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Upper Record Values from Extended Exponential Distribution

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Some recurrence relations are established for the single and product moments of upper record values for the extended exponential distribution by Nadarajah and Haghighi (2011) as an alternative to the gamma, Weibull, and the exponentiated exponential distributions. Recurrence relations for negative moments and quotient moments of upper record values are also obtained. Using relations of single moments and product moments, means, variances, and covariances of upper record values from samples of sizes up to 10 are tabulated for various values of the shape parameter and scale parameter. A characterization of this distribution based on conditional moments of record values is presented.

Keywords: Record values, single moment, product moment, recurrence relations, extended exponential distribution, conditional expectation, characterization

Introduction

Record values are used in many statistical applications, statistical modeling and inference involving data pertaining to weather, athletic events, economics, life testing studies, and so on. Examples from *Guinness World Records* include the fastest time taken to recite the periodic table of the elements, shortest ever tennis matches both in terms of number of games and duration of time, or fastest indoor marathon. Several attempts are taken to make records, but records are made only when the attempt is a success. Usually, there is no data on all of the attempts made to break the record. Regarding the distributional properties of record-breaking data see Chandler (1952), Resnick (1973), Shorrock (1973), Glick (1978), Nevzorov (1987), Ahsanullah (1995), Balakrishnan and Ahsanullah (1993, 1994), Grunzień and Szynal (1997), and Arnold, Balakrishnan, and Nagaraja (1992, 1998).

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The extended exponential has a decreasing probability function like an exponential distribution, but its mode at zero. It also allows for increasing, decreasing, and constant hazard rates like a Weibull distribution or an exponentiated exponential distribution. This distribution has explicit expressions of its survival function and failure rate function (see Nadarajah & Haghighi, 2011).

A random variable X is said to have an extended exponential distribution if its probability density function (pdf) is of the form

$$\mathrm{f}\left(x;\alpha,\lambda\right)=\alpha\lambda\left(1+\lambda x\right)^{\alpha-1}\mathrm{e}^{1-\left(1+\lambda x\right)^{\alpha}},\quad x>0,\alpha,\lambda>0\,, \tag{1}$$

and its corresponding cumulative distribution function (cdf) is

$$\mathrm{F}\left(x;\alpha,\lambda\right)=1-\mathrm{e}^{1-\left(1+\lambda x\right)^{\alpha}},\quad x>0,\alpha,\lambda>0\,. \tag{2}$$

The hazard rate function is given by

$$\mathrm{h}\left(x;a,\lambda\right)=\alpha\lambda\left(1+\lambda x\right)^{\alpha-1},\quad x>0,\alpha,\lambda>0\,. \tag{3}$$

The properties of this distribution were studied by Nadarajah and Haghighi (2011). They obtained moment and maximum likelihood estimators of the distribution. This distribution is a particular member of the three-parameter generalized power Weibull distribution, introduced by Nikulin and Haghighi (2006). This distribution is a special case of the Gurvich, Dibenedetto, and Rande (1997) class as $\mathrm{F}(x) = 1 - \exp(-a\mathrm{G}(x))$, where $\mathrm{G}(x)$ is a monotonically increasing function of x with the only limitation $\mathrm{G}(x) \geq 0$.

The concept of recurrence relations has its own importance. They are useful in reducing the number of operations necessary to obtain a general form for the function under consideration. Furthermore, they are used in characterizing distributions, which is an important area and permits the identification of the population distribution from the properties of the sample. Recurrence relations and identities have attained importance as they reduce the amount of direct computation, time, and labor. Recurrence relations for single and product moments of k record values from Weibull, Pareto, generalized Pareto, Burr, exponential, and Gumble distribution are derived by Pawlas and Szynal (1998, 1999, 2000). Kumar (2015, 2016), Kumar, Jain, and Gupta (2015), and Kumar, Kumar, Saran, and Jain (2017) established recurrence relations for moments of k^{th} record values for generalized

Rayleigh, Dagum, type-I generalized half logistic, and Kumaraswamy-Burr III distribution, respectively.

Some of the application areas of the extended exponential distribution include rainfall data analysis, earthquake frequency analysis, reliability, and survival analysis. In these and other application areas, the primary interest is in prediction of future events: what would be the magnitude of future rainfall, the magnitude of a future earthquake, etc. These predictions can be based on moments of record values.

