Introducing Arbitrage

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Abstract

While arbitrage is an integral part of the study of derivatives and of corporate finance, it appears also in money and banking and in international economics. Students find it difficult to develop intuition concerning arbitrage situations and trying to learn how to apply the concept at the same time increases the difficulty. I separate the two problems for them by introducing students to the notion of arbitrage first in a simple context. There they can develop some facility in the use of the ideas before plunging into the harder study of more complicated ideas. Starting with two securities, which can be converted one for the other at some date in the future, I add complications (including state-contingent pay-offs) until students can see how any two assets which are even temporarily "the same" can give rise to arbitrage opportunities if they are not equal in price today and how a complex security might be thought of as a bundle of simpler securities and therefore can then be priced as the sum of the component prices. I also point out how each example shows not only how to execute an arbitrage strategy but how much can sometimes go wrong.

Introduction

The concept of arbitrage pervades financial economics, not just the study of derivatives. In the world of perfect markets certain relationships between underlying assets and the securities derived from them must exist or else there is an arbitrage opportunity. We normally teach pricing rules for options and futures by appealing to no-arbitrage theorems. The Modigliani-Miller propositions depend on arbitrage arguments. The relationship between spot and forward exchange rates for currencies depends on covered interest arbitrage. In the term structure of interest rates forward rates are those which would prevent arbitrage in the world of perfect and complete markets.

The simultaneous introduction of the notion of arbitrage and its application to the subject at hand is usually heavy going. At least in my experience what the students find is not simultaneous reinforcement of the two kinds of concepts but overload. The applications are filled with "light bulb concepts" which make no sense to a student until the light bulb turns on and then seem simple. It makes the subject more difficult still that students must become accustomed to arbitrage at the same time as they learn about specifics of the subject matter of the course.

Separating the two kinds of learning, the nature of arbitrage and the subject of the course, provides easier access to both. Derivatives and investments textbooks, like [Chance 2004] pg. 10 or [Sharpe, Alexander and Bailey 1999] pg. 284, typically provide a one-paragraph introduction to arbitrage and move right along to the use of the concept in option pricing. [Strong 2002] pg. 100-101 provides almost two pages and cites the folk wisdom that finance is "the study of arbitrage" but still provides no practice.

I start the derivatives course with a unit on arbitrage in an artificially simple situation and give my students enough practice with arbitrage so that they develop some facility with solving problems using the concept. I use this same unit in money and

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banking classes when we get to foreign exchange. This unit takes a week or two and pays dividends throughout the semester. A similar though shorter unit helps corporate finance students appreciate the proofs of the Modigliani-Miller propositions. Finally, consideration of derivatives-related disasters is helped by previous introduction of the conditions necessary for economic arbitrage (a costless, riskless profit), whose absence can lead "not quite" arbitrage to blow up.

The idea for this separation came from my experience long ago when I began calculus in college. The calculus course for mathematics majors is especially difficult because it typically combines difficult mathematics with an introduction to rigorous mathematical reasoning. Mine was an honors course with full Weierstrassian epsilon-delta rigor, and it began with a stroke of luck. That luck seemed at first to be bad but later proved to have been good. The textbooks did not arrive in the bookstore until the third week of the term, so the instructor used that time teaching the class about inequalities, and we students developed some facility manipulating the symbols and proving propositions about them. Later when we were asked to "find δ given ε such that if $|x-a|<\delta$, then $|f(x)-f(a)|<\varepsilon$" the practice we had already put in learning to manipulate inequalities made learning much easier.

Using this insight from my own past I take some time to get students used to finding arbitrage profits. First I define arbitrage as "a costless, riskless profit" and show a few simple examples where traders can benefit from a violation of the Law of One Price, buying low and simultaneously selling high without having to keep open a short position. The class agrees that it would be a good thing to find an arbitrage opportunity, but that it seems too good to be true. I introduce the world of perfect markets and show that the Law of One Price must hold, and that there will be no arbitrage opportunities there, and we agree that even in the real world it must hold most of the time at least for financial instruments when there is short selling.

