Teaching the Time Value of Money: Advantages of the Continuous Compounding (Exponential) Model

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It is vitally important that business students understand the conceptual meaning of the time value of money (TVM) and how to solve TVM related business problems. Most textbooks explain TVM using only discrete compounding. We argue that incorporating continuous (or exponential) compounding into the curriculum can be advantageous. The exponential model allows students to develop a better conceptual understanding of TVM problems. It allows instructors to visually demonstrate a number of important TVM concepts. And finally, the exponential model allows students to understand and solve problems from the natural world. Student reaction to the incorporation of continuous compounding into the curriculum is generally positive.

INTRODUCTION

The time value of money (TVM) is central to modern finance. Therefore, TVM is discussed in all introductory finance textbooks. However, the majority of textbook authors only use discrete (annual, monthly, daily etc.) compounding periods when teaching TVM. We believe the incorporation of continuous compounding into TVM pedagogy will improve students' intuitive understanding of important TVM concepts. Introducing continuous compounding into the curriculum allows students to gain insight into TVM problems rather than simply learning the sequence of buttons to push on a financial calculator when faced with a specific TVM problem.

There are a number of areas where taking a continuous compounding view of the situation can aid student understanding. For example, a simple exponential curve to the left and right of the origin can be viewed as an indifference curve intuitively relating past, present, and future values for a particular interest/discount rate. In addition to showing the future value of an annuity, the area under an exponential curve to the right of the origin shows a visually compelling relationship between the amount of money contributed by the investor, the interest earned, and the value of compounding over long time periods. An exponential curve to the left of the origin visually shows the relationship between the amount of money borrowed and the total interest paid on a loan. And finally, there are many problems relating to the natural world (carbon-14 dating problems and intravenous drug dosing, for example) that are easily solved if the student is familiar with continuous compounding. Each of these areas will be described in detail in this paper.

The next section describes how TVM is covered in financial textbooks. There are then four sections describing the advantages of the continuous compounding model in detail. The article concludes with suggestions on how to incorporate continuous compounding into the curriculum and a description of the generally positive reception of students to continuous compounding.

TEXTBOOK TREATMENT OF TIME VALUE OF MONEY

We did a brief survey of corporate finance textbooks in order to better understand the conventional approach to teaching the time value of money. This was not intended to be a comprehensive survey of the prominent texts in the field. Rather, the intent of the survey was to find out the extent to which continuous compounding and growing annuities are covered in current textbooks. Results of the survey are shown in Table 1.

Table 1 shows that seven of ten texts surveyed do not mention continuous compounding at all. Eight of the ten texts do not discuss how to calculate the future value of a growing annuity. Half the texts surveyed do not include continuous compounding or the future value of growing annuities. The Brealey, Myers, Allen graduate level textbook, Principles of Corporate Finance 8th edition, contains a more extended treatment of continuous compounding than the other texts. Interestingly, Brealey et. al. include two separate tables in an appendix that serve a similar function to Table 3 in this paper.

We believe that the omission of continuous compounding and the future value of growing annuities from most textbooks leaves room for improvement in the TVM pedagogy for most professors. Therefore, the intent of this paper is to explain how including continuous compounding in the TVM pedagogy can improve student comprehension of important TVM concepts.

The Exponential Curve as an Indifference Curve

If two cash flows occur at different points in time, they must be translated to the same point in time in order to be compared. That point is usually either the present or the future, but it could be any point in time. With discrete compounding, the familiar equation $FV = PV*(1 + r)t$ is used to do the actual calculation. In this equation r is the discrete interest rate and t is the number of discrete periods.