Record Values and Preliminaries

Let $\{X_n, n \geq 1\}$ be a sequence of identically independently distributed (i.i.d.) random variables with cdf $\mathrm{F}(x)$ and pdf $\mathrm{f}(x)$. The j^{th} order statistic of a sample $(X_1, X_2, \ldots, X_n)$ is denoted by $X_{j:n}$. For fixed $k \geq 1$, we define the sequence $\{\mathrm{U}^{(k)}(n), n \geq 1\}$ of k^{th} lower record times of $X_1, X_2, \ldots$ as follows:

$$\mathrm{U}^{(k)}(1) = 1,$$
$$\mathrm{U}^{(k)}(n+1) = \min\left\{ j > \mathrm{U}^{(k)}(n),\, X_{j:j+k-1} > X_{\mathrm{U}^{(k)}(n):\mathrm{U}^{(k)}(n)+k-1} \right\}.$$

The sequences $\left\{Y_n^{(k)}, n \geq 1\right\}$ with $Y_n^{(k)} = X_{\mathrm{U}^{(k)}(n):\mathrm{U}^{(k)}(n)+k-1}$, $n = 1, 2, \ldots$, are called the sequences of k^{th} upper record values of $\{X_n, n \geq 1\}$. For convenience, we shall also take $Y_0^{(k)} = 0$. Note that for $k = 1$, we have $Y_1^{(1)} = X_{\mathrm{U}(n)}$, $n \geq 1$, i.e. record values of $\{X_n, n \geq 1\}$. The joint pdf of k^{th} upper record values $Y_1^{(k)}, Y_2^{(k)}, \ldots, Y_n^{(k)}$ can be given as the joint pdf of k^{th} upper record values of $\{X_n, n \geq 1\}$ by

$$\mathrm{f}_{Y_1^{(k)}, Y_2^{(k)}, \ldots, Y_n^{(k)}}(y_1, y_2, \ldots, y_n) = k^n \left(\prod_{i=1}^{n-1} \frac{\mathrm{f}(y_i)}{\bar{\mathrm{F}}(y_i)} \right) \left[\bar{\mathrm{F}}(y_n)\right]^{k-1} \mathrm{f}(y_n), \quad y_1 > y_2 > \ldots > y_n,$$

where $\bar{\mathrm{F}}(x) = 1 - \mathrm{F}(x)$. In view of above equation, the marginal pdf of $X_n^{(k)}$, $n \geq 1$, is given by

$$\mathrm{f}_{X_n^{(k)}}(x) = \frac{k^n}{(n-1)!} \left[-\ln\left(\bar{\mathrm{F}}(x)\right)\right]^{n-1} \left[\bar{\mathrm{F}}(x)\right]^{k-1} \mathrm{f}(x), \tag{4}$$

and the joint pdf of $X_m^{(k)}$ and $X_n^{(k)}$, $1 \le m < n$, $n \ge 1$, is given by

$$\begin{aligned} \mathrm{f}_{X_m^{(k)},X_n^{(k)}}(x,y) = {} & \frac{k^n}{(m-1)!(n-m-1)!}\left[-\ln\left(\bar{\mathrm{F}}(x)\right)\right]^{m-1} \\ & \times\left[-\ln\left(\bar{\mathrm{F}}(y)\right)+\ln\left(\bar{\mathrm{F}}(y)\right)\right]^{n-m-1}\left[\bar{\mathrm{F}}(y)\right]^{k-1}\frac{\mathrm{f}(x)}{\bar{\mathrm{F}}(x)}\mathrm{f}(y) \end{aligned} \tag{5}$$

for $x > y$. Denote

$$\mu_{n:k}^{(r)} = \mathrm{E}\left(\left(X_n^{(k)}\right)^r\right) = \int_{-\infty}^{\infty} x^r\, \mathrm{f}_{X_n^{(k)}}(x)\,dx, \quad r,n = 1,2,\ldots$$

$$\mu_{m,n:k}^{(r,s)} = \mathrm{E}\left(\left(X_m^{(k)}\right)^r\left(X_n^{(k)}\right)^s\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{x} x^r y^s\, \mathrm{f}_{X_m^{(k)},X_n^{(k)}}(x,y)\,dydx,$$

$$1 \le m \le n-1, r,s = 1,2,\ldots$$

$$\mu_{m,n:k}^{(r,0)} = \mathrm{E}\left(\left(X_m^{(k)}\right)^r\right) = \mu_{m:k}^{(r)}, \quad r = 1,2,\ldots$$

$$\mu_{m,n:k}^{(0,s)} = \mathrm{E}\left(\left(X_n^{(k)}\right)^s\right) = \mu_{n:k}^{(s)}, \quad r = 1,2,\ldots$$

Relations for Single and Product Moments of Record Values

Relations for Single Moments

For the extended exponential distribution in (1),

$$\mathrm{f}(x;\alpha,\lambda) = \alpha\sum_{u=0}^{\alpha-1}\binom{\alpha-1}{u}\lambda^{u+1}x^u\,\bar{\mathrm{F}}(x;\alpha,\lambda). \tag{6}$$

