present value of the interest also are developed. In addition, the relationships for equity cash flows, required equity reinvestment, the present value of equity, and the present value of the taxes for the levered firm are provided. These relationships are used later in the paper to develop models for the present value of the tax shields resulting from using debt.

A. Free Cash Flows and Their Present Values

For growing perpetuities, the free cash flows for the firm are

$$FCF_{t+1} = (1 - T_C) \cdot EBIT_{t+1} - g \cdot BVC_t \tag{7}$$

where FCF_{t+1} is the free cash flow for time t + 1, T_C is the corporate income tax rate, $EBIT_{t+1}$ is the earnings before interest and taxes for time t + 1, g is the constant perpetual growth rate, and BVC_t is the book value of capital at time t. The constant return on capital before taxes is defined as

$$ROC = \frac{EBIT_{t+1}}{BVC_t}. \tag{8}$$

The constant reinvestment rate for capital is defined as

$$b_U = \frac{g \cdot BVC_t}{EBIT_{t+1}} \tag{9}$$

which implies that

$$g \cdot BVC_t = b_U \cdot EBIT_{t+1} \tag{10}$$

and

$$FCF_{t+1} = (1 - T_C) \cdot EBIT_{t+1} - b_U \cdot EBIT_{t+1}. \tag{11}$$

b_U denotes the reinvestment rate for the unlevered firm. The growth rate is related to the reinvestment rate for capital and the return on capital by

$$g = b_U \cdot ROC = \frac{g \cdot BVC_t}{EBIT_{t+1}} \cdot \frac{EBIT_{t+1}}{BVC_t} \tag{12}$$

and the reinvestment rate for capital is related to the growth rate and the return on capital by

$$b_U = \frac{g}{ROC} \tag{13}$$

which shows that the free cash flow is

$$FCF_{t+1} = (1 - T_C) \cdot EBIT_{t+1} - \left(\frac{g}{ROC}\right) \cdot EBIT_{t+1}. \tag{14}$$

The present value of the free cash flows at time t is

$$V_{FCF,t} = \frac{(1 - T_C) \cdot EBIT_{t+1} - \left(\frac{g}{ROC}\right) \cdot EBIT_{t+1}}{R_U - g} = \left[(1 - T_C) - \left(\frac{g}{ROC}\right)\right] \cdot V_{EBIT,t} \tag{15}$$

where the present value of the earnings before interest and taxes is given by

$$V_{EBIT,t} = \frac{EBIT_{t+1}}{R_U - g}. \tag{16}$$

The taxes for the unlevered firm for time t + 1 are

$$TAX_{U,t+1} = T_C \cdot EBIT_{t+1} \tag{17}$$

and their present value at time t is

$$V_{TAX_{U,t}} = \frac{T_C \cdot EBIT_{t+1}}{R_U - g}. \tag{18}$$

B. Debt Cash Flows and Their Present Values

For growing perpetuities, the debt cash flows for the debt holders are

$$DCF_{t+1} = INT_{t+1} - g \cdot BVD_t \tag{19}$$

where DCF_{t+1} is the debt cash flow for time t + 1, INT_{t+1} is the interest for time t + 1, g is the constant perpetual growth rate, and BVD_t is the book value of debt at time t. The constant return on debt is defined as

$$ROD = \frac{INT_{t+1}}{BVD_t}. \tag{20}$$

The constant reinvestment rate for debt is defined as

$$b_D = \frac{g \cdot BVD_t}{INT_{t+1}} \tag{21}$$

where b_D denotes the reinvestment rate for the debt holders. The growth rate is related to the reinvestment rate for debt and the return on debt by

$$g = b_D \cdot ROD = \left(\frac{g \cdot BVD_t}{INT_{t+1}}\right) \cdot \left(\frac{INT_{t+1}}{BVD_t}\right) = \frac{g \cdot BVD_t}{BVD_t} \tag{22}$$

which implies that

$$g \cdot BVD_t = b_D \cdot INT_{t+1} \tag{23}$$

and

$$DCF_{t+1} = INT_{t+1} - b_D \cdot INT_{t+1} = (1 - b_D) \cdot INT_{t+1}. \tag{24}$$

Since the reinvestment rate for debt is related to the growth rate and the return on debt by

$$b_D = \frac{g}{ROD}, \tag{25}$$

$$DCF_{t+1} = \left[1 - \left(\frac{g}{ROD}\right)\right] \cdot INT_{t+1}. \tag{26}$$

The present value of the debt cash flows at time t is

$$V_{DCF,t} = \frac{DCF_{t+1}}{R_D - g} = \frac{\left[1-\left(\frac{g}{ROD}\right)\right] \cdot INT_{t+1}}{R_D - g} = \left[1-\left(\frac{g}{ROD}\right)\right] \cdot \left(\frac{INT_{t+1}}{R_D - g}\right) = \left[1-\left(\frac{g}{ROD}\right)\right] \cdot V_{INT,t} \tag{27}$$

where

$$V_{INT,t} = \frac{INT_{t+1}}{R_D - g} \tag{28}$$

is the present value of the interest. When ROD = R_D,

$$V_{DCF,t} = \left(\frac{ROD - g}{ROD}\right) \cdot \left(\frac{INT_{t+1}}{R_D - g}\right) = \frac{INT_{t+1}}{R_D} \tag{29}$$

and $BVD_t = V_{DCF,t}$. $V_{DCF,t}$ is the present value of the firm's debt at time t.