Table 1: Textbook Treatment of Continuous Compounding and the Future Value of Growing Annuities

Textbook	A	B	C	D
Brealey, Myers, Marcus: Fundamentals of Corporate Finance 4th ed.	X			
Keown, Martin, Petty, Scott: Foundations of Finance 4th ed.	X			
Brigham and Houston: Fundamentals of Financial Management 8th ed.	X			
Beck, Demarzo, Harford: Fundamentals of Corporate Finance 1st ed.	X			
Ross, Westerfield, Jordan: Essentials of Corporate Finance 6th ed.	X			
Welch: Corporate Finance, An Introduction 1st ed.		X		
Berk, Demarzo: Corporate Finance: The Core 1st ed.		X		
Gitman: Principles of Managerial Finance 11th ed.			X	
Megginson, Smart: Introduction to Corporate Finance 1st ed.			X	
Brealey, Myers, Allen: Principles of Corporate Finance 8th ed.				X

A) No mention of continuous compounding or future value of growing annuities.
B) No mention of continuous compounding, gives discrete solution to future value of a growing annuities.
C) Includes a brief section on continuous compounding, no mention of future value of growing annuities.
D) More details on continuous compounding, no mention of future value of growing annuities

With continuous compounding, the equation becomes FV = PV * e^{it}, where i is the continuous rate of interest and t is the length of time. When the discrete compounding period is short (daily for instance) the differences between the future values given by the two equations is small. However, the advantage of the continuous compounding version is that it allows us to view an exponential curve as an indifference curve between cash flows past, present, and future. For example, consider the exponential curve for 10% compounded continuously shown in Figure 1.

Figure 1. Exponential Indifference Curve of one US dollar for r = 10%

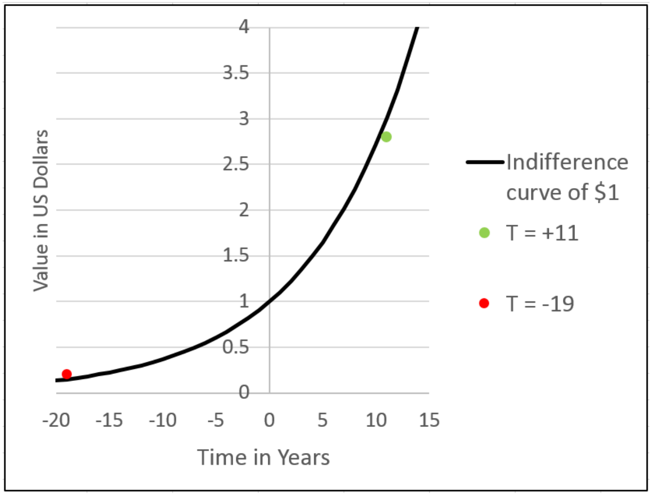


Figure 1 shows that 15 cents 19 years ago [$e^{(-0.1*19)} = e^{-1.9} = 0.15$] is equivalent to $1 today, which is equivalent to $3.00 eleven years in the future [$e^{(.1*11)} = e^{1.1} = 3.0$]. The exponential curve gives us a picture of the bank balance at various points in the past and future that is equivalent to $1 today. Since all these different amounts are equivalent, we can easily compare amounts at different points in time. The exponential curve can be seen as a model of the fact that we expect compensation for any delay, and conversely that an amount of money invested at any point will grow over time. The compensation is the amount of interest that could be earned during the period of the delay. In general, we prefer points above the indifference curve to points below the indifference curve. For example, we would prefer receiving 20 cents 19 years ago to receiving $2.90 eleven years in the future, because 20 cents is above the indifference curve and the $2.90 is below it.

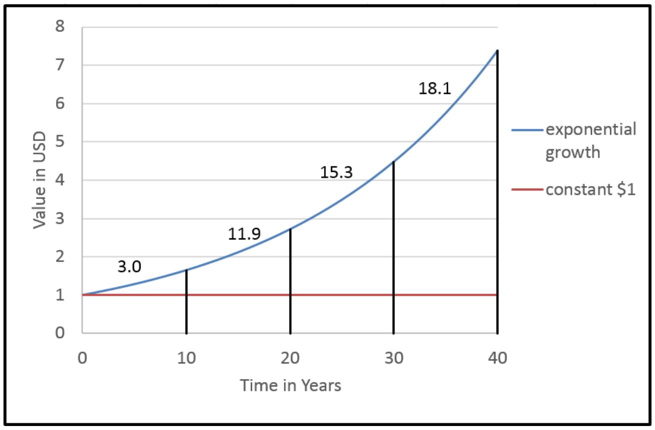
Annuity Future Value with Continuous Compounding

Compound interest is an extremely powerful financial concept. The total investment return increases significantly with time when interest is compounded. One practical application related to this concept is that students should begin saving for retirement as soon as they start working. It is vitally important for students to understand that putting off saving for retirement dramatically reduces the future value of their retirement account and their potential post-retirement standard of living. Therefore, it is important for professors who teach the time value of money to be able to drive home to their students that there are big differences between saving for retirement for only twenty years versus saving for thirty or forty years.