The relation in (6) will be exploited to derive recurrence relations for the moments of record values for the extended exponential distribution. Establish the explicit expression for single moment of k^{th} record values $\mathrm{E}\left(\left(X_n^{(k)}\right)^r\right)$. Using (4),

$$\mu_{n:k}^{(r)} = \frac{k^n}{(n-1)!}\int_0^\infty x^r \left[-\ln\left(\bar{\mathrm{F}}(x)\right)\right]^{n-1}\left[\bar{\mathrm{F}}(x)\right]^{k-1} \mathrm{f}(x)\,dx. \tag{7}$$

By setting $t = -\ln[\bar{\mathrm{F}}(x)]$ in (7) and simplifying,

$$\mu_{n:k}^{(r)} = \frac{k^n}{\lambda^r (n-1)!}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^{p+r}\,\Gamma\left(\frac{p}{\alpha}+1\right)\Gamma(r+1)\Gamma(n+q)}{k^q\,p!\,q!\,\Gamma\left(\frac{p}{\alpha}+1-q\right)\Gamma(r+1-q)}, \tag{8}$$

where $\Gamma(a)$ denotes the complete gamma function defined by $\Gamma(a) = \int_0^\infty t^{a-1}\mathrm{e}^{-t}dt$. Specially, the first moment (mean) of the n^{th} record value is

$$\mu_n = \mu_n^{(1)} = \varphi(n,\alpha,\lambda,1), \tag{9}$$

where

$$\varphi(n,\alpha,\lambda,a) = \frac{1}{\lambda^a (n-1)!}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^{p+a}\,\Gamma\left(\frac{p}{\alpha}+1\right)\Gamma(a+1)\Gamma(n+q)}{k^q\,p!\,q!\,\Gamma\left(\frac{p}{\alpha}+1-q\right)\Gamma(a+1-q)}.$$

In addition, the variance of U_n is

$$\sigma_n^2 = \mu_n^{(2)} - (\mu_n)^2 = \varphi(n,\alpha,\lambda,2) - \left[\varphi(n,\alpha,\lambda,1)\right]^2. \tag{10}$$

Remark 1. For $k = 1$ in (8), deduce the explicit expression for single moments of upper record values for the extended exponential distribution.

Recurrence relations for single moments of k^{th} upper record values for cdf (2) are derived in the following theorem:

Theorem 1. For the extended exponential distribution given in (1) with fixed parameters $\alpha, \lambda > 0$, $k, n = 1, 2, \ldots$, and the convention that $\mu_{n:k}^{(0)} = 1$ and $\mu_{0:k}^{(r)} = 0$,

$$\mu_{n:k}^{(r)} = \alpha k \sum_{u=0}^{\alpha-1} \binom{\alpha-1}{u} \frac{\lambda^{u+1}}{(r+u+1)} \left[\mu_{n:k}^{(r+u+1)} - \mu_{n-1:k}^{(r+u+1)} \right]. \tag{11}$$

Proof. For $n \geq 1$ and r = 0, 1, 2,…, from (4),

$$\mu_{n:k}^{(r)} = \frac{k^n}{\alpha(n-1)!} \sum_{u=0}^{\alpha-1} \binom{\alpha-1}{u} \lambda^{u+1} \int_0^\infty \left[-\ln\left(\bar{\mathrm{F}}(x)\right) \right]^{n-1} \left[\bar{\mathrm{F}}(x) \right]^k dx \tag{12}$$

Integrating by parts, using $[-\ln(\bar{\mathrm{F}}(x))]^{n-1}[\bar{\mathrm{F}}(x)]^k$ for differentiation and the rest of the integrand for integration, we get the relation given in (11).

Remark 2. Setting $k = 1$ in (11), we can deduce the recurrence relation for single moments of upper record values for the extended exponential distribution.

Relations for Product Moments

Using (5), the explicit expression for the product moments of k^{th} upper record values $\mu_{m,n:k}^{(r,s)}$ can be obtained as

$$\mu_{m,n:k}^{(r,s)} = \frac{k^n}{(m-1)!(n-m-1)!} \int_0^\infty x^r \left[-\ln\left(\bar{\mathrm{F}}(x)\right) \right]^{m-1} \frac{\mathrm{f}(x)}{\bar{\mathrm{F}}(x)} \mathrm{G}(x)\, dx, \tag{13}$$

where

$$\mathrm{G}(x) = \int_x^\infty y^s \left[-\ln\left(\bar{\mathrm{F}}(y)\right) + \ln\left(\bar{\mathrm{F}}(x)\right) \right]^{n-m-1} \left[\bar{\mathrm{F}}(y) \right]^{k-1} \mathrm{f}(y)\, dy. \tag{14}$$

By setting $w = -\ln(\bar{\mathrm{F}}(y)) + \ln(\bar{\mathrm{F}}(x))$ in (14), we obtain

$$\mathrm{G}(x) = \sum_{p=0}^\infty \sum_{q=0}^\infty \frac{(-1)^{p\left(\frac{1}{\alpha}+1\right)+q+s} \Gamma\left(\frac{p}{\alpha}+1\right) \Gamma(s+1) \Gamma(n+q) \left[\bar{\mathrm{F}}(x) \right]^k \left[-\ln \bar{\mathrm{F}}(x) \right]^q \Gamma(n-m+q)}{k^{n-m+q} p! q! \Gamma\left(\frac{p}{\alpha}+1-q\right) \Gamma(s+1-p) \lambda^2}$$

