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A New Generalized Family of Distributions for Lifetime Data

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A new class of continuous distributions called the generalized Burr X-G family is introduced. Some special models of the new family are provided. Some of its mathematical properties including explicit expressions for the quantile and generating functions, ordinary and incomplete moments, order statistics and Rényi entropy are derived. The maximum likelihood is used for estimating the model parameters. The flexibility of the generated family is illustrated by means of two applications to real data sets.

Keywords: Burr X distribution, generating function, maximum likelihood, order statistic, Rényi entropy

Introduction

The statistics literature contains many generalized families of distributions which have been proposed through introducing one or more shape parameters to the classical distributions. These families have been extensively and used in modeling data in several applied areas such as engineering, economics, medical sciences and finance, among others. However, applied areas such as finance, lifetime analysis, and insurance require more flexible forms of these distributions.

Gupta et al. (1998) defined the exponentiated-G (exp-G) family by raising the cumulative distribution function (CDF) to a positive power parameter. Some other classes are: the beta-G class presented by Eugene et al. (2002), the Kumaraswamy-G family introduced by Cordeiro and de Castro (2011), exponentiated generalized-G family proposed by Cordeiro et al. (2013), the Lomax generator due to Cordeiro et al. (2014), the beta transmuted-H family studied by Afify, Yousof, and Nadarajah (2017) and the exponentiated Weibull-H family due to Cordeiro, Afify, Yousof, Pescim, and Aryal (2017).

A new family of distributions is introduced based on the Burr X (BX) distribution. Based on the T-X family pioneered by Alzaatreh et al. (2013), we construct a new generator so-called the generalized Burr X-G (GBX-G) family and give a comprehensive description of some of its mathematical properties.

The probability density function (PDF) and CDF of the BX distribution are given (for $x > 0$) by

$$g(x) = 2\beta x \exp(-x^2) [1 - \exp(-x^2)]^{\beta-1} \quad \text{and} \quad G(x) = [1 - \exp(-x^2)]^\beta,$$

respectively, where β is a positive shape parameter.

Let $g(x; \boldsymbol{\varphi})$ and $G(x; \boldsymbol{\varphi})$ denote the PDF and CDF of the baseline model with parameter vector $\boldsymbol{\varphi}$. Then, the CDF of the GBX-G family becomes

$$\begin{aligned} F(x; \alpha, \beta, \boldsymbol{\varphi}) &= 2\beta \int_0^{\frac{G(x; \boldsymbol{\varphi})^\alpha}{1-G(x; \boldsymbol{\varphi})^\alpha}} t \exp(-t^2) [1 - \exp(-t^2)]^{\beta-1} dt \\ &= \left\{ 1 - \exp \left[- \left(\frac{G(x; \boldsymbol{\varphi})^\alpha}{1-G(x; \boldsymbol{\varphi})^\alpha} \right)^2 \right] \right\}^\beta. \end{aligned} \quad (1)$$

The PDF of the GBX-G is given by

$$\begin{aligned} f(x; \alpha, \beta, \boldsymbol{\varphi}) &= \frac{2\alpha\beta g(x; \boldsymbol{\varphi}) G(x; \boldsymbol{\varphi})^{2\alpha-1}}{[1-G(x; \boldsymbol{\varphi})^\alpha]^3} \exp \left\{ - \left[\frac{G(x; \boldsymbol{\varphi})^\alpha}{1-G(x; \boldsymbol{\varphi})^\alpha} \right]^2 \right\} \\ &\quad \times \left\{ 1 - \exp \left[- \left(\frac{G(x; \boldsymbol{\varphi})^\alpha}{1-G(x; \boldsymbol{\varphi})^\alpha} \right)^2 \right] \right\}^{\beta-1} \end{aligned} \quad (2)$$

where α and β are positive shape parameters. For $\alpha = 1$, the GBX-G family reduces to the Burr X-G (BX-G) class proposed by Yousof et al. (2017). The two additional parameters can allow us to study the tail behavior of the density (2) with more flexibility. Further, the GBX-G class due to its flexibility in accommodating all forms of the hazard rate function (HRF), as shown in the next section, becomes an important class to be used in many applications.

The GBX-G Special Models

The GBX-Exponential (GBXEx) Distribution

Consider the PDF $g(x) = ae^{-ax}$, for $x > 0$, of the exponential (Ex) distribution with positive parameter a . Then, the PDF of the GBXEx model is given by

$$f(x) = \frac{2\alpha\beta a e^{-ax} (1-e^{-ax})^{2\alpha-1}}{\left[1-(1-e^{-ax})^\alpha\right]^3} \exp\left\{-\left[\frac{(1-e^{-ax})^\alpha}{1-(1-e^{-ax})^\alpha}\right]^2\right\} \times \left(1 - \exp\left\{-\left[\frac{(1-e^{-ax})^\alpha}{1-(1-e^{-ax})^\alpha}\right]^2\right\}\right)^{\beta-1}$$

For $\alpha = 1$, the GBXEx model reduces to the BXEx model with parameters β and a . The PDF and HRF plots of the GBXEx model are shown in Figure 1. These plots reveal that the PDF of this model can be reversed J-shape, symmetric, unimodal, left skewed or right skewed. The HRF of the GBXEx distribution can be J-shape, reversed J-shape, decreasing (DFR), increasing (IFR) or bathtub (BT) failure rate shapes.