The typical example of a problem of this type is the investor who puts money into an Individual Retirement

Account (IRA) annually. The question is how much money will be in the IRA at retirement, given the amount of money invested each year and the annual rate of return. This class of problem involves calculating the future value of an annuity. Although the power of compounding can be demonstrated with discrete compounding examples by calculating annuity future values for varying years of investment, we suggest that the continuous compounding model provides a visually compelling example of the power of compounding over time. For example, Figure 2 shows the exponential curve associated with continuous compounding at 5%.

Figure 2. Continuously-compounded total value and payment value.



The area under an exponential curve represents the future value of an annuity. Figure 2 shows that the total area under the curve can be divided into two parts. The rectangle bounded by 0 and 1 on the y-axis and t on the x-axis represents the money put into the account by the investor over a given time period. For example, for the first 10 years at $1 per year, $10 has been put in by the investor. That is the left-most rectangle in Figure 2. The area between the top of that rectangle and the exponential curve represents the interest earned for the first 10 years. In this case, that interest earned is $31. Even without numbers, the picture of the continuous compounding model shown in Figure 2 demonstrates the relationship between the money invested and the interest accumulated over time.

Figure 2 also illustrates the dramatic effect of compounding over time. For example, when the investment period increases from 20 to 30 years, the rectangle between 20 and 30 years represents the $10 put in by the investor over those 10 years. The $25 (below the curve and above the rectangle) represents the interest earned over that ten-year period. The $25 interest earned during this third block of 10 years is more than twice the $11 interest earned during the second block of 10 years. This visualization provides students with a compelling example of the power of compounding over time. If money is allowed to compound over 40 years rather than 30 years the difference between the money contributed and interest earned is even more dramatic. The $48 interest earned from year 30 to year 40 dwarfs the $10 contributed by the investor during those 10 years. Illustrations like Figure 2 can help students more fully comprehend and internalize just how powerful compound interest can be over time.

Another important aspect of compounding is that relatively small changes in interest rates make a dramatic difference in future values, especially over long time periods. The continuous compounding model in Figure 3 shows how much difference a change in the interest rate can make in the future value of an annuity.

Figure 3: Exponential growth at 5% and 10%

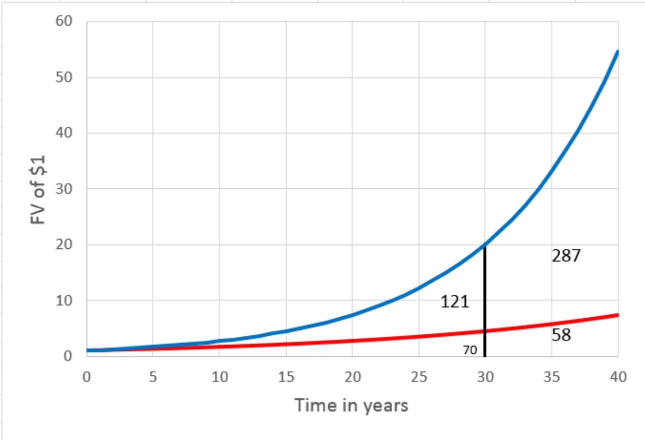


Figure 3 shows the difference in the future value of a $1 per year annuity for interest rates of 5% and 10%. The investor's $40 contribution over the 40 years is not shown because the vertical scale in Figure 3 is so large that the interest earned completely overshadows the $1 per year contributed by the investor. As in Figure 2, the future value of the annuity is the area between the curve and the x-axis from time zero to any given time. The difference in future value between 5% and 10% interest is most striking when the compounding period is more than 20 years. For example, the area between the two curves from 30 to 40 years shows that the same annuity of $1 per year will earn $287 more for those last 10 years at 10% rather than at 5%. In addition, Figure 3 shows that the future value of the annuity at 10% for 40 years consists of four distinct parts:

1) The $70 on hand after 30 years at 5%, plus
2) an additional $121 obtained by raising the interest rate to 10% for the first 30 years, plus
3) an additional $58 obtained by the continuation of the 5% annuity for another 10 years, plus
4) the additional $287 obtained by raising the interest rate to 10% for the final 10 years.