C. Equity Cash Flows and Their Present Values

The equity cash flows for growing perpetuities for the equity holders are

$$ECF_{t+1} = (1 - T_C) \cdot (EBIT_{t+1} - INT_{t+1}) - g \cdot BVE_t \tag{30}$$

where ECF_{t+1} is the equity cash flow for time t + 1, T_C is the corporate income tax rate, $EBIT_{t+1}$ is the earnings before interest and taxes for time t + 1, INT_{t+1} is the interest for time t + 1, g is the constant perpetual growth rate, and BVE_t is the book value of equity at time t. Since

$$BVE_t = BVC_t - BVD_t, \tag{31}$$

$$ECF_{t+1} = (1 - T_C) \cdot (EBIT_{t+1} - INT_{t+1}) - g \cdot (BVC_t - BVD_t). \tag{32}$$

Since

$$g \cdot BVC_t = b_U \cdot EBIT_{t+1} \tag{33}$$

and

$$g \cdot BVD_t = b_D \cdot INT_{t+1}, \tag{34}$$

$$ECF_{t+1} = (1-T_C)\cdot(EBIT_{t+1} - INT_{t+1}) - (b_U \cdot EBIT_{t+1} - b_D \cdot INT_{t+1}). \tag{35}$$

Since the reinvestment rate for capital is related to the growth rate and the return on capital by

$$b_U = \frac{g}{ROC} \tag{36}$$

and the reinvestment rate for debt is related to the growth rate and the return on debt by

$$b_D = \frac{g}{ROD}, \tag{37}$$

the equity cash flow is given by

$$ECF_{t+1} = \left[(1-T_C) - \left(\frac{g}{ROC}\right)\right]\cdot EBIT_{t+1} - \left[(1-T_C) - \left(\frac{g}{ROD}\right)\right]\cdot INT_{t+1}. \tag{38}$$

The present value of the equity cash flows at time t is

$$V_{ECF,t} = \frac{ECF_{t+1}}{R_L - g} = \frac{\left[(1-T_C) - \left(\frac{g}{ROC}\right)\right]\cdot EBIT_{t+1} - \left[(1-T_C) - \left(\frac{g}{ROD}\right)\right]\cdot INT_{t+1}}{R_L - g} \tag{39}$$

which also equals

$$V_{ECF,t} = \left[(1-T_C) - \left(\frac{g}{ROC}\right)\right]\cdot\left(\frac{EBIT_{t+1}}{R_U - g}\right) - \left[(1-T_C) - \left(\frac{g}{ROD}\right)\right]\cdot\left(\frac{INT_{t+1}}{R_D - g}\right). \tag{40}$$

The present value of the growing perpetuity of earnings before interest and taxes and the present value of the growing perpetuity of interest payments, the income tax rate, the reinvestment rate for capital [$b_U = (g / ROC)$], and the reinvestment rate for debt [$b_D = (g / ROD)$]determine the value of the equity for the levered firm.

$$V_{ECF,t} = \left[(1-T_C) - \left(\frac{g}{ROC}\right)\right]\cdot V_{EBIT,t} - \left[(1-T_C) - \left(\frac{g}{ROD}\right)\right]\cdot V_{INT,t} \tag{41}$$

The taxes for the levered firm for time t + 1 are

$$TAX_{L,t+1} = T_C \cdot (EBIT_{t+1} - INT_{t+1}) \tag{42}$$

and their present value at time t is

$$V_{TAX_L,t} = \frac{TAX_{L,t+1}}{R_L - g} = \frac{T_C \cdot (EBIT_{t+1} - INT_{t+1})}{R_L - g} = T_C \cdot (V_{EBIT,t} - V_{INT,t}) . \tag{43}$$

The present value of the tax shields is the difference between the present values of two different cash flows each with its own risk, the taxes paid by the unlevered firm and the taxes paid by the levered firm. Using relationships between cash flows and present values shows that the present value of the tax shields can be calculated as the present value of the tax savings from the interest.

$$V_{TAX_U,t} - V_{TAX_L,t} = T_C \cdot V_{EBIT,t} - T_C \cdot (V_{EBIT,t} - V_{INT,t}) = T_C \cdot V_{INT,t} = \frac{T_C \cdot INT_{t+1}}{R_D - g} = V_{TS,t} . \tag{44}$$

IV. An Option Valuation Framework for the Firm

Option valuation analysis can be applied to securities that are claims or options on the value of the firm (Merton 1974). The underlying asset for the contingent claim is the value of the firm's operating assets. Debt and equity are valued as contingent claims on the value of the firm's operating assets. Suppose the firm has only two types of claims on the value of the firm: equity and zero-coupon debt which prohibits payment of dividends until after the face value of the debt is paid at maturity. The zero-coupon debt will be rolled over perpetually (paid off and reissued) creating an infinite stream of interest expenses for the firm and interest income for the debt holders. Assume that there are no taxes. Let the value of the underlying asset at time t for the option pricing model be denoted by V_t, the value of the equity at time t for the option pricing model be denoted by E_t., and the value of the debt at time t for the option pricing model be denoted by D_t. Conceptually, V_t in the option pricing model equals $V_{EBIT,t}$ in the cash flow model for growing perpetuities, E_t in the option pricing model equals $V_{EBIT,t}$ minus $V_{INT,t}$ in the cash flow model for growing perpetuities, and D_t equals $V_{INT,t}$ in the cash flow model for growing perpetuities. Let the face value of the debt for the option pricing model be denoted by B and the time until maturity for the debt be denoted by T. If the value of the firm's operating assets ($V_T = V_{EBIT,T}$) exceeds or equals the face value of the debt when the debt matures ($V_T \geq B$), the debt holders will receive the face value of the debt (B) and the equity holders will receive the residual ($V_T - B$). If the value of the firm's operating assets is less than the face value of the debt when the debt matures ($V_T < B$), the equity holders will exercise their limited liability option and default on the promised debt payment and surrender the firm to the debt holders. In this analysis, default on the firm's debt and surrender of the firm to the debt holders does not create any costs paid to third parties. The debt holders will receive the value of the firm's operating assets (V_T) and the equity holders will receive nothing. The table shows the payoffs when the debt matures for this case.