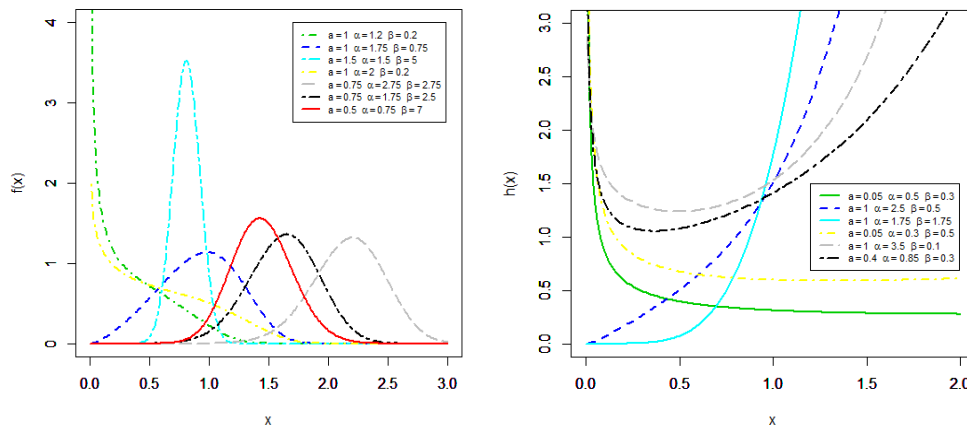


Figure 1. Plots of the PDF and HRF of the GBXEx distribution

The GBX-Weibull (GBXW) Distribution

Consider the PDF $g(x) = abx^{b-1}e^{-ax^b}$, for $x > 0$, of the Weibull (W) distribution with positive parameters a and b . Then, the PDF of the GBXW model is given by

$$f(x) = \frac{2\alpha\beta abx^{b-1}e^{-ax^b}(1-e^{-ax^b})^{2\alpha-1}}{\left[1-G(x;\varphi)^\alpha\right]^3} \exp\left\{-\left[\frac{(1-e^{-ax^b})^\alpha}{1-(1-e^{-ax^b})^\alpha}\right]^2\right\} \\ \times \left(1 - \exp\left\{-\left[\frac{(1-e^{-ax^b})^\alpha}{1-(1-e^{-ax^b})^\alpha}\right]^2\right\}\right)^{\beta-1}$$

Some special cases of the GBXW distribution are: the GBXEx distribution for $b = 1$; we obtain the the GBX-Rayleigh (GBXR) distribution for $\beta = 2$; the BXW distribution (Yousof et al., 2017) for $\alpha = 1$; the BXEx distribution for $\alpha = b = 1$; and the BXR model for $\alpha = 1$ and $\beta = 2$. The plots in Figure 2 show some possible shapes of the PDF and HRF of the GBXW distribution. These plots in Figure 2 show that the PDF of the GBXW can be reversed J-shape, symmetric, left skewed, right skewed, or unimodal shapes. One can see that, the HRF can be J-shape, reversed J-shape, DFR, IFR or BT failure rate shapes.

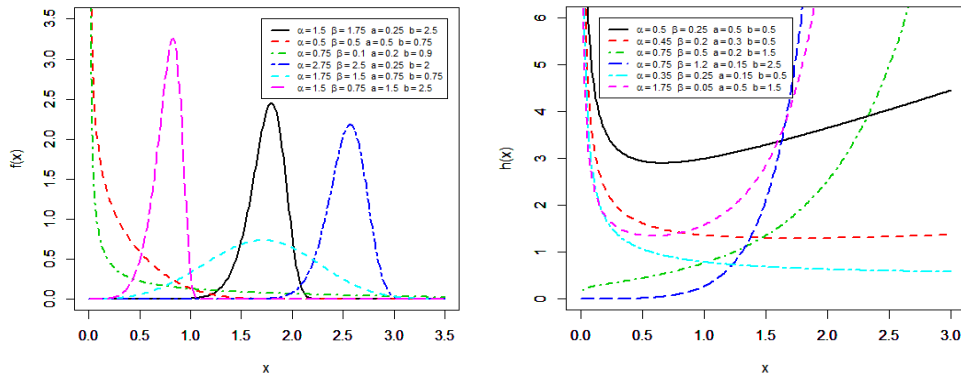


Figure 2. Plots of the PDF and HRF of the GBXW distribution

The GBX-Lomax (GBXLx) Distribution

The PDF of the Lomax (Lx) distribution with positive parameters a and b is $g(x) = (a/b)[1+x/b]^{-a-1}$ for $x > 0$. Then, the PDF of the GBXLx distribution becomes

$$f(x) = \frac{2\alpha\beta a \left[1 - \left(1 - [1+x/b]^{-a}\right)^\alpha\right]^{-3}}{b[1+x/b]^{a+1} \left(1 - [1+x/b]^{-a}\right)^{1-2\alpha}} \exp\left\{-\left[\frac{\left(1 - [1+x/b]^{-a}\right)^\alpha}{1 - \left(1 - [1+x/b]^{-a}\right)^\alpha}\right]^2\right\} \\ \times \left(1 - \exp\left\{-\left[\frac{\left(1 - [1+x/b]^{-a}\right)^\alpha}{1 - \left(1 - [1+x/b]^{-a}\right)^\alpha}\right]^2\right\}\right)^{\beta-1}$$

Plots of the density function of the GBXLx distribution are displayed in Figure 3. Figure 3 shows that the PDF of this model can be reversed J-shape, symmetric, concave down, unimodal, right skewed, or left skewed. The GBXLx HRF can be DFR, IFR, upside down bathtub (UBT), or BT failure rate shapes.