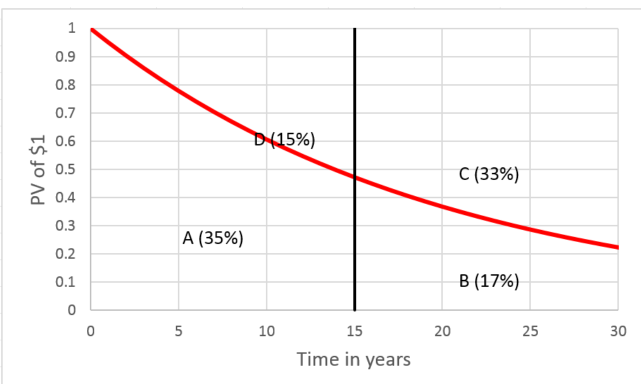
The different areas in Figure 3 are proportional to the dollar amounts. Even if the numbers are not shown, the student will get an intuitive feeling for what happens when the length of time and interest rates are changed, if the instructor uses a figure similar to Figure 3 when teaching annuity future value concepts.

Annuity Present Value with Continuous Compounding

Another very common class of TVM problems involves finding the present value of an annuity. Calculating the payment on a loan is typical of this class of problem. For example, most people need a loan to buy a home. Therefore, it is important for students to be able to calculate a monthly payment given the terms of a mortgage. However, the ability to simply calculate a monthly payment and compare it to monthly income does not enable students to fully comprehend several important financial concepts associated with long-term loans such as mortgages. Two aspects of long-term borrowing that are important for students to understand include (1) the relationship between equity [loan principal] and interest paid on the mortgage over the life of the loan, and (2) the relative amount of equity versus interest paid in each monthly payment. Students need to understand these concepts so that they can make wise financial decisions regarding a home purchase. As with future value of annuity problems, the continuous compounding model can be used to visually illustrate these concepts.

For example, Figure 4 shows exponential decay at a 5% rate. This is equivalent to borrowing money at an interest rate of 5% compounded continuously.

Figure 4: Exponential Decay at 5% (numbers are % of total payments)



In Figure 4 (as in the future value of an annuity case shown in Figures 2 and 3) the area rectangle, the height of which is 1 and the width of which is time on the horizontal axis, to the right of 1.0 to any given point in time represents the total amount paid on the loan over its life (in this case, $1 per year times 30 years = $30). This means rectangle consists of that areas A+B+C+D. is the amount of money paid by the borrower over a 30- year loan. The area under the exponential curve represents the amount of money borrowed or the loan amount. Therefore, area A+BD is the amount of money borrowed on a 30-year mortgage.

Figure 4 reveals an important aspect of long-term loans. It illustrates the relationship between the amount of money borrowed and the interest paid on the loan. In this 5% example, visual inspection shows that the interest on the loan (area B+C+D) is about roughly equal to the amount of money borrowed (area A+BD). To be precise, Table 3 shows that area B+C+D is 48% of area A+B+C+D. The sobering fact is:, if you borrow money at 5% for 30 years, you buy one house and pay for two. The surprisingly large (to most borrowers) amount of interest paid when borrowing long-term is compellingly illustrated when clearly and simply shown by the relative areas composing in the form of Figure 4. It is useful for professors to demonstrate this concept as shown in Figure 4 because it is important for students to understand just how much interest they will pay when they borrow money for a house.