On substituting the above expression of G(x) into (13) and simplifying the resulting equation, obtain

$$\mu_{m,n:k}^{(r,s)} = \frac{1}{\lambda^{s+r}(m-1)!(n-m-1)!}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^{p\left(\frac{1}{\alpha}+1\right)+2q+s+r+u+m-1}}{p!q!u!\Gamma\left(\frac{p}{\alpha}+1-q\right)}$$
$$\times\frac{\Gamma\left(\frac{p}{\alpha}+1\right)\Gamma(s+1)\Gamma(r+1)\Gamma(n-m+q)}{k^{q-m}p!q!\Gamma(s+1-p)\Gamma(r+1-u)}\frac{\partial^{m+q-1}}{\partial k^{m+q-1}}\mathrm{B}\left(\frac{u}{\lambda}+1,k\right) \tag{15}$$

where $\mathrm{B}(a, b)$ denotes the complete beta function defined by $\mathrm{B}(a, b) = \Gamma(a)\Gamma(b) / \Gamma(a + b)$. As a check, put $s = 0$ in (15) and use (8), $\mu_{m,n:k}^{(r,0)} = \mu_n^{(r)}$.

For simplicity, we denote the $(1, 1)^{\text{th}}$ moment of U_m and U_n, which are also called the simple product moment of these records, by $\mu_{m,n}$. The simple product moments are used for evaluating the covariances:

$$\sigma_{m,n} = \operatorname{cov}(U_m, U_n) = \mu_{m,n} - \mu_m\mu_n .$$

Remark 3. Setting $k = 1$ in (15), deduce the explicit expression for product moments of record values from the extended exponential distribution.

Making use of (6), drive recurrence relations for product moments of k^{th} upper record values:

Theorem 2. For $1 \le m \le n - 2$ and $r, s = 0, 1, 2,\ldots$,

$$\mu_{m,n:k}^{(r,s)} = \alpha\sum_{u=0}^{\alpha-1}\binom{\alpha-1}{u}\frac{k\lambda^{u+1}}{(s+u+1)}\left[\mu_{m,n:k}^{(r,s+1)} - \mu_{m,n-1:k}^{(r,s+1)}\right], \tag{16}$$

and for $m \ge 2$ and $r, s = 0, 1, 2,\ldots$,

$$\mu_{m,m+1:k}^{(r,s)} = \alpha\sum_{u=0}^{\alpha-1}\binom{\alpha-1}{u}\frac{k\lambda^{u+1}}{(s+u+1)}\left[\mu_{m,m+1:k}^{(r,s+1)} - \mu_{m:k}^{(r+s+1)}\right]. \tag{17}$$

Proof. From equation (5) for $1 \le m \le n - 2$ and $r, s = 0, 1, 2,\ldots$, on using (6),

$$\mu_{m,n:k}^{(r,s)} = \frac{k^n}{(m-1)!(n-m-1)!}\int_0^{\infty} x^r\left[-\ln\left(\overline{\mathrm{F}}(x)\right)\right]^{m-1}\frac{\mathrm{f}(x)}{\overline{\mathrm{F}}(x)}\mathrm{I}(x)\,dx \tag{18}$$

where

$$\mathrm{I}(x)=\alpha\sum_{u=1}^{\alpha-1}\binom{\alpha-1}{u}\lambda^{u+1}\int_x^{\infty}y^{s+u}\left[-\ln\left(\bar{\mathrm{F}}(y)\right)+\ln\left(\bar{\mathrm{F}}(x)\right)\right]^{n-m-1}\left[\bar{\mathrm{F}}(y)\right]^{k}\mathrm{f}(y)dy.$$

Integrating I(x) by parts,

$$\begin{aligned}\mathrm{I}(x)=-\alpha\sum_{u=1}^{\alpha-1}\binom{\alpha-1}{u}\frac{\lambda^{u+1}}{(s+u+1)}\\ \times\Big\{(n-m-1)\int_x^{\infty}y^{s+u+1}\left[-\ln\left(\bar{\mathrm{F}}(y)\right)+\ln\left(\bar{\mathrm{F}}(x)\right)\right]^{n-m-2}\left[\bar{\mathrm{F}}(y)\right]^{k-1}\mathrm{f}(y)dy\\ +k\int_x^{\infty}y^{s+u+1}\left[-\ln\left(\bar{\mathrm{F}}(y)\right)+\ln\left(\bar{\mathrm{F}}(x)\right)\right]^{n-m-1}\left[\bar{\mathrm{F}}(y)\right]^{k-1}\mathrm{f}(y)dy\Big\}\end{aligned}$$

Substituting this expression in (18) and simplifying, it leads to (16). Proceeding in a similar manner for the case of $n = m + 1$, the recurrence relation (17) can be established.