	Default (Out of the Money)	No Default (At or In the Money)
V_T	$< B$	$\geq B$
D_T	V_T	B
E_T	0	$V_T - B$
$D_T + E_T$	V_T	V_T

A. Value of the Equity

When the debt matures, the equity will be worth the greater of $V_T - B$ or zero:

$$E_T = \max(V_T - B, 0). \tag{45}$$

The equity for this type of levered firm is a European call option on the value of the firm's operating assets (V_T). The exercise price is equal to the face value of the zero-coupon debt (B). The time until expiration is equal to the time to maturity for the debt (T). The present value of the equity is given by the Black-Scholes solution for a European call option (Black and Scholes 1973; Merton 1973).

$$E_0 = V_0 \cdot N(d_1) - B \cdot e^{-r \cdot T} \cdot N(d_2), \tag{46}$$

$$d_1 = \frac{\ln\left(\frac{V_0}{B}\right) + \left[r + \left(\frac{\sigma^2}{2}\right)\right] \cdot T}{\sigma \cdot \sqrt{T}}, \tag{47}$$

and

$$d_2 = d_1 - \sigma \cdot \sqrt{T}. \tag{48}$$

N(d) is the cumulative normal probability distribution and r (equals R_F) is the risk-free rate of return. The stockholders have given ownership of the firm to the debt holders until the debt matures while keeping a call option to buy the firm back by paying the face value of the debt when the debt matures. V_0 for this option pricing model equals the value of $V_{EBIT,0}$ and E_0 for this option pricing model gives the value of $V_{EBIT,0}$ minus $V_{INT,0}$.

B. Value of the Risky Debt

When the debt matures, the debt holders will receive the smaller of the value of the firm's operating assets or the face value of the debt.

$$D_T = \min(V_T, B). \tag{49}$$

The value of the firm's debt is also contingent on the value of the firm's operating assets. The present value of the risky debt is given by the Black-Scholes solution for the equity and the identity requiring that $V_0 = D_0 + E_0$ in this case.

$$D_0 = V_0 - E_0 = V_0 - V_0 \cdot N(d_1) + B \cdot e^{-r \cdot T} \cdot N(d_2) = V_0 \cdot [1 - N(d_1)] + B \cdot e^{-r \cdot T} \cdot N(d_2). \tag{50}$$

D_0 for this option pricing model gives the value of $V_{INT,0}$.

C. Value of the Firm with Risky Debt

The present value of the firm with risky debt, VF_0, is equal to the sum of the present value of the equity plus the present value of the debt.

$$VF_0 = D_0 + E_0 \tag{51}$$

$$VF_0 = V_0\,[1 - N(d_1)] + B\,e^{-rT}\,N(d_2) + V_0\,N(d_1) - B\,e^{-rT}\,N(d_2) = V_0\,. \tag{52}$$

In this situation, the use of debt affects the value of the equity and the value of the debt, but does not affect the value of the firm.

V. Valuation with Corporate Taxes

The valuation expressions from the cash flow model for growing perpetuities and the values for E_0 and D_0 produced by the option pricing model along with the requirement that $V_0 = V_{EBIT,0}$ are used to determine the value of the firm's equity when there are corporate income taxes, the value of the firm's debt when there are corporate income taxes, the value of the firm when there are corporate income taxes, and the value of the firm's tax shields when there are corporate income taxes.

A. Value of the Equity when there are Corporate Income Taxes

When there are corporate income taxes, the value of the equity is equal to the present value of the equity cash flows and is given by

$$V_{ECF,0} = \left[(1-T_C) - \left(\frac{g}{ROC}\right)\right] \cdot V_{EBIT,0} - \left[(1-T_C) - \left(\frac{g}{ROD}\right)\right] \cdot V_{INT,0}\,. \tag{53}$$

The value of $V_{INT,0}$ is given by D_0 in the Black-Scholes option pricing model indicating that

$$V_{ECF,0} = \left[(1-T_C) - \left(\frac{g}{ROC}\right)\right] \cdot V_0 - \left[(1-T_C) - \left(\frac{g}{ROD}\right)\right] \cdot D_0\,. \tag{54}$$

Substituting the Black-Scholes solution for D_0 into the expression for the value of the equity, $V_{ECF,0}$, gives

$$V_{ECF,0} = \left[(1-T_C) - \left(\frac{g}{ROC}\right)\right] \cdot V_0 - \left[(1-T_C) - \left(\frac{g}{ROD}\right)\right] \cdot \{V_0 \cdot [1 - N(d_1)] + B \cdot e^{-r \cdot T} \cdot N(d_2)\}\,. \tag{55}$$