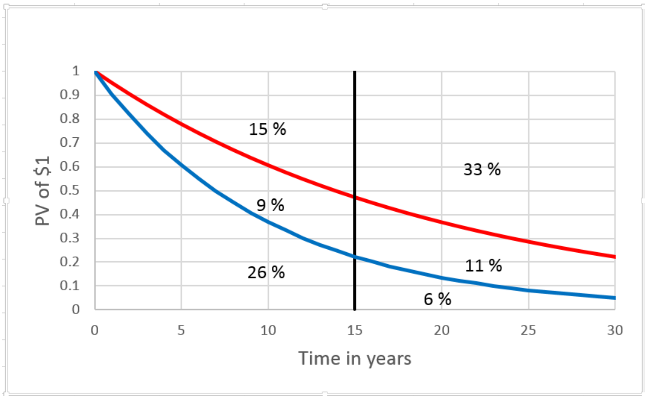
Figure 4 also dramatically illustrates the advantage of borrowing money for 15 years over borrowing money for 30 years (assuming the borrower can will make the same monthly payments on the higher payments on the 15- year mortgage as on the 30-year mortgage). Area B+C+D is the total interest paid on a 30-year loan and area DB is the interest paid on a 15-year loan. Area D [15] is much smaller than half the area of C [33]. Put another way, the total interest paid on a 15-year loan is much less than the interest paid on a 30-year loan. Area DB is approximately 30% of area A+DB. This means that only 30% of the total amount paid on a 15 year loan goes toward interest as opposed to about half of total payments going toward interest on a 30 year loan. In addition, Area B+C represents all the money the borrower won't have to pay if he or she borrows for 15 years rather than 30 years.

Of course, with the same monthly payments, a 15-year mortgage will not get you as much house as a 30-year mortgage would. You can only buy a house priced at 35/(35+17) = 67% of a house with the 30-year mortgage. But is that not a small price to pay for being free from mortgage payments for the next 15 years after the mortgage is paid off? As an aside, we believe that lenders should be required to show something like Figure 4 to all borrowers when they make a loan. That's really truth in lending, and would probably result in an increased percentage of 15-year rather than 30-year mortgages. Of course, the lending industry would vigorously oppose such a requirement.

Figure 4 also visually shows the relationship between interest paid versus equity paid in each monthly payment. In the 5% loan case, the early payments on the loan are approximately 22% equity and 78% interest. This can be seen by looking at the value of the exponential curve at 30 years. The curve therefore shows what fraction of the payment is interest versus equity over the life of the loan. It takes 16 years of payments before the payment is half interest and half equity. In addition, Figure 4 shows that the interest payments during the first 15 years of the mortgage are 33% of the total payments over the 30 year life of the mortgage, and then decline to 15% in the last 15 years. As before, the advantage of the continuous compounding model is that it shows these relationships visually and therefore makes them easier for students to understand.

A diagram like Figure 4 can also be used to show the dramatic difference that the interest rate can make when borrowing long-term. Figure 5 shows the exponential decay curves for 5% and 10% interest rates.

Figure 5: Exponential Decay at 5% and 10% (numbers are % of total payments)



As in Figure 4, the area under the curves in Figure 5 represents the amount borrowed and the area above the curves represents the interest paid on the loan. Figure 5 shows that the amount of interest paid on a 10% loan is dramatically more than the interest paid on a 5% loan. In fact, the interest paid on a 10% loan is about 68% [33+15+11+9] of the total payments versus 48% [15+33] of the total payments on a 5% loan. In terms we used earlier, when borrowing at 10%, the borrower buys one house and pays for three (versus paying for two when borrowing at 5%). Figure 5 also shows that interest paid on a 15-year mortgage is dramatically smaller than the interest paid on a 30-year mortgage when borrowing at 10%. This shows that the advantage of borrowing for 15 years rather than 30 years increases with increasing interest rates. An important advantage of the continuous compounding model over the discrete compounding model is that the continuous model allows these important concepts to be illustrated visually.

Non-Business Related Applications of the Exponential Model

We believe that a major advantage of the exponential growth model is that the model helps students to understand and solve many problems related to the natural world. Students are human beings first and business students second. Therefore, as important as it is for them to be able to understand and solve business-related TVM problems, it is equally important for them to develop an understanding and interest in the natural world that we are all a part of. Teaching them how to solve TVM problems using the continuous compounding model gives them an important tool that will also enable them to conceptually understand and solve many natural world problems.