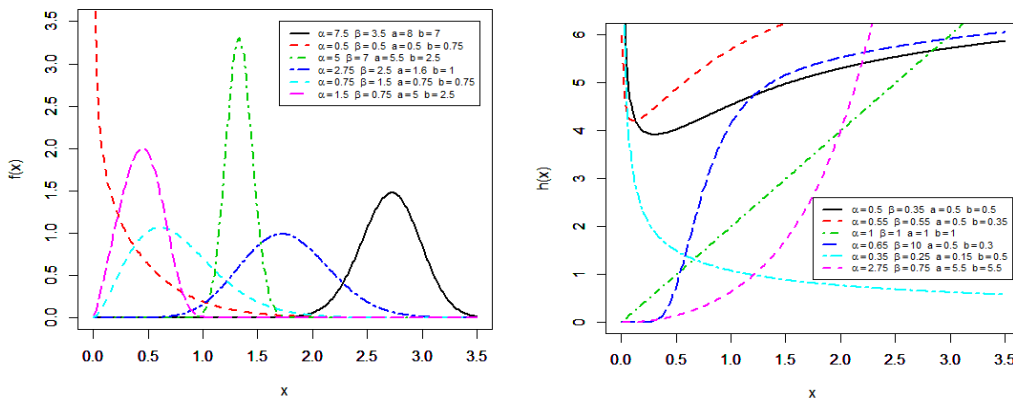


Figure 3. Plots of the PDF and HRF of the GBXLx distribution

Properties of the GBX-G Family

Linear Representation

A useful representation for the GBX-G PDF will now be presented. Consider the power series

$$(1-z)^q = \sum_{l=0}^{\infty} (-1)^l \binom{q}{l} z^l. \quad (3)$$

By applying (3) to (2),

$$f(x) = \frac{2\alpha\beta g(x;\boldsymbol{\varphi})G(x;\boldsymbol{\varphi})^{2\alpha-1}}{\left[1-G(x;\boldsymbol{\varphi})^\alpha\right]^3} \sum_{l=0}^{\infty} (-1)^l \binom{\beta-1}{l} \exp\left[-\frac{(l+1)G(x;\boldsymbol{\varphi})^{2\alpha}}{\left[1-G(x;\boldsymbol{\varphi})^\alpha\right]^2}\right]. \quad (4)$$

Applying the exponential series to Equation (4),

$$f(x) = 2\alpha\beta g(x;\boldsymbol{\varphi}) \sum_{l,j=0}^{\infty} \frac{(-1)^{l+j}}{j!(l+1)^{-j}} \binom{\beta-1}{l} \frac{G(x;\boldsymbol{\varphi})^{2\alpha(j+1)-1}}{\left[1-G(x;\boldsymbol{\varphi})^\alpha\right]^{2j+3}}.$$

Consider the series expansion

$$(1-z)^q = \sum_{k=0}^{\infty} (-1)^k \binom{-q}{k} z^k. \quad (5)$$

Applying the series expansion (5) to Equation (4) gives

$$f(x) = 2\alpha\beta g(x;\boldsymbol{\varphi}) \sum_{l,j,k=0}^{\infty} \frac{(-1)^{l+j+k}}{j!(l+1)^{-j}} \binom{\beta-1}{l} \binom{-2j-3}{k} G(x;\boldsymbol{\varphi})^{\alpha(k+2j+2)-1}.$$

Thus, the PDF of the GBX-G class reduces to

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$$f(x) = \sum_{j,k=0}^{\infty} d_{j,k} h_{\alpha(k+2j+2)}(x), \quad (6)$$

where $h_{\alpha}(x) = \alpha g(x)G(x)^{\alpha-1}$ is the exp-G PDF with power parameter α and

$$d_{j,k} = 2\beta \sum_{l=0}^{\infty} \frac{(-1)^{l+j+k} (l+1)^j (\beta-1) \binom{-2j-3}{k}}{j!(k+2j+2) \binom{l}{k}}.$$

Based on Equation (6), several mathematical properties of the GBX-G family can be derived from those of the exp-G family.

Ordinary and Incomplete Moments

Let $Y_{\alpha(k+2j+2)}$ denote a random variable having the exp-G distribution with power parameter $\alpha(k+2j+2)$.

The r^{th} moment of X follows from (6) as

$$\mu'_r = E(X^r) = \sum_{j,k=0}^{\infty} d_{j,k} E(Y_{\alpha(k+2j+2)}^r).$$

The s^{th} incomplete moment of X can be derived from (6) as

$$m_s(z) = \int_{-\infty}^z x^s f(x) dx = \sum_{j,k=0}^{\infty} d_{j,k} \int_{-\infty}^z x^s h_{\alpha(k+2j+2)}(x) dx. \quad (7)$$

The first incomplete moment of X can be obtained from (7) as

$$m_1(z) = \sum_{j,k=0}^{\infty} d_{j,k} J_{\alpha(k+2j+2)}(z),$$

where

$$J_{\alpha(k+2j+2)}(z) = \int_{-\infty}^z x^s h_{\alpha(k+2j+2)}(x) dx$$

is the first incomplete moment of the exp-G distribution. It can be computed analytically or numerically from the baseline quantile function (QF) $Q_G(u) = G^{-1}(u)$.

Quantile and Generating Functions

The QF of the GBX-G distribution follows, by inverting (1), as

$$Q(u) = Q_G(u) \left\{ \frac{[-\ln(1-u^{1/\beta})]^{1/2}}{1+[-\ln(1-u^{1/\beta})]^{1/2}} \right\}^{\frac{1}{\alpha}}, \quad (8)$$

where $Q_G(u) = G^{-1}(u)$ is the QF of the baseline G distribution and $u \in (0, 1)$.