B. Value of the Debt when there are Corporate Income Taxes

The value of the firm's debt with corporate income taxes is also contingent on the value of the firm. When there are corporate income taxes, the value of the debt is equal to the present value of the debt cash flows and is given by

$$V_{DCF,0} = \left[1 - \left(\frac{g}{ROD}\right)\right] \cdot V_{INT,0} \cdot \tag{56}$$

The value of $V_{INT,0}$ is given by D_0 in the Black-Scholes option pricing model. Substituting the Black-Scholes solution for D_0 into the expression for the value of the debt, $V_{DCF,0}$, gives

$$V_{DCF,0} = \left[1 - \left(\frac{g}{ROD}\right)\right] \cdot \{V_0 \cdot [1 - N(d_1)] + B \cdot e^{-r \cdot T} \cdot N(d_2)\} \tag{57}$$

for the value of risky debt when there are corporate income taxes.

C. Value of the Firm when there are Taxes

The present value of the levered firm when there are taxes is equal to the present value of the equity cash flows plus the present value of the debt cash flows so that

$$V_{ECF+DCF,0} = \left[(1 - T_C) - \left(\frac{g}{ROC}\right)\right] \cdot V_{EBIT,0} - \left[(1 - T_C) - \left(\frac{g}{ROD}\right)\right] \cdot V_{INT,0} + \left[1 - \left(\frac{g}{ROD}\right)\right] \cdot V_{INT,0} \tag{58}$$

which indicates that

$$V_{ECF+DCF,0} = \left[(1 - T_C) - \left(\frac{g}{ROC}\right)\right] \cdot V_{EBIT,0} + T_C \cdot V_{INT,0} \cdot \tag{59}$$

Using the Black-Scholes option pricing model solution for the value of $V_{INT,0}$ and V_0 for the value of $V_{EBIT,0}$ indicates that

$$V_{ECF+DCF,0} = \left[(1 - T_C) - \left(\frac{g}{ROC}\right)\right] \cdot V_0 + T_C \cdot \{V_0 \cdot [1 - N(d_1)] + B \cdot e^{-r \cdot T} \cdot N(d_2)\}. \tag{60}$$

$V_{ECF+DCF,0}$ gives the value of the levered firm when there are corporate income taxes. The value of the unlevered firm when there are corporate income taxes is equal to the present value of the free cash flows and is given by

$$V_{FCF,0} = \left[(1 - T_C) - \left(\frac{g}{ROC}\right)\right] \cdot V_{EBIT,0} = \left[(1 - T_C) - \left(\frac{g}{ROC}\right)\right] \cdot V_0 \,. \tag{61}$$

D. Present Value of the Tax Shields for the Firm

The taxes for the unlevered firm for time t + 1 are

$$TAX_{U,t+1} = T_C \cdot EBIT_{t+1} \tag{62}$$

and their present value at time 0 is

$$V_{TAX_U,0} = T_C \cdot V_{EBIT,0}. \tag{63}$$

Using V_0 as the value of $V_{EBIT,0}$ gives

$$V_{TAX_U,0} = T_C \cdot V_0 \tag{64}$$

for the present value of the taxes for the unlevered firm at time 0. The taxes for the levered firm for time t + 1 are

$$TAX_{L,t+1} = T_C \cdot (EBIT_{t+1} - INT_{t+1}) \tag{65}$$

and their present value at time 0 is

$$V_{TAX_L,0} = T_C \cdot (V_{EBIT,0} - V_{INT,0}). \tag{66}$$

The present value of the taxes for the levered firm is equal to the present value of the earnings before interest and taxes less the present value of the interest multiple by the corporate income tax rate. Since the value of $V_{EBIT,0}$ minus $V_{INT,0}$ is given by the solution for E_0 from the Black-Scholes option pricing model,

$$V_{TAX_L,0} = T_C \cdot E_0. \tag{67}$$

The present value of the tax shields is equal to the present value of the taxes for the unlevered firm minus present value of the taxes for the levered firm.

$$V_{TS,0} = V_{TAX_U,0} - V_{TAX_L,0} = T_C \cdot V_0 - T_C \cdot E_0 = T_C \cdot V_0 - T_C(V_0 - D_0) = T_C \cdot D_0. \tag{68}$$

Substituting the solution from the Black-Scholes option pricing model for D_0 gives

$$V_{TS,0} = V_{TAX_U,0} - V_{TAX_L,0} = T_C \cdot \{V_0 \cdot [1 - N(d_1)] + B \cdot e^{-r \cdot T} \cdot N(d_2)\} \tag{69}$$

for the present value of the tax shields. The present value of the tax shields is also equal to the present value of the levered firm minus present value of the unlevered firm.

$$V_{TS,0} = V_{ECF+DCF,0} - V_{FCF,0}. \tag{70}$$

Using the expressions for the value of the levered firm when there are corporate income taxes and the value of the unlevered firm when there are corporate income taxes indicates that

$$V_{TS,0} = \left[(1-T_C)-\left(\frac{g}{ROC}\right)\right]\cdot V_0 + T_C\cdot\{V_0\cdot[1-N(d_1)]+B\cdot e^{-r\cdot T}\cdot N(d_2)\} - \left[(1-T_C)-\left(\frac{g}{ROC}\right)\right]\cdot V_0. \tag{71}$$

Simplifying this expression for the present value of the tax shields gives

$$V_{TS,0} = V_{ECF+DCF,0} - V_{FCF,0} = T_C\cdot D_0 = T_C\cdot\{V_0\cdot[1-N(d_1)]+B\cdot e^{-r\cdot T}\cdot N(d_2)\}. \tag{72}$$

Both ways of calculating the present value of the tax shields show that the present value of the tax shields equals the income tax rate times the solution from the Black-Scholes option pricing model for D_0. The present value of the tax shields should be calculated as the present value of the tax savings resulting from the risky debt.