Remark 4. Setting $k = 1$ in (16) and (17), deduce the recurrence relation for the product moments of upper record values from the extended exponential distribution.

Recurrence Relations for the Negative Moments

Let $X_{\mathrm{U}(1)}$, $X_{\mathrm{U}(1)}$,… be the sequence of the upper record values from the extended exponential distribution. The recurrence relation for the negative moments is given in the following theorem.

Theorem 3. For $n \geq 1$ and s > 1,

$$\mathrm{E}\left(\frac{1}{X_{\mathrm{U}(n)}^{s}}\right)=\alpha\sum_{u=0}^{\alpha-1}\binom{\alpha-1}{u}\frac{k\lambda^{u+1}}{(u-s+1)}\left\{\mathrm{E}\left(\frac{1}{X_{\mathrm{U}(n)}^{s-u-1}}\right)-\mathrm{E}\left(\frac{1}{X_{\mathrm{U}(n-1)}^{s-u-1}}\right)\right\}. \qquad (19)$$

Proof. From (3), and for $n \geq 1$ and s > 1,

$$E\left(\frac{1}{X_{U(n)}^{s}}\right)=\frac{k^{n}}{(n-1)!}\int_{x}^{\infty}x^{-s}\left[-\ln\left(\bar{F}(x)\right)\right]^{n-1}\left[\bar{F}(x)\right]^{k-1}f(x)\,dx$$
$$=\alpha\sum_{u=0}^{\alpha-1}\binom{\alpha-1}{u}\frac{k^{n}\lambda^{u+1}}{(n-1)!}\int_{x}^{\infty}x^{u-s}\left[-\ln\left(\bar{F}(x)\right)\right]^{n-1}\left[\bar{F}(x)\right]^{k}f(x)\,dx \tag{20}$$

Integrating (20) by parts and simplifying the resultant expression, the result given in (19) is obtained.

Remark 5. Setting $k = 1$ in (19), deduce the recurrence relation for the negative moments of upper record values from the extended exponential distribution.

Recurrence Relations for the Quotient Moments

Let $X_{U(1)}, X_{U(1)},\ldots$ be the sequence of the upper record values from the extended exponential distribution. Then the recurrence relation for the quotient moments is given by the following theorem:

Theorem 5. For $1 \le m \le n$ and for $r \ge 1$ and $s \ge 1$,

$$E\left(\frac{X_{U(m:k)}^{r}}{X_{U(n:k)}^{s}}\right)=\alpha\sum_{u=0}^{\alpha-1}\binom{\alpha-1}{u}\frac{k\lambda^{u+1}}{(u-s-1)}\left\{E\left(\frac{X_{U(m:k)}^{r}}{X_{U(n)}^{s-u+1}}\right)-E\left(\frac{X_{U(m:k)}^{r}}{X_{U(n-1)}^{s-u+1}}\right)\right\}. \tag{21}$$

Proof. From (5), and for $1 \le m \le n$, $r \ge 1$, and $s \ge 1$,

$$E\left(\frac{X_{U(m:k)}^{r}}{X_{U(n:k)}^{s}}\right)=\int_{0}^{\infty}\int_{x}^{\infty}\frac{x^{r}}{y^{2}}f_{X_{m}^{(k)},X_{n}^{(k)}}(x,y)\,dy\,dx$$
$$=\frac{k^{n}}{(m-1)!(n-m-1)!}\int_{0}^{\infty}x^{r}\left[-\ln\left(\bar{F}(x)\right)\right]^{m-1}\frac{f(x)}{\bar{F}(x)}h(x)\,dx \tag{22}$$

where

$$h(x)=\int_{x}^{\infty}y^{-s}\left[-\ln\left(\bar{F}(y)\right)+\ln\left(\bar{F}(x)\right)\right]^{n-m-1}\left[\bar{F}(y)\right]^{k}dy\,.$$

From the differential equation (6),

$$\mathrm{h}(x)=\alpha\sum_{u=0}^{\alpha-1}\binom{\alpha-1}{u}\lambda^{u+1}\int_x^\infty y^{u-s}\left[-\ln\left(\bar{\mathrm{F}}(y)\right)+\ln\left(\bar{\mathrm{F}}(x)\right)\right]^{n-m-1}\left[\bar{\mathrm{F}}(y)\right]^k dy \,. \quad (23)$$

Integrating (23) by parts and using (22), the result given in (21) is obtained.