The moment generating function (MGF) of the GBX-G class follows from Equation (6) as

$$\begin{aligned} M(t) &= \sum_{j,k=0}^{\infty} d_{j,k} M_{\alpha(k+2j+2)}(t) \\ &= \sum_{j,k=0}^{\infty} \alpha(k+2j+2) d_{j,k} \tau(t, \alpha(k+2j+2)-1) \end{aligned}$$

where $M_{\alpha(k+2j+2)}(t)$ is the MGF of $Y_{\alpha(k+2j+2)}$ and

$$\tau(t, k+j+\alpha-1) = \int_0^1 \exp[t Q_G(u)] u^{\alpha(k+2j+2)-1} du.$$

Then, the MGF of X can be derived from the exp-G MGF.

Order statistics

Order statistics are important in several areas of statistical theory and practice. Let X_1, \dots, X_n be a random sample from the GBX-G class. The PDF of $X_{i:n}$ is given by

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(x)^{j+i-1}. \quad (9)$$

where $B(\cdot, \cdot)$ is the beta function. Based on Equations (1) and (2), one can write

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$$f(x)F(x)^{j+i-1} = \frac{2\alpha\beta g(x;\varphi)G(x;\varphi)^{2\alpha-1}}{\left[1-G(x;\varphi)^\alpha\right]^3} \exp\left\{-\left[\frac{G(x;\varphi)^\alpha}{1-G(x;\varphi)^\alpha}\right]^2\right\} \\ \times \left\{1 - \exp\left[-\left(\frac{G(x;\varphi)^\alpha}{1-G(x;\varphi)^\alpha}\right)^2\right]\right\}^{\beta(j+i)-1}$$

Applying the power series (3), the above equation becomes

$$f(x)F(x)^{j+i-1} = \frac{2\alpha\beta g(x;\varphi)G(x;\varphi)^{2\alpha-1}}{\left[1-G(x;\varphi)^\alpha\right]^3} \sum_{l=0}^{\infty} (-1)^l \binom{\beta(j+i)-1}{l} \\ \times \exp\left[-\frac{(l+1)G(x;\varphi)^{2\alpha}}{\left[1-G(x;\varphi)^\alpha\right]^2}\right]$$

Using the exponential series, we obtain

$$f(x)F(x)^{j+i-1} = 2\alpha\beta g(x;\varphi) \sum_{l,s=0}^{\infty} \frac{(-1)^{l+s}}{s!(l+1)^{-s}} \binom{\beta(j+i)-1}{l} \frac{G(x;\varphi)^{2\alpha(s+1)-1}}{\left[1-G(x;\varphi)^\alpha\right]^{2s+3}}.$$

After applying the expansion (5), the last equation reduces to

$$f(x)F(x)^{j+i-1} = 2\alpha\beta g(x;\varphi) \sum_{l,s,k=0}^{\infty} \frac{(-1)^{l+s+k}}{s!(l+1)^{-s}} \binom{\beta(j+i)-1}{l} \\ \times \binom{-2s-3}{k} G(x;\varphi)^{\alpha(k+2s+2)-1} \quad (10)$$

Combining Equations (9) and (10), the PDF of $X_{i:n}$ follows as

$$f_{i:n}(x) = \sum_{s,k=0}^{\infty} m_{s,k} h_{\alpha(k+2s+2)}(x), \quad (11)$$

where $h_{\alpha(k+2j+2)}(x)$ is the exp-G density with power parameter $\alpha(k+2s+2)$ and

$$m_{s,k} = \sum_{l=0}^{\infty} \sum_{j=0}^{n-i} \frac{2\beta(-1)^{j+l+s+k} (l+1)^s}{s!(k+2s+2)B(i, n-i+1)} \binom{n-i}{j} \binom{\beta(j+i)-1}{l} \binom{-2s-3}{k}.$$

Hence, the PDF of the GBX-G order statistics is a linear combination of exp-G densities. Equation (11) reveal that the properties of $X_{i:n}$ can be derived from those properties of $Y_{\alpha(k+2j+2)}$.

Rényi Entropy

The Rényi entropy of a random variable X represents a measure of variation of the uncertainty. The Rényi entropy is given by

$$I_{\eta}(X) = (1-\eta)^{-1} \log \left(\int_{-\infty}^{\infty} f(x)^{\eta} dx \right) \quad \eta > 0 \quad \text{and} \quad \eta \neq 1$$

Using the PDF (2),

$$\begin{aligned} f(x)^{\eta} &= \frac{(2\alpha\beta)^{\eta} g(x; \boldsymbol{\varphi})^{\eta} G(x; \boldsymbol{\varphi})^{\eta(2\alpha-1)}}{[1-G(x; \boldsymbol{\varphi})^{\alpha}]^{3\eta}} \exp \left\{ -\eta \left[\frac{G(x; \boldsymbol{\varphi})^{\alpha}}{1-G(x; \boldsymbol{\varphi})^{\alpha}} \right]^2 \right\} \\ &\times \left\{ 1 - \exp \left[- \left(\frac{G(x; \boldsymbol{\varphi})^{\alpha}}{1-G(x; \boldsymbol{\varphi})^{\alpha}} \right)^2 \right] \right\}^{\eta(\beta-1)} \end{aligned}$$

Using the expansion (3),

$$\begin{aligned} f(x)^{\eta} &= \frac{(2\alpha\beta)^{\eta} g(x; \boldsymbol{\varphi})^{\eta} G(x; \boldsymbol{\varphi})^{\eta(2\alpha-1)}}{[1-G(x; \boldsymbol{\varphi})^{\alpha}]^{3\eta}} \sum (-1)^l \binom{\eta(\beta-1)}{l} \\ &\times \exp \left\{ -(l+\eta) \left[\frac{G(x; \boldsymbol{\varphi})^{\alpha}}{1-G(x; \boldsymbol{\varphi})^{\alpha}} \right]^2 \right\} \end{aligned}$$