The Law of One Price is not enforced by the Financial Police. Violations of the law lead to opportunities to buy low and sell high. Alert investors watch for these opportunities, and when they take advantage of such an opportunity they force the low price up and the high price down until the opportunity vanishes. I tell students that we teach them about arbitrage for two reasons, first to prove the pricing theorems and second so that when they see an arbitrage opportunity some day they can take advantage of it. They won't have much time to figure out how, so they should get some practice during my course. If they succeed, they can send my university an alumni contribution; but of course arbitrage is a zero-sum game, and if they buy high and sell low I won't want to hear about it. Later examples show that identifying the high and low securities is not always easy.

I then give the students practice at spotting arbitrage opportunities with a sequence of increasingly difficult examples. In each one there are two securities linked together by a two-way conversion provision at some date in the future. It is this "date with destiny" as Myron Scholes put it in the derivatives class I took some years ago, that produces the arbitrage opportunity, and a conversion privilege is like real world conversion feature of a convertible bond and much easier to understand than the relationship which must hold between a stock and its options at the moment of expiration. First I work out several problems, and then the students do some in imitation of them for homework. Then I work out more complicated examples, and the students do some in imitation of them.

The Simplest Example

The simplest example starts with two securities, *A* and *B*. They may be held long or sold short and provide no cash throw-off to their owners. What binds them together is a conversion privilege; one year from today (at $t = 1$) the conversion window will open for one hour, and each unit of *A* may be exchanged costlessly for one unit of *B* and *vice versa*. If now (at $t = 0$) the price A_0 of *A* is \$40.00 per unit, what must be the price of *B* to avoid an arbitrage opportunity? Students think for a while and conclude (possibly with my help) that *B* should sell for \$40.00 per unit also. Then I ask why this must be so.

I show them that if *B* costs only \$38.00 per unit, then by buying *B* and selling *A* short they may obtain a positive \$2.00 cash flow now without any use of their own funds, and then we look forward to the end of the year when the conversion window is open. At that point I show them that they can exchange the unit of *B* they own for a unit of *A* they may then return to the lender of *A* and cover the short position. I also point out that if institutional rules prevent the short position being held open through $t = 1$ then the arbitrage is no longer riskless, and if traders have to post margin money, then it is no longer quite costless.

We check that this operation passes the three tests for an arbitrage opportunity:

(1) It requires no investment.
(2) It produces a positive cash flow.
(3) There is no risk, either now or later.

We also note (with some prodding from me) that the size of the arbitrage profit per unit if it is taken in the present is the same as the size of the mispricing in the present, and that since *B* was mispriced relative to *A* by \$2.00, the positive \$2.00 cash flow at $t = 0$ shows we are doing the right thing. This also leads to the observation that large enough transactions costs would defeat the arbitrage. If for example the price difference were only \$0.02 per unit and the transaction cost \$0.03 per unit there would be no opportunity.

This simple exchange provision serves to tie the prices of the two securities together until it has expired. We also notice that at any time $0 \leq t \leq 1$, we must see $A_t = B_t$ or else the same arbitrage opportunity exists. That makes the end of the year the date with destiny that holds out an arbitrage profit whenever the correct relationship between *A* and *B* is violated.

Making the Example Harder

Once students have imitated this example with different prices and different conversion ratios (one unit of *A* for two units of *B* for example) I introduce cash throw-off. What will happen to the no-arbitrage price of *B* (given the \$40.00 price for *A*) if each unit of *B* will pay its owner a risk-free \$2.00 per unit dividend in six months with a risk-free interest rate of 2% per half-year? We agree that *B* is certainly more desirable now than *A* and therefore ought to cost more. If necessary, I ask, "If you could have either *A* or *B* for the same price, would you be indifferent or would you prefer *B*?" I invoke the Law of One Price and separate *B* into its two components. We agree that without the dividend B would cost \$40.00 per unit, and we calculate the present value of the \$2.00 as \$1.96 and identify that as the value today of the right to receive the dividend when it is paid. I then claim that B should sell for \$41.96. This is the price of a synthetic unit of B made up of one unit of A and one risk-free, zero-coupon bond with payoff \$2.00 at six months. Not all students agree immediately, but we go over the arbitrage opportunities if B sells for less than \$41.96 or "too low" or else more than \$41.96 or "too high".