E. Beta Coefficient and Appropriate Discount Rate for Risky Debt and Risky Tax Shields

The present value of the risky debt in the Black-Scholes option pricing model, D_0, is equal the present value of risk-free debt less the present value of a European put option with an exercise price of B and time until expiration equal to T.

$$D_0 = B\cdot e^{-rT} - \{V_0\cdot[1-N(d_1)] + B\cdot e^{-rT}\cdot[1-N(d_2)]\} = B\cdot e^{-rT} - P_0. \tag{73}$$

The appropriate discount rate for the risky debt and the risky tax shields is the discount rate for a portfolio consisting of a long position in risk-free debt and a short position in a European put option. The beta coefficient from the capital asset pricing model for a portfolio is equal to the value-weighted average of the beta coefficients for the components of the portfolio. The beta coefficient for the appropriate discount rate for the risky debt and the risky tax shields is given by

$$\beta_D = \left(\frac{B\cdot e^{-rT}}{D_0}\right)\cdot\beta_{RF} - \left(\frac{P_0}{D_0}\right)\cdot\beta_P = \left(\frac{B\cdot e^{-rT}}{D_0}\right)\cdot 0 - \left(\frac{B\cdot e^{-rT}-D_0}{D_0}\right)\cdot\beta_P = -\left(\frac{B\cdot e^{-rT}-D_0}{D_0}\right)\cdot\beta_P \tag{74}$$

where β_D is the beta coefficient for risky debt, β_{RF} is the beta coefficient for risk-free debt, and β_P is the beta coefficient for the put option. Since the beta coefficient for risk-free debt equals zero, the beta coefficient for the risky debt and the risky tax shields depends on the beta coefficient for the European put option and the difference in the present values of the risk-free debt, $B\ e^{-rT}$, and the risky debt, D_0 (the reduction in the present value of debt due to its risk). When the present value of the firm's risky debt equals the present value of risk-free debt ($D_0 = B\ e^{-rT}$), the beta coefficient for risk-free debt equals zero and the appropriate discount rate for the risky debt and the risky tax shields is the risk-free rate (Modigliani and Miller 1963). This will be approximately true for small relative amounts of debt. For larger amounts for debt, the appropriate discount rate for the risky debt and the risky tax shields is determined by the beta coefficient for the European put option and the difference in the present values of the risk-free debt and the risky debt. The beta coefficient for the European put option is given by (Jarrow and Rudd 1983)

$$\beta_P = \left(\frac{V_0}{P_0}\right)\cdot[N(d_1)-1]\cdot\beta_U = \left(\frac{V_0}{B\cdot e^{-rT}-D_0}\right)\cdot[N(d_1)-1]\cdot\beta_U. \tag{75}$$

where β_U is the beta coefficient for the firm's unlevered equity. For very large relative amounts of debt, the beta coefficient for the European put option equals minus one times the beta coefficient for the firm's operating assets (unlevered equity). The beta coefficient for the risky debt and the risky tax shields is given by

$$\beta_D = -\left(\frac{B\cdot e^{-rT}-D_0}{D_0}\right)\cdot\left(\frac{V_0}{B\cdot e^{-rT}-D_0}\right)\cdot[N(d_1)-1]\cdot\beta_U = \left(\frac{V_0}{D_0}\right)\cdot[1-N(d_1)]\cdot\beta_U \tag{76}$$

When all of the firm's financing is provided by debt [$D_0 = V_0$ and $N(d_1) = 0$], the beta coefficient for the risky debt and the risky tax shields is equal to the beta coefficient for the firm's operating assets (unlevered equity) and the appropriate discount rate for the risky debt and the risky tax shields is the cost of unlevered equity (Fernandez 2002 and 2004). For intermediate relative amounts of debt, the beta coefficient for the risky debt and the risky tax shields increases from zero to the beta coefficient for the firm's operating assets (unlevered equity) as the relative amount of debt increases from relatively low levels to relatively high levels. The appropriate discount rate for the risky debt and the risky tax shields increases from the risk-free rate (Modigliani and Miller 1963) to the unlevered cost of equity (Fernandez 2002 and 2004) as the relative amount of debt increases from relatively low levels to relatively high levels. In general, the appropriate discount rate for the tax shields from using debt is the required rate of return for risky debt which is somewhere between the risk-free rate and the unlevered cost of equity (Myers 1974).