Using the exponential series and the expansion (5),

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$$f(x)^\eta = (2\alpha\beta)^\eta \sum_{l,j,k=0}^{\infty} \frac{(-1)^{l+j+k} (l+\eta)^j}{j!} \binom{\eta(\beta-1)}{l} \\ \times \binom{-2j-3\eta}{k} g(x; \boldsymbol{\varphi})^\eta G(x; \boldsymbol{\varphi})^{\alpha(2j+k+2\eta)-\eta}$$

Then, the Rényi entropy of the GBX-G family reduces to

$$I_\eta(X) = (1-\eta)^{-1} \log \left[\sum_{j,k=0}^{\infty} w_{j,k} \int_{-\infty}^{\infty} g(x; \boldsymbol{\varphi})^\eta G(x; \boldsymbol{\varphi})^{\alpha(2j+k+2\eta)-\eta} dx \right],$$

where

$$w_{j,k} = (2\alpha\beta)^\eta \sum_{l=0}^{\infty} \frac{(-1)^{l+j+k} (l+\eta)^j}{j!} \binom{\eta(\beta-1)}{l} \binom{-2j-3\eta}{k}.$$

Maximum Likelihood Estimation

Consider the estimation of the unknown parameters of the GBX-G class from complete samples only by maximum likelihood. Let X_1, \dots, X_n be a random sample from this family with parameters α , β , and $\boldsymbol{\varphi}$. Then, the log-likelihood function of $\boldsymbol{\theta} = (\alpha, \beta, \boldsymbol{\varphi}^T)^T$, where $\boldsymbol{\theta}$ is the $p \times 1$ parameter vector, is given by

$$\ell(\boldsymbol{\theta}) = n \log(2\alpha) + n \log(\beta) + \log g(x_i; \boldsymbol{\varphi}) + (2\alpha - 1) \log G(x_i; \boldsymbol{\varphi}) \\ - 3 \log \left[1 - G(x_i; \boldsymbol{\varphi})^\alpha \right] - \left[\frac{G(x_i; \boldsymbol{\varphi})^\alpha}{1 - G(x_i; \boldsymbol{\varphi})^\alpha} \right]^2 \\ + (\beta - 1) \log \left\{ 1 - \exp \left[- \left(\frac{G(x_i; \boldsymbol{\varphi})^\alpha}{1 - G(x_i; \boldsymbol{\varphi})^\alpha} \right)^2 \right] \right\}$$

The score vector components,

$$\mathbf{U}(\boldsymbol{\theta}) = \frac{\partial \ell}{\partial \boldsymbol{\theta}} = (U_\alpha, U_\beta, U_{\boldsymbol{\varphi}_k})^T,$$

are

$$\begin{aligned}
U_\alpha &= \frac{n}{\alpha} + 2 \sum_{i=1}^n \log G(x_i; \boldsymbol{\varphi}) + 3 \sum_{i=1}^n \frac{G(x_i; \boldsymbol{\varphi})^\alpha \log G(x_i; \boldsymbol{\varphi})}{1 - G(x_i; \boldsymbol{\varphi})^\alpha} \\
&\quad - \sum_{i=1}^n \frac{2 G(x_i; \boldsymbol{\varphi})^{2\alpha} [1 - 2 G(x_i; \boldsymbol{\varphi})^\alpha] \log G(x_i; \boldsymbol{\varphi})}{[1 - G(x_i; \boldsymbol{\varphi})^\alpha]^3} \\
&\quad + (\beta - 1) \sum_{i=1}^n \frac{2 [1 - 2 G(x_i; \boldsymbol{\varphi})^\alpha] \exp \left[- \left(\frac{G(x_i; \boldsymbol{\varphi})^\alpha}{1 - G(x_i; \boldsymbol{\varphi})^\alpha} \right)^2 \right] \log G(x_i; \boldsymbol{\varphi})}{G(x_i; \boldsymbol{\varphi})^{-2\alpha} [1 - G(x_i; \boldsymbol{\varphi})^\alpha]^3 \left\{ 1 - \exp \left[- \left(\frac{G(x_i; \boldsymbol{\varphi})^\alpha}{1 - G(x_i; \boldsymbol{\varphi})^\alpha} \right)^2 \right] \right\}} \\
U_\beta &= \frac{n}{\beta} + \sum_{i=1}^n \log \left\{ 1 - \exp \left[- \left(\frac{G(x_i; \boldsymbol{\varphi})^\alpha}{1 - G(x_i; \boldsymbol{\varphi})^\alpha} \right)^2 \right] \right\},
\end{aligned}$$

and

$$\begin{aligned}
U_{\varphi_k} &= \sum_{i=1}^n \frac{g'(x_i; \boldsymbol{\varphi})}{g(x_i; \boldsymbol{\varphi})} + (2\alpha - 1) \sum_{i=1}^n \frac{G'(x_i; \boldsymbol{\varphi})}{G(x_i; \boldsymbol{\varphi})} + 3\alpha \sum_{i=1}^n \frac{G'(x_i; \boldsymbol{\varphi})^{\alpha-1}}{1 - G(x_i; \boldsymbol{\varphi})^\alpha} \\
&\quad - 2\alpha \sum_{i=1}^n \frac{G'(x_i; \boldsymbol{\varphi})^{\alpha-1} \left\{ [1 - G(x_i; \boldsymbol{\varphi})^\alpha] G'(x_i; \boldsymbol{\varphi})^\alpha - G(x_i; \boldsymbol{\varphi})^{2\alpha} \right\}}{[1 - G(x_i; \boldsymbol{\varphi})^\alpha]^3} \\
&\quad + (\beta - 1) \sum_{i=1}^n \frac{\left\{ [1 - G(x_i; \boldsymbol{\varphi})^\alpha] G'(x_i; \boldsymbol{\varphi})^\alpha - G(x_i; \boldsymbol{\varphi})^{2\alpha} \right\} \exp \left[- \left(\frac{G(x_i; \boldsymbol{\varphi})^\alpha}{1 - G(x_i; \boldsymbol{\varphi})^\alpha} \right)^2 \right]}{G'(x_i; \boldsymbol{\varphi})^{1-\alpha} [1 - G(x_i; \boldsymbol{\varphi})^\alpha]^3 \left\{ 1 - \exp \left[- \left(\frac{G(x_i; \boldsymbol{\varphi})^\alpha}{1 - G(x_i; \boldsymbol{\varphi})^\alpha} \right)^2 \right] \right\}}
\end{aligned}$$