Suppose that B costs only \$41.00, too low by \$0.96, so we buy a B and sell an A, which is too high if B is too low. Now we also have to track what happens to the dividend. If we buy B we have the right to receive the risk-free \$2.00 in six months, and we can borrow \$1.96 against that \$2.00. This makes the complete instructions at $t = 0$ "Buy a unit of B, borrow the present value of the dividend, and sell a unit of A short. At six months, collect the \$2.00 dividend and repay the loan. Finally at the end of the year, convert the unit of B to a unit of A and cover the short position." The cash flow at time 0 is

$$CF_0 = -\$41.00 + \$1.96 + \$40.00 = \$0.96 .$$

Not only is this positive, but it is the right size. Calculating the cash flow at $t = 0$ shows students when they have found the correct instructions, so they know whether they are doing it right before we go over everything again in class. We then check cash flows at time $t = 0.5$ when we collect the dividend and pay back the loan and also at $t = 1$ when we convert and cover the short. That these cash flows are both zero shows that there is no risk.

Adding Uncertainty

Once the students have learned some technique from dissecting the B into its A and bond components, I ask them what would happen if the \$2.00 dividend were contingent on some not yet known outcome. Suppose that the owner of a unit of B only gets the \$2.00 if before the six-month point Arsenal Football Club have won the Champions League. Now we can discuss market completeness. If there is a market for dollars to be paid in six months if Arsenal are Champions, then we can once again dissect B into a copy of A and a bet on the Gunners. Once we know the price of a \$1.00 bet on Arsenal to win is \$0.12 paid now we can once again say that to avoid arbitrage B must sell for the sum of the price of A and the price of two \$1.00 bets. Now the proper price of a unit of B is

$$B_0 = A_0 + 2(\$0.12) = \$40.00 + \$0.24 = \$40.24 .$$

This uses the law of one price and introduces the process of checking for arbitrage in different states of the world in the future (at six months).

Suppose B sells for \$41.00, too high by \$0.76. Sell B and buy A, of course, but what do we do about the bet? If we sell B short and Arsenal win we will owe the owner \$2.00, so we will have to buy (place) two \$1.00 bets now. The cash flow at time 0 is

$$CF_0 = B_0 - A_0 - 2(\$0.12) = \$41.00 - \$40.00 - \$0.24 = \$0.76$$

leading us to suspect we have done it right. Now we have to look forward to six months and ask whether we have any risk, whether Arsenal have won or not. If the Gunners have lost, we do not collect on our bet and do not owe anything to the owner of the borrowed *B*; if they have won, we both collect and pay out the $2.00. Either way our cash flow is zero at six months. Looking farther forward to the end of the year, we once again convert and cover the short position so that our cash flow is zero at time *1*.

If *B* sells today for $40.02, too low by $0.22, we reverse everything, buying *B*, selling *A* and selling two $1.00 bets on the Gunners. Our cash flow at time 0 is

$$CF_0 = -B_0 + A_0 + 2(\$0.12) = -\$40.02 + \$40.00 + \$0.24 = \$0.22,$$

which is once again positive and the right size. At six months if the Gunners have lost, we do not collect as *B* owners, but do not need to pay off the bets. If the Gunners have won we both collect and pay the $2.00. At the end of the year we convert and cover.

If the market for bets on the Gunners does not exist we can construct such a bet by buying a unit of *B* and selling a unit of *A* thus completing the market. We can then discuss opportunities to profit from completing the market in the real world.

The same kind of simple context can be used to introduce hedging with two states of the world at the end of the first time period. Keeping the context simple is important for most undergraduate economics and business students who find continuous time daunting and who remember little or nothing of whatever they may have learned about exponential and logarithmic functions in high school or the "quantitative methods" course in college. I remind them that the normal distribution can be thought of as binomial with many branchings and skip over the problem of deciding what properties of the binomial process pass through the limiting process to the normal distribution.

Conclusion

Developing a facility for seeing how a compound security may be separated into pieces and priced to avoid arbitrage gives students a chance to concentrate on the details of the particular securities they will be studying in the bulk of the derivatives class. Furthermore, it may happen that some of them will see arbitrage opportunities some day and exploit them. My university can use the donations.

References

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