VI. Effects of Using Debt when there are Taxes

The values of parameters shown in Table 1 are used to illustrate relationships between the relative amount of debt used and the probability of default, the beta coefficient for risky debt and risky tax shields, the required rate of return for risky debt and risky tax shields, and the present value of the tax shields. **All of the tables and figures used to show these relationships are presented at the end of this paper.**

A. Probability of Default, Beta Coefficient for Risky Debt and Risky Tax Shields, and Required Rate of Return for Risky Debt and Risky Tax Shields

Table 2 shows the relationships between the relative amount of debt used and the probability of default, the beta coefficient for risky debt and risky tax shields, and the required rate of return for risky debt and risky tax shields. In this table, the relative amount of debt is measured by D_0 / V_0 where D_0 is the solution for risky debt from the Black-Scholes option pricing model. For the constant growth firm, D_0 represents $V_{INT,0}$ and V_0 represents $V_{EBIT,0}$ which is the present value of the underlying asset for the contingent claims. The relative amount of debt is increased by increasing the face value of the debt (B) in the Black-Scholes option pricing model which increases D_0. Table 2 shows how the probability of default [$1 - N(d_1)$] increases from zero to one as the relative amount of debt used increases. Table 2 and Figure 1 show how the beta coefficient for risky debt and the risky tax shields increases from zero to β_U as the relative amount of debt used increases. Table 2 and Figure 2 show how the required rate of return for risky debt and the risky tax shields increases from the risk-free rate to the required rate of return for unlevered equity as the relative amount of debt increases. The information provided in Table 2, Figure 1, and Figure 2 illustrate the general nature of the relationships between the relative amount of debt used and the probability of default, the beta coefficient for risky debt and the risky tax shields, and the required rate of return for risky debt and risky tax shields.

B. Present Value of the Tax Shields

Table 3 shows the relationships between the relative amount of debt used and the probability of default and the present value of the tax shields. Figure 3 show the relationship between the relative amount of debt used and the present value of the tax shields. As the relative amount of debt increases from zero to 100 percent, the probability of default increases from zero to one and the present value of the tax shields increases from zero to $T_C D_0$ where $D_0 = V_{INT,0}$ is the solution for risky debt from the Black-Scholes option pricing model.

VII. Summary and Conclusions

The finance literature contains several different models for the appropriate discount rate for the tax shields and the present value of the tax shields resulting from the use of debt. This research uses an option valuation framework to develop models for the appropriate discount rate for the risky tax shields, the present value of the tax shields, and the probability of default resulting from the use of debt. Relationships between cash flows and their present values for firms with constant, perpetual growth are developed so that the reinvestment required to support the growth is incorporated in the expressions. Analysis using option valuation theory and the relationships between cash flows and their present values provides a framework for unifying the major apparently dissimilar results for the appropriate discount rate for the tax shields and the present value of the tax shields in the finance literature. This research indicates that if all of the firm's financing is provided by debt, the beta coefficient for the risky debt and the risky tax shields is equal to the beta coefficient for the firm's operating assets (unlevered equity) and the appropriate discount rate for the risky debt and the risky tax shields is the cost of unlevered equity (Fernandez 2002 and 2004). For this boundary condition, the tax shields should be discounted using the unlevered cost of equity. For intermediate relative amounts of debt, the beta coefficient for the risky debt and the risky tax shields increases from zero to the beta coefficient for the firm's operating assets (unlevered equity) as the relative amount of debt increases from relatively low levels to relatively high levels and the appropriate discount rate for the risky debt and the risky tax shields increases from the risk-free rate (Modigliani and Miller 1963) to the unlevered cost of equity as the relative amount of debt increases from relatively low levels to relatively high levels (Myers 1974). This study shows that the appropriate discount rate for the tax shields is always the appropriate discount rate for risky debt and this rate changes as the relative amount of debt changes so that the risk-free rate is the appropriate discount rate for low levels of relative debt and the unlevered cost of equity is the appropriate discount rate for extremely high levels of relative debt. The appropriate discount rate for the tax shields is not a constant rate. It is a discount rate that is a function of the relative amount of debt utilized. It can be the risk-free rate and it can be the unlevered cost of equity, but these are extremes (boundary conditions).

Table 1. Parameters Used in the Illustrative Example

r	0.06
σ	0.35
T	1.00
V_0	100
R_U	0.11
β_U	1.00
RP_M	0.05
T_C	0.35
ROC	0.25
g	0.04

Table 2. Probability of Default, Beta Coefficient, and Required Rate of Return for Risky Debt and Risky Tax Shields