Corollary 1. For $m \geq 1$, $r \geq 1$, and $s \geq 1$,

$$\mathrm{E}\left(\frac{X_{\mathrm{U}(m:k)}^{r}}{X_{\mathrm{U}(m+1:k)}^{s}}\right)=\alpha\sum_{u=0}^{\alpha-1}\binom{\alpha-1}{u}\frac{k\lambda^{u+1}}{(u-s-1)}\left\{\mathrm{E}\left(\frac{X_{\mathrm{U}(m:k)}^{r}}{X_{\mathrm{U}(m+1:k)}^{s-u+1}}\right)-\mathrm{E}\left(X_{\mathrm{U}(m:k)}^{(r+s-u-1)}\right)\right\}.$$

Proof. Upon substituting $n = m + 1$ in (21) and simplifying, obtain the quotient moment (21).

Remark 6. Setting $k = 1$ in (21), deduce the recurrence relation for the quotient moments of upper record values from the extended exponential distribution.

Characterization

For upper record values, let $\mathrm{L}(a, b)$ stand for the space of all integrable functions on (a, b). A sequence $(\mathrm{h}_n) \subset \mathrm{L}(a, b)$ is called complete on $\mathrm{L}(a, b)$ if, for all functions $\mathrm{g} \in \mathrm{L}(a, b)$, the condition

$$\int_a^b \mathrm{g}(x)\mathrm{h}_n(x)\,dx = 0, \quad n \in \mathrm{N},$$

implies $\mathrm{g}(x) = 0$ almost everywhere (a.e.) on (a, b). Start with the following result of Lin (1986, p. 595):

Proposition. Let n_0 be any fixed non-negative integer, $-\infty \leq a < b \leq \infty$, and $\mathrm{g}(x) \geq 0$ an absolutely continuous function with $\mathrm{g}'(x) \neq 0$ a.e. on (a, b). Then the sequence of functions $\{(\mathrm{g}(x))^n e^{-\mathrm{g}(x)}, n \geq n_0\}$ is complete on $\mathrm{L}(a, b)$ iff $\mathrm{g}(x)$ is strictly monotone on (a, b).

Using the above Proposition, a stronger version of Theorem 1 is obtained. Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. continuous random variables with cdf $\mathrm{F}(x)$ and pdf $\mathrm{f}(x)$. Let $X_{\mathrm{L}(n)}$ be the n^{th} upper record value; then the conditional pdf of $X_{\mathrm{U}(n)}$, given $X_{\mathrm{U}(m)} = x$ and in view of (4) and (5), is

$$\mathrm{f}\left(X_{\mathrm{U}(n)} \mid X_{\mathrm{U}(m)}=x\right)=\frac{1}{(n-m-1)!}\left[-\ln\bar{\mathrm{F}}(y)+\ln\bar{\mathrm{F}}(x)\right]^{n-m-1}\frac{\mathrm{f}(y)}{\bar{\mathrm{F}}(x)}. \qquad (24)$$

Theorem 5. Let X be an absolutely continuous random variable with cdf F(x) and pdf f(x) on the support (0, ∞). Then, for $m < n$,

$$\mathrm{E}\left[X_{\mathrm{U}(n)} \mid X_{\mathrm{U}(m)}=x\right]=\frac{1}{(n-m-1)!}\sum_{p=0}^{\infty}\frac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(n-m+p)}{p!\,\Gamma\left(\frac{1}{\alpha}+1-p\right)(1+\lambda x)^{\alpha p-1}}-1 \qquad (25)$$

if and only if $\mathrm{F}(x;\alpha,\lambda)=1-\mathrm{e}^{1-(1+\lambda x)^{\alpha}}$, $x > 0$, $\alpha, \lambda > 0$.

Proof. From (24),

$$\mathrm{E}\left[X_{\mathrm{U}(n)} \mid X_{\mathrm{U}(m)}=x\right]=\frac{1}{(n-m-1)!}\int_{x}^{\infty} y\left[\ln\left(\frac{\bar{\mathrm{F}}(x)}{\bar{\mathrm{F}}(y)}\right)\right]^{n-m-1}\frac{\mathrm{f}(y)}{\bar{\mathrm{F}}(x)}dy. \qquad (26)$$

By setting $t = \ln(\bar{\mathrm{F}}(x) / \bar{\mathrm{F}}(y))$ from (2) in (26),

$$\mathrm{E}\left[X_{\mathrm{U}(n)} \mid X_{\mathrm{U}(m)}=x\right]=\frac{1}{(n-m-1)!}\sum_{p=0}^{\infty}\frac{\Gamma\left(\frac{1}{\alpha}+1\right)}{p!\,\Gamma\left(\frac{1}{\alpha}+1-p\right)(1+\lambda x)^{\alpha p-1}}\times\int_{0}^{\infty}t^{n-m+p-1}\mathrm{e}^{-t}dt-\frac{1}{(n-m-1)!}\int_{0}^{\infty}t^{n-m+p-1}\mathrm{e}^{-t}dt$$