For example, radioactive decay is one feature of the natural world that can be understood and modeled using an exponential function. To illustrate, strontium-90 is one of the most deadly products of a nuclear explosion. It decays (gives off radioactivity) at a rate of 2.4% per year. A common question concerning radioactive decay is to ask how much of the radioactive material will remain after some period of time. In this case we can ask what percentage of the original strontium is left after 100 years? Although this problem does not appear to be similar to a time value of money problem, it is actually analogous to finding the present value of a future cash flow. In TVM terms, the future value is equal to $e^{100x\ -0.024}$ = .0907. Therefore, there will be 9% of the original amount of strontium-90 left in 100 years.

Some radioactive isotopes decay at very low rates. Carbon-14 (famously used for dating samples of old matter) is a good example of a slowly decaying isotope. Scientists prefer to talk about these kinds of isotopes in half-lives rather than very small decay rates. Carbon-14 has a half-life of 5,720 years. The exponential model allows students to understand how scientists use the decay of carbon-14 to date ancient matter. For example, a tiny piece of a Dead Sea scroll was found to have 80% of its original carbon-14 left. How old is the scroll?

In order to answer this question, students need to understand that the term half-life refers to the time it takes for half of the original amount of material to disappear (or half to remain). In this case, since there is more than half left, namely 80%, we know that the scroll is less than 5,720 years old. As in the strontium-90 problem, we recognize this as a decay problem, and solve it in two steps. First we find how fast carbon-14 decays, and then using that information we can find the age of the scroll. In TVM terms, the first step is to calculate the interest rate that yields half the amount in 5,720 years. So we solve as; 0.5 = e 5,720x i. Therefore i = ln(.5) / 5,720 = -0.0001212. Given this decay rate, the time it takes to have 80% left is found by; 0.8 = et x .-0.0001212 . Therefore t = ln(.8) / -0.0001212 which is 1,841. In other words, the scroll is approximately 1,800 years old.

Implementation in the Classroom

The previous sections of this paper are primarily concerned with the theory associated with continuous compounding, and theoretical advantages of the continuous model compared with the discrete compounding model. We believe these are interesting topics in themselves, but for educators it is also important to be able to bring the theory into the classroom, and to be able to ascertain if the inclusion of continuous compounding concepts improves student comprehension of TVM principles. Therefore, the practical aspects of using continuous compounding are addressed next. However, it's important to remember that everyone's teaching style is different and what follows is just one possible way to incorporate continuous compounding into the curriculum. It is not necessarily the only way, or the best way, to put these ideas into practice in the classroom.

Our general approach is to use Figures 1-5 as aids to improve the student's understanding of broad TVM principles rather than a way to solve specific TVM problems. We do this primarily because our textbook addresses TVM in discrete compounding terms, so it would be awkward to show students how to solve problems using the continuous compounding version of the formulas while their textbook uses the discrete compounding version. An additional benefit is that financial calculators are designed to solve problems with discrete compounding, so textbook descriptions of how to use a financial calculator are directly analogous to classroom demonstrations of how to do the problems.

This is how we do it in practice. Following the general textbook practice, we start by showing students how to solve for the present value and future value of single cash flows with annual compounding. The single cash flow problems are then extended to non-annual compounding periods. Multiple cash flow problems with non-constant cash flows are demonstrated next. Then, in contrast to most textbooks, we introduce a growing annuity as a special case of multiple cash flows which can be easily solved for lengthy payment periods using the discrete version of the growing annuity formula. The advantage of this approach is that traditional annuities and perpetuities can then be seen as special cases of a growing annuity. Students are shown and asked to solve all of these types of problems on a financial calculator.

However, just because students can solve for the future value of an annuity or calculate a mortgage payment does not mean that they fully comprehend the power of compounding over time, or how much interest a borrower pays over the life of a 30-year loan. So after the students have mastered how to solve specific TVM problems, we use Figures 1-5 to drive these compounding and interest ideas home. But before showing the diagrams, we have the students calculate the numbers they will see in the diagrams using discrete compounding techniques. In Figure 2, for example, we have the students calculate the interest earned over the first three ten-year periods (the numbers 3, 11, and 25 in the figure), assuming a 5% interest rate with daily compounding. We then show Figure 2 but reveal the figure in ten-year increments starting with the first ten years. This allows the students to see that their calculations of the interest earned each ten year period approximately matches what is shown in the figure. The advantage of this pedagogical technique is that the figure gives an easily understood visual representation of how much the interest earned increases within each ten year period. This is a visually compelling illustration that demonstrates the power of compounding over time. Students may not fully understand the power of compounding if they've just calculated numbers and do not give any thought to their significance. A similar procedure is followed for the other figures.