where

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$$g'(x_i; \boldsymbol{\varphi}) = \frac{\partial g(x_i; \boldsymbol{\varphi})}{\partial \varphi_k} \quad \text{and} \quad G'(x_i; \boldsymbol{\varphi}) = \frac{\partial G(x_i; \boldsymbol{\varphi})}{\partial \varphi_k}.$$

Setting the nonlinear system of equations $U_\alpha = U_\beta = 0$ and $U_{\varphi_k} = 0$ and solving them simultaneously yields the MLE $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\boldsymbol{\varphi}}^T)^T$. This can be implemented by adopting nonlinear optimization methods such as the quasi-Newton algorithm to maximize ℓ numerically.

Applications

The flexibility of the GBXEx and GBXW distributions will be illustrated using two real data sets. The first data consist of 74 observations on gauge lengths of 20 mm (Kundu & Raqab, 2009). The second data set corresponds to the exceedances of flood peaks (in m³/s) of the Wheaton River near Carcross in Yukon Territory, Canada. This data set contains 72 exceedances for the years 1958-1984, rounded to one decimal place (Choulakian & Stephens, 2001). The GBXEx and GBXW distributions will be compared with some competitive models listed in Table 1. To compare the fitted distributions, we consider some measures namely: Cramér-Von Mises (CVM), Anderson-Darling (AD) and Kolmogorov Smirnov (KS) and its p -value (PV). The results in this section are obtained using the R PROGRAM.

Table 1. The competitive models of the GBXEx and GBXW distributions

Distribution	Author(s)
Exponential (Ex)	
Weibull Lindley (WLi)	Asgharzadeh et al. (2018)
Exponentiated transmuted Burr X (ETBX)	Afify, Nofal, and Ebraheim (2015)
Beta Ex (BEx)	Nadarajah and Kotz (2006a)
Transmuted generalized Ex (TGEx)	Khan et al. (2017)
Generalized Lindley (GLi)	Nadarajah et al. (2011)
Exponentiated Ex (EEx)	Nadarajah and Kotz (2006b)
Alpha power Ex (APEX)	Mahdavi and Kundu (2016)
Weibull (W)	Weibull (1951)
Kumaraswamy complementary W geometric (KCWG)	Afify, Cordeiro, Butt, et al. (2017)
Marshall-Olkin additive W (MOAW)	Afify, Cordeiro, Yousof, Saboor, and Ortega (2018)
Alpha logarithmic transformed W (ALTW)	Nassar et al. (2018)
Lindley W (LiW)	Cordeiro, Afify, Yousof, Cakmakyapan, and Ozel (2017)
Weibull Burr XII (WBXII)	Afify, Cordeiro, Ortega, et al. (2018)
Transmuted complementary W geometric (TCWG)	Afify, Nofal, and Butt (2014)
Kumaraswamy transmuted W (KTW)	Afify, Cordeiro, Yousof, Alzaatreh, and Nofal (2016)

Table 2. MLEs, SEs (in parentheses) and goodness-of-fit measures for gauge lengths data

Model	Estimates			CVM	AD	KS	PV
GBXEx	1.563	2.366	0.449	0.027	0.213	0.058	0.962
(α, β, a)	(1.221)	(1.810)	(0.194)				
LiW	5.739	0.374	0.001	0.027	0.242	0.067	0.888
(α, β, λ)	(0.508)	(0.007)	(0.063)				
ETBX	24.820	0.707	-0.958	0.292	0.063	0.387	0.059
$(\alpha, \beta, \lambda, \delta)$	(20.970)	(0.041)	(0.190)	(0.268)			
BEx	24.317	92.490	0.094	0.087	0.573	0.068	0.880
(a, b, λ)	(3.988)	(154.900)	(0.142)				
TGEx	90.150	2.215	-0.697	0.175	1.122	0.084	0.668
$(\alpha, \lambda, \theta)$	(38.880)	(0.187)	(0.206)				
GLi	64.830	2.291		0.203	1.318	0.092	0.546
(α, a)	(23.420)	(0.176)					
EEx	89.430	2.019		0.217	1.405	0.095	0.512
(α, λ)	(32.470)	(0.171)					
APEX	1938073.000	1.258		0.115	0.752	0.191	0.008
(α, λ)	(16777.000)	(0.054)					
Ex	0.403			0.087	0.574	0.449	0.000
(λ)	(0.046)						

Provided in Table 2 are MLEs, their corresponding standard errors (SEs) (in parentheses) and the values of CVM, AD, KS, PV measures for the GBXEx model and other fitted models. Some plots of the fitted GBXEx pdf and other fitted PDFs are displayed in Figure 4. The PP plots of the fitted models are shown in Figure 5. These plots also indicate that the GBXEx model yields the best fit to the glass fibers data compared to the other models.