Simplifying the above expression, derive the relation given in (25). To prove the sufficient part, we have from (24) and (25)

$$\frac{1}{(n-m-1)!}\int_{x}^{\infty} y\left[-\ln\bar{\mathrm{F}}(y)+\ln\bar{\mathrm{F}}(x)\right]^{n-m-1}\mathrm{f}(y)dy=\bar{\mathrm{F}}(x)\mathrm{H}_{n-m}(x), \qquad (27)$$

where

$$\mathrm{H}_{n-m}(x)=\frac{1}{(n-m-1)!}\sum_{p=0}^{\infty}\frac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(n-m+p)}{p!\Gamma\left(\frac{1}{\alpha}+1-p\right)(1+\lambda x)^{\alpha p-1}}-1.$$

Differentiating (27) both sides with respect to *x*,

$$\frac{\mathrm{f}(x;\alpha,\lambda)}{\overline{\mathrm{F}}(x;\alpha,\lambda)}=\frac{\mathrm{H}'_{n-m}(x)}{\left[\mathrm{H}_{n-m-1}(x)-\mathrm{H}_{n-m}(x)\right]}=\alpha\lambda(1+\lambda x)^{\alpha-1},$$

which proves $\mathrm{F}(x;\alpha,\lambda)=1-\mathrm{e}^{1-(1+\lambda x)^{\alpha}}$, $x > 0$, α, $\lambda > 0$.

Theorem 6. Let X be a non-negative random variable having an absolutely continuous distribution function F(x) with F(0) = 0 and 0 < F(x) < 1 for all $x > 0$. Then

$$\mu_{n:k}^{(r)}=\alpha k\sum_{u=0}^{\alpha-1}\binom{\alpha-1}{u}\frac{\lambda^{u+1}}{(r+u+1)}\left[\mu_{n:k}^{(r+u+1)}-\mu_{n-1:k}^{(r+u+1)}\right] \tag{28}$$

if and only if $\mathrm{F}(x;\alpha,\lambda)=1-\mathrm{e}^{1-(1+\lambda x)^{\alpha}}$, $x > 0$, α, $\lambda > 0$.

Proof. The necessary part follows immediately from equation (11). However, if the recurrence relation in equation (28) is satisfied, then on using equation (4),

$$\begin{aligned}
&\frac{k^n}{(n-1)!}\int_0^{\infty}x^r\left[\overline{\mathrm{F}}(x)\right]^{k-1}\left[-\ln\overline{\mathrm{F}}(x)\right]^{n-1}\mathrm{f}(x)\,dx\\
&=-\alpha\sum_{u=0}^{\alpha-1}\binom{\alpha-1}{u}\frac{k^{n+1}\lambda^{u+1}}{(r+u+1)(n-2)!}\\
&\qquad\times\int_0^{\infty}x^{r+u+1}\left[\overline{\mathrm{F}}(x)\right]^{k-1}\left[-\ln\overline{\mathrm{F}}(x)\right]^{n-2}\mathrm{f}(x)\,dx\\
&+\alpha\sum_{u=0}^{\alpha-1}\binom{\alpha-1}{u}\frac{k^{n+1}\lambda^{u+1}}{(r+u+1)(n-1)!}\int_0^{\infty}x^{r+u+1}\left[\overline{\mathrm{F}}(x)\right]^{k-1}\left[-\ln\overline{\mathrm{F}}(x)\right]^{n-1}\mathrm{f}(x)\,dx
\end{aligned} \tag{29}$$

Integrating by parts, the first integral on the right-hand side of equation (29), and simplifying the resulting expression

$$\frac{\mathrm{f}(x)}{\overline{\mathrm{F}}(x)} = \alpha \sum_{u=0}^{\alpha-1} \binom{\alpha-1}{u} \lambda^{u+1} x^{u} ,$$

which proves

$$\mathrm{F}(x;\alpha,\lambda) = 1 - \mathrm{e}^{1-(1+\lambda x)^{\alpha}}, \quad x > 0, \alpha, \lambda > 0 .$$

Numerical Results

In Table 1, the values of means for $n = 1, 2, \ldots, 10$, $\alpha = 1\ (1)\ 3$, and $\lambda = 0.5$, 1.0 are presented. Observe the means are decreasing with respect to n but increasing with respect to α. In Table 2, the variances and covariances for different of values m and n and for $\alpha = 1\ (1)\ 3$ and $\lambda = 0.5$, 1.0 are reported. Observe the variances and covariances are decreases with respect to both α and λ.