D_0 / V_0	$1 - N(d_1)$	β_D	R_D
0.000	0.000	0.000	0.060
0.009	0.000	0.000	0.060
0.011	0.000	0.000	0.060
0.013	0.000	0.000	0.060
0.016	0.000	0.000	0.060
0.019	0.000	0.000	0.060
0.024	0.000	0.000	0.060
0.028	0.000	0.000	0.060
0.034	0.000	0.000	0.060
0.040	0.000	0.000	0.060
0.048	0.000	0.000	0.060
0.055	0.000	0.000	0.060
0.064	0.000	0.000	0.060
0.073	0.000	0.000	0.060
0.083	0.000	0.000	0.060
0.093	0.000	0.000	0.060
0.105	0.000	0.000	0.060
0.117	0.000	0.000	0.060
0.129	0.000	0.000	0.060
0.143	0.000	0.000	0.060
0.157	0.000	0.000	0.060
0.172	0.000	0.000	0.060
0.187	0.000	0.000	0.060
0.204	0.000	0.000	0.060
0.221	0.000	0.000	0.060
0.238	0.000	0.000	0.060
0.257	0.000	0.000	0.060
0.276	0.000	0.000	0.060
0.295	0.000	0.000	0.060
0.316	0.000	0.001	0.060
0.337	0.001	0.002	0.060
0.359	0.001	0.003	0.060
0.381	0.002	0.004	0.060
0.405	0.003	0.007	0.060
0.428	0.005	0.011	0.061
0.453	0.007	0.017	0.061
0.478	0.011	0.024	0.061
0.503	0.017	0.033	0.062
0.529	0.024	0.045	0.062
0.554	0.033	0.060	0.063
0.580	0.045	0.078	0.064
0.606	0.060	0.099	0.065
0.632	0.078	0.123	0.066
0.658	0.099	0.150	0.067
0.683	0.123	0.180	0.069
0.708	0.150	0.212	0.071
0.731	0.181	0.247	0.072
0.754	0.214	0.283	0.074
0.776	0.250	0.322	0.076
0.797	0.288	0.361	0.078
0.817	0.328	0.401	0.080
0.836	0.369	0.441	0.082
0.853	0.410	0.481	0.084
0.869	0.452	0.520	0.086
0.884	0.494	0.559	0.088
0.898	0.535	0.596	0.090
0.910	0.575	0.632	0.092
0.921	0.614	0.666	0.093
0.931	0.650	0.699	0.095
0.940	0.685	0.729	0.096
0.948	0.718	0.757	0.098
0.955	0.749	0.784	0.099
0.962	0.777	0.808	0.100
0.967	0.803	0.830	0.102
0.972	0.827	0.850	0.103
0.976	0.848	0.869	0.103
0.980	0.867	0.885	0.104
0.983	0.885	0.900	0.105
0.986	0.900	0.913	0.106
0.988	0.914	0.925	0.106
0.990	0.926	0.935	0.107
0.992	0.937	0.945	0.107
0.993	0.946	0.953	0.108
0.994	0.954	0.960	0.108
0.995	0.961	0.966	0.108
0.996	0.967	0.971	0.109
0.997	0.972	0.975	0.109
0.997	0.976	0.979	0.109
0.998	0.980	0.982	0.109
0.998	0.983	0.985	0.109
0.998	0.986	0.988	0.109
0.999	0.988	0.990	0.109
0.999	0.990	0.991	0.110
0.999	0.992	0.993	0.110
0.999	0.993	0.994	0.110
0.999	0.995	0.995	0.110
1.000	0.995	0.996	0.110
1.000	0.996	0.997	0.110
1.000	0.997	0.997	0.110
1.000	0.997	0.998	0.110
1.000	0.998	0.998	0.110
1.000	0.998	0.998	0.110
1.000	0.999	0.999	0.110
1.000	0.999	0.999	0.110
1.000	0.999	0.999	0.110
1.000	0.999	0.999	0.110
1.000	0.999	0.999	0.110
1.000	1.000	1.000	0.110
1.000	1.000	1.000	0.110

Table 3. Present Value of the Tax Shields

D_0 / V_0	$1 - N(d_1)$	$V_{TS,0}$	D_0 / V_0	$1 - N(d_1)$	$V_{TS,0}$	D_0 / V_0	$1 - N(d_1)$	$V_{TS,0}$
0.000	0.000	0.000	0.405	0.003	14.162	0.980	0.867	34.293
0.009	0.000	0.330	0.428	0.005	14.994	0.983	0.885	34.402
0.011	0.000	0.379	0.453	0.007	15.846	0.986	0.900	34.496
0.013	0.000	0.453	0.478	0.011	16.715	0.988	0.914	34.576
0.016	0.000	0.552	0.503	0.017	17.601	0.990	0.926	34.645
0.019	0.000	0.676	0.529	0.024	18.498	0.992	0.937	34.703
0.024	0.000	0.824	0.554	0.033	19.404	0.993	0.946	34.752
0.028	0.000	0.997	0.580	0.045	20.315	0.994	0.954	34.794
0.034	0.000	1.195	0.606	0.060	21.226	0.995	0.961	34.829
0.040	0.000	1.417	0.632	0.078	22.131	0.996	0.967	34.858
0.048	0.000	1.665	0.658	0.099	23.027	0.997	0.972	34.883
0.055	0.000	1.937	0.683	0.123	23.907	0.997	0.976	34.903
0.064	0.000	2.233	0.708	0.150	24.766	0.998	0.980	34.920
0.073	0.000	2.555	0.731	0.181	25.601	0.998	0.983	34.935
0.083	0.000	2.901	0.754	0.214	26.405	0.998	0.986	34.946
0.093	0.000	3.271	0.776	0.250	27.176	0.999	0.988	34.956
0.105	0.000	3.667	0.797	0.288	27.909	0.999	0.990	34.964
0.117	0.000	4.087	0.817	0.328	28.602	0.999	0.992	34.971
0.129	0.000	4.532	0.836	0.369	29.253	0.999	0.993	34.976
0.143	0.000	5.002	0.853	0.410	29.861	0.999	0.995	34.981
0.157	0.000	5.496	0.869	0.452	30.424	1.000	0.995	34.984
0.172	0.000	6.016	0.884	0.494	30.943	1.000	0.996	34.987
0.187	0.000	6.559	0.898	0.535	31.418	1.000	0.997	34.990
0.204	0.000	7.128	0.910	0.575	31.850	1.000	0.997	34.992
0.221	0.000	7.721	0.921	0.614	32.242	1.000	0.998	34.993
0.238	0.000	8.339	0.931	0.650	32.594	1.000	0.998	34.995
0.257	0.000	8.982	0.940	0.685	32.910	1.000	0.999	34.996
0.276	0.000	9.649	0.948	0.718	33.191	1.000	0.999	34.996
0.295	0.000	10.341	0.955	0.749	33.440	1.000	0.999	34.997
0.316	0.000	11.058	0.962	0.777	33.659	1.000	0.999	34.998
0.337	0.001	11.798	0.967	0.803	33.852	1.000	0.999	34.998
0.359	0.001	12.563	0.972	0.827	34.020	1.000	1.000	34.999
0.381	0.002	13.351	0.976	0.848	34.166	1.000	1.000	35.000