Provided in Table 3 are the MLEs of the model parameters and the values of CVM, AD, KS, PV statistics for Wheaton River data. The plots of the fitted GBXW density and other fitted densities are displayed in Figure 6. Figure 7 displays the PP plots of the GBXW distribution and other competitive models. These plots reveal that the GBXW distribution yields the best fit and it can be considered a very competitive model to other fitted distributions.

GENERALIZED BURR X-G FAMILY

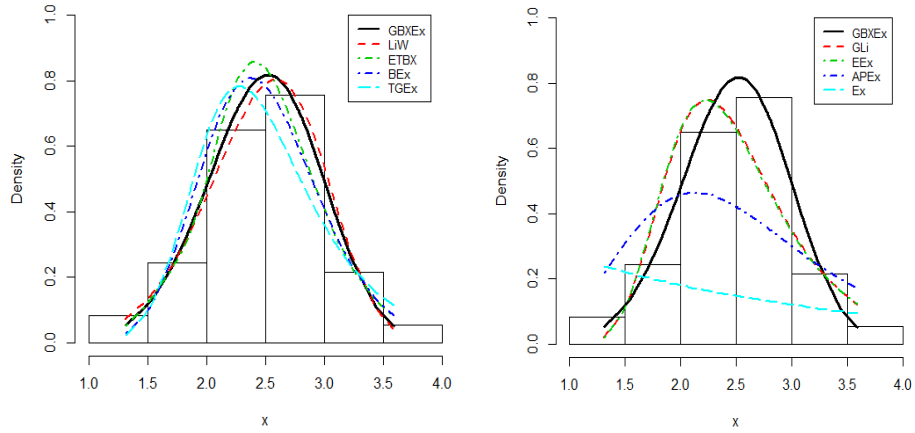


Figure 4. The fitted GBXEx PDF and other PDFs for data set I

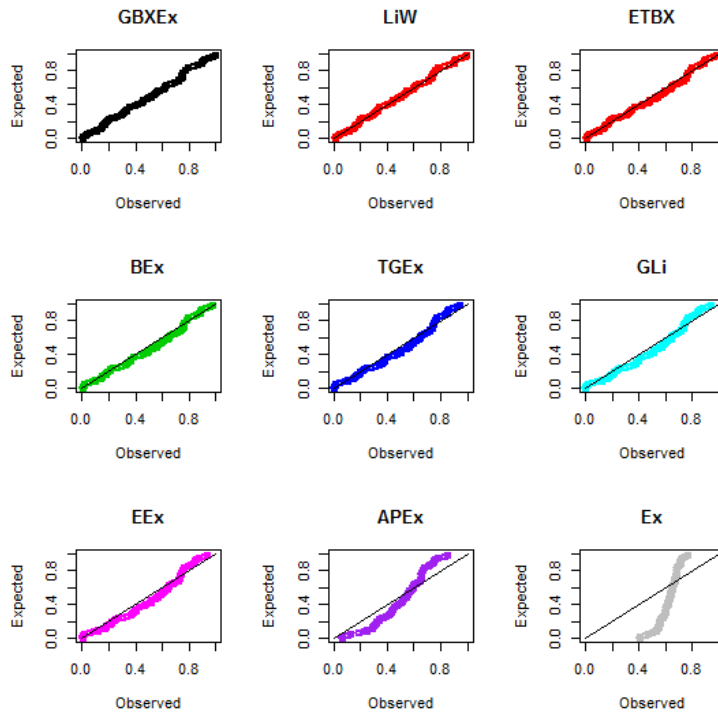
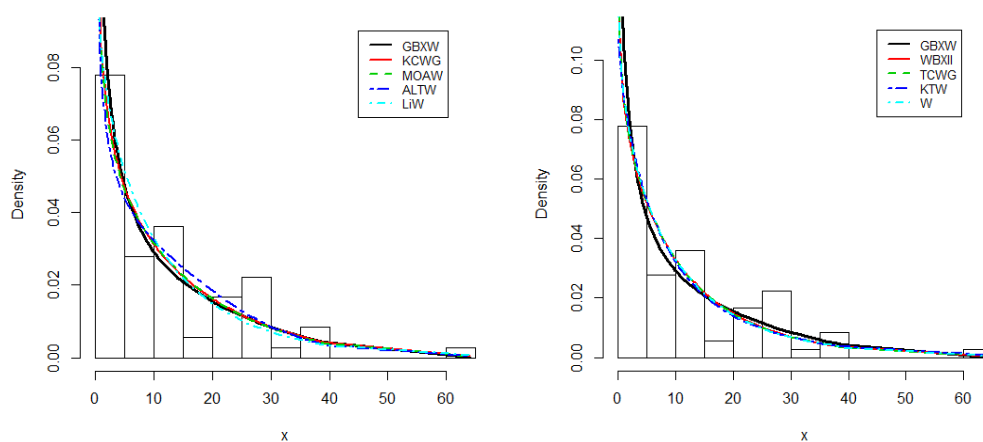


Figure 5. PP plots of the GBXEx distribution and other models for data set I

Table 3. MLEs, SEs (in parentheses), and goodness-of-fit measures for Wheaton river data