Table 1. Means of record statistics from (8) for $k = 1$

	$\lambda = 0.5$			$\lambda = 1.0$		
n	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
1	0.895563	1.791126	2.686689	0.352834	0.705668	1.058502
2	0.664068	1.328135	1.992203	0.199384	0.398769	0.598153
3	0.477761	0.955521	1.433282	0.107730	0.215459	0.323189
4	0.336661	0.673323	1.009984	0.056642	0.113284	0.169926
5	0.233775	0.467551	0.701326	0.029272	0.058543	0.087815
6	0.160622	0.321244	0.481866	0.014959	0.029918	0.044877
7	0.029140	0.219019	0.328529	0.007589	0.015178	0.022767
8	0.074238	0.148476	0.222713	0.003831	0.007663	0.011494
9	0.050114	0.100229	0.150343	0.001928	0.003856	0.005784
10	0.033724	0.067447	0.101171	0.000968	0.001936	0.002905

Table 2. Variances and covariances of record statistics

		$\lambda = 0.5$			$\lambda = 1.0$		
m	n	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
1	1	9.028346	8.315561	6.775821	3.648953	2.078890	1.693955
1	2	5.267103	4.930445	4.218403	2.055083	1.232611	1.054601
1	3	4.134393	3.916293	3.452603	1.060843	0.979073	0.863151
1	4	3.986392	3.479149	3.118780	0.965401	0.869787	0.779695
1	5	3.749502	3.261027	2.946530	0.957263	0.815257	0.736633
1	6	3.520278	3.143586	2.848129	0.893164	0.785897	0.712032
1	7	3.319361	3.077308	2.787888	0.820892	0.769327	0.696972
1	8	3.125013	3.038634	2.749184	0.812618	0.759659	0.687296
1	9	3.107148	3.015473	2.723474	0.806329	0.753868	0.680869
2	2	1.212775	1.591575	0.963569	0.303194	0.397894	0.240892
2	3	1.353388	1.035388	0.718258	0.338347	0.258847	0.179560
2	4	1.392593	0.864081	0.634816	0.348148	0.216020	0.158704
2	5	1.407665	0.790269	0.597443	0.351916	0.197567	0.149361
2	6	1.414247	0.754026	0.578157	0.353562	0.188506	0.144539
2	7	1.417306	0.735006	0.567199	0.354326	0.183751	0.141800
2	8	1.418774	0.724603	0.560513	0.354694	0.181151	0.140128
2	9	1.419492	0.718736	0.556216	0.354873	0.179684	0.139054
3	3	0.071090	0.084961	0.044993	0.284359	0.339844	0.179972
3	4	0.297832	0.258112	0.153716	0.074458	0.064528	0.038429
3	5	0.302410	0.228511	0.143493	0.075603	0.057128	0.035873
3	6	0.304325	0.215197	0.138739	0.076081	0.053799	0.034685
3	7	0.305199	0.208620	0.136260	0.076300	0.052155	0.034065
3	8	0.305616	0.205204	0.134846	0.076404	0.051301	0.033712
3	9	0.305819	0.203369	0.133981	0.076455	0.050842	0.033495
4	4	0.091612	0.083490	0.040061	0.022903	0.020873	0.010015
4	5	0.093130	0.071017	0.036969	0.023282	0.017754	0.009242
4	6	0.093731	0.065853	0.035655	0.023433	0.016463	0.008914
4	7	0.093999	0.063425	0.035024	0.023500	0.015856	0.008756
4	8	0.094126	0.062212	0.034689	0.023531	0.015553	0.008672
4	9	0.094187	0.061584	0.034490	0.023547	0.015396	0.008624
5	5	0.033398	0.022933	0.009890	0.008349	0.005733	0.002472
5	6	0.033594	0.020897	0.009503	0.008399	0.005224	0.002376
5	7	0.033680	0.019980	0.009330	0.008420	0.004995	0.002332
5	8	0.033720	0.019536	0.009244	0.008430	0.004884	0.002311
5	9	0.033739	0.019311	0.009198	0.008435	0.004828	0.002299
6	6	0.013079	0.006776	0.002589	0.003270	0.001694	0.000647
6	7	0.013107	0.006428	0.002539	0.003277	0.001607	0.000635
6	8	0.013120	0.006263	0.002516	0.003280	0.001566	0.000629
6	9	0.013126	0.006181	0.002504	0.003281	0.001545	0.000626
7	7	0.005369	0.002096	0.000701	0.001342	0.000524	0.000175
7	8	0.005373	0.002034	0.000694	0.001343	0.000509	0.000174
7	9	0.005369	0.002004	0.000691	0.001344	0.000501	0.000173
8	8	0.002278	0.000667	0.000193	0.000569	0.000167	0.000048
8	9	0.002258	0.000656	0.000183	0.000570	0.000164	0.000043

Conclusion

The upper record values from the extended exponential model were considered, and exact explicit expressions were obtained, as well as recurrence relations for the single and product moments of record values. Recurrence relations for negative moments and quotient moments of upper record values were also obtained. The recurrence relations obtained in the paper allows us to evaluate the means, variances, and covariances of all upper record values for all sample sizes in a simple recursive manner.

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