Figure 1
Beta Coefficient for Risky Debt and Risky Tax Shields

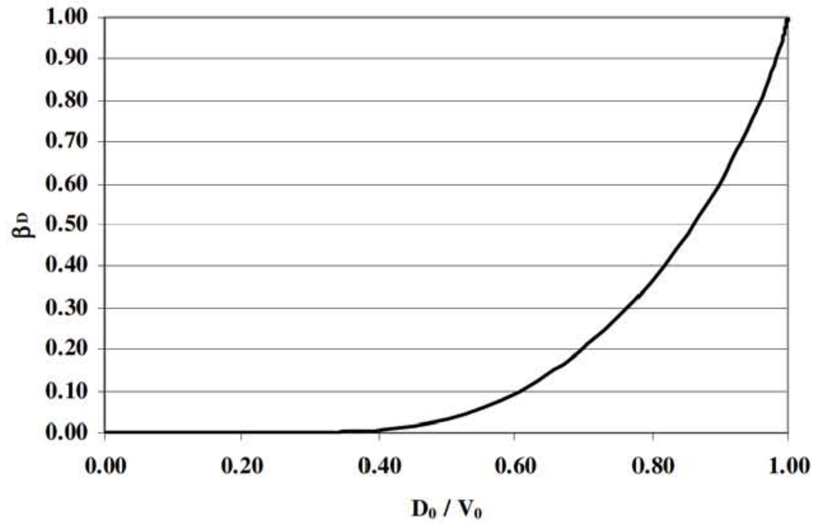


Figure 2
Required Rate of Return for Risky Debt and Risky Tax Shields

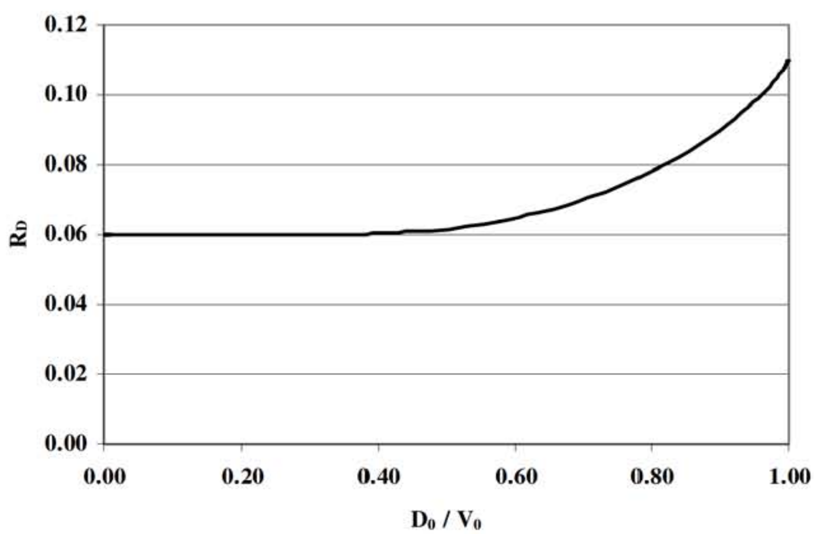
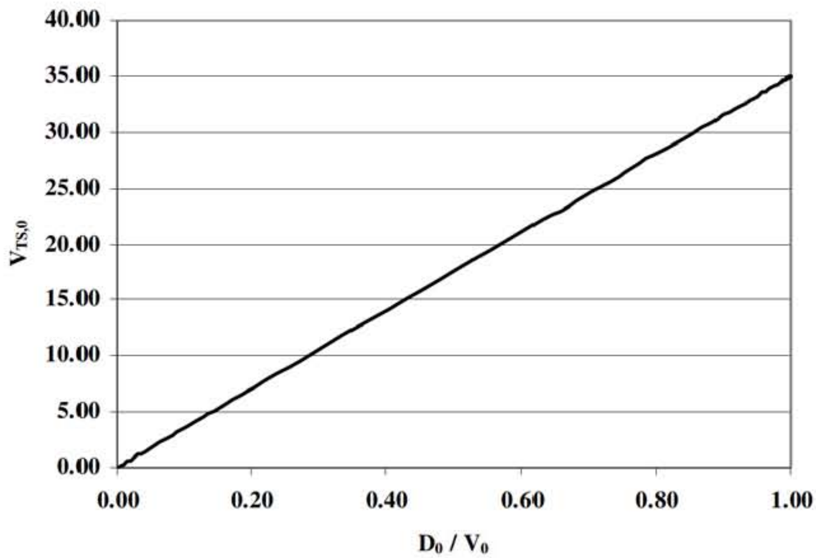


Figure 3
Present Value of the Tax Shields



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