Student Reaction

Immediately after the presentation of the diagrams we asked students in two separate sections to provide written responses to the following question in order to determine if the figures were helpful or not. The question asked was: Please explain how and/or why the continuous compounding diagrams did (or did not) help your understanding of TVM concepts. The results are shown in Table 2.

Table 2: Student Responses to the Question, "Please explain how and/or why the continuous compounding diagrams did (or did not) help your understanding of TVM concepts."

Class section	1	2
Class size	46	46
Responses		
Positive	32	13
"Didn't really help me"	4	6
Unclear	6	7

Before addressing the results shown in Table 2, a natural question to ask is why is the response rate in Section 2 so much lower than the response rate in Section 1? Section 1 students were asked to write their responses immediately following the presentation and before they left the classroom. There was not enough time at the end of class to do this in Section 2, so Section 2 students were asked to answer the question as a homework assignment. As the responses were not graded or required, Section 2 had a much lower response rate. We've taken this as a lesson learned for future classroom surveys.

All of the student quotes below are unedited in order to avoid misrepresenting what was written. Readers can draw their own conclusions with respect to grammar. Table 2 shows that the majority of students thought the figures were helpful. Many of the positive responses came from students who said they are visual learners. These students argued that the diagrams helped them to better understand the concepts being discussed. The following quotes are typical of this category of response, "The continuous compounding diagrams helped my understanding of TVM concepts because I am a visual learner. The graphs make things much easier for me to conceptualize in my head." Another student wrote, "The continuous compounding diagrams helped me understand the time value of money concept because it gave me a visual representation comparing the different rates side by side rather than just a number. Numbers alone can be harder to digest and visualize so seeing those did help." Some students said they understood TVM concepts before but that the diagrams clarified things for them. For example, another student wrote, "I understood it mostly before but I don't think I fully understood the great effect it had on TVM." A similar point is illustrated by the following quote, "Doubling the interest rate DOES NOT mean doubling interest paid." Two students found the diagrams so compelling that they will implement these ideas in their lives. One said, "For the rest of my life I will try to shorten loans and lengthen investment time, ha ha." Another wrote, "I plan to start investing early and plan to pay off loans early."

Of course, not all of the students thought the figures were helpful. Most of these students said they already knew the concepts or thought it was better to do the calculations themselves. For example, one wrote; "The continuous compounding diagrams didn't really help me at all because I have a pretty firm grasp of TVM concepts and the ability for it to grow." Another said, "They did not help me understand TVM concepts because, although I know how to read a graph, I understand TVM concepts better when I actually perform the problem." Two students found the diagrams confusing and one thought the figures reduced understanding. One said, "They confused me way too much. I thought I understood the concepts before and after those diagrams I feel lost." Unfortunately, this student did not come after class to ask for clarification. Another wrote, "I thought it was confusing. I didn't understand it."

The final category of response in Table 2 is referred to as an unclear response. An unclear response is one that could not be placed in either of the other two categories. A single example will suffice. This student wrote, "I think the diagrams helped in the sense of planning out your work and solving. But to truly understand how to solve the equation, I needed to figure out what the question was asking me. Then put my information in a diagram."

CONCLUSION

This paper describes how the inclusion of continuous compounding theory into finance pedagogy can improve student comprehension of important time value of money concepts. Student surveys after exposure to the ideas in this paper found that the majority of students thought the continuous compounding concepts described above did increase their understanding of the power of compounding and the very large amount of interest paid on long-term loans. We encourage finance professors to consider incorporating continuous compounding diagrams into their time value of money lesson plans.

Endnote

1. It is not the purpose of this paper to describe how this $3 interest amount is calculated. All of the numbers in Figures 1-5 are taken from (or can be calculated from) the data in Table 3. Table 3 is presented without explanation but the interested reader can find a complete description of how to do the calculations in Chapter 3 of Thomsen (1991).

REFERENCE

Thomsen, C. T. (1991). The Accounting Model (2nd ed.). Fresno, CA: BIP

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