Model	Estimates				CVM	AD	KS	PV
GBXW	28.162	0.217	1.824	0.207	0.071	0.422	0.082	0.716
(α, β, a, b)	(4.617)	(0.124)	(0.514)	(0.067)				
KCWG	0.236	1.258	0.025	0.612	5.077	0.094	0.577	0.466
$(\alpha, \beta, \gamma, a, b)$	(1.041)	(1.028)	(0.031)	(0.510)	(20.710)			
MOAW	0.002	0.725	0.150	1.191	1.714	0.101	0.623	0.473
$(\alpha, \beta, \gamma, \delta, \theta)$	(0.002)	(0.195)	(0.217)	(1.755)	(0.265)			
ALTW	111.050	0.656	0.574		0.100	0.636	0.111	0.332
(α, β, λ)	(485.800)	(0.137)	(0.604)					
LiW	0.042	0.868	2.406		0.131	0.756	0.106	0.385
(α, β, θ)	(0.078)	(0.104)	(3.358)					
WBXII	0.411	1.245	0.060	1.954	0.133	0.766	0.106	0.391
(α, β, a, b)	(1.041)	(1.179)	(0.214)	(4.651)				
TCWG	0.866	0.876	0.000	0.094	0.135	0.776	0.107	0.38
$(\alpha, \beta, \lambda, \sigma)$	(0.812)	(0.168)	(0.445)	(0.051)				
KTW	0.013	0.207	(0.466)	4.625	90.680	0.156	0.874	0.101
$(\alpha, \beta, \lambda, a, b)$	(0.016)	(0.135)	(1.619)	(7.303)	(153.800)			
W	0.901	0.109			0.137	0.785	0.105	0.402
(a, b)	(0.085)	(0.030)						

**Figure 6.** The fitted GBXW PDF and other PDFs for data set II

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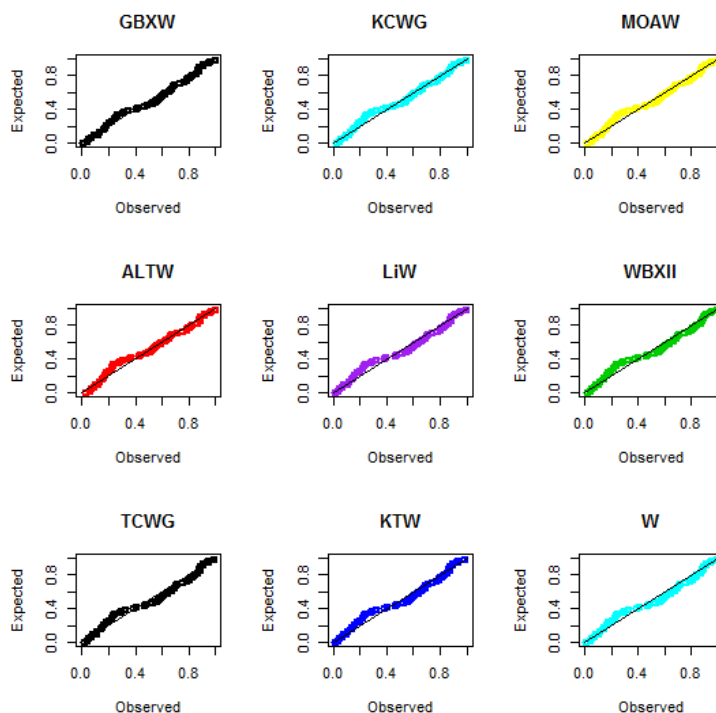


Figure 7. PP plots of the GBXW model and other models for data set II

Simulation

A Monte Carlo simulation study is provided for the GBXEx model to verify the finite sample behavior of the MLEs of its parameters a , α , and β . There were 1000 samples generated from the GBXEx model using the inversion method with different sample sizes, where $n = 20, 50, 100,$ and 200 , and with different parameter combinations, where $a = (0.50, 0.75)$, $\alpha = (0.50, 1.50, 250)$ and $\beta = (0.50, 0.75, 1.75, 2.00)$.

Table 4 lists the mean values of estimates, mean square errors (MSE) and bias of the MLEs of the model parameters. The values in Table 4 indicates that the MSEs and biases decrease when the sample size increases.

Table 4. Estimates, bias and MSE of a , α , and β of the GBXEx distribution

Parameter	n	Estimate	MSE	Bias
$a = 0.50$	20	0.508	0.039	0.008
$\alpha = 1.50$		1.684	1.619	0.184
$\beta = 0.50$		0.710	0.291	0.210
	50	0.495	0.018	-0.041
		1.364	0.580	-0.136
		0.657	0.086	0.157
	100	0.437	0.010	-0.063
		1.223	0.202	-0.277
		0.645	0.046	0.145
	200	0.430	0.007	-0.070
		1.181	0.147	-0.319
		0.638	0.032	0.138
$a = 0.75$	20	0.806	0.126	0.056
$\alpha = 1.50$		2.057	3.991	0.557
$\beta = 0.75$		1.040	0.585	0.290
	50	0.725	0.051	-0.025
		1.547	0.972	0.047
		0.946	0.193	0.196
	100	0.672	0.027	-0.078
		1.301	0.294	-0.199
		0.948	0.115	0.198
	200	0.658	0.016	-0.092
		1.221	0.144	-0.279
		0.930	0.055	0.180
$a = 0.75$	20	0.767	0.037	0.017
$\alpha = 2.50$		2.850	3.228	0.350
$\beta = 1.75$		2.213	1.109	0.463
	50	0.718	0.015	-0.032
		2.363	0.888	-0.137
		2.108	0.485	0.358
	100	0.702	0.008	-0.048
		2.200	0.353	-0.030
		2.081	0.248	0.331
	200	0.695	0.005	-0.055
		2.145	0.245	-0.355
		2.064	0.175	0.314

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Table 4 (continued).

Parameter	n	Estimate	MSE	Bias
$\lambda = 0.50$	20	0.731	0.297	0.231
$\alpha = 0.50$		0.929	1.168	0.429
$\beta = 2.00$		2.395	3.564	0.395
$\lambda = 0.50$	50	0.605	0.081	0.105
$\alpha = 0.50$		0.645	0.193	0.145
$\beta = 2.00$		2.062	1.145	0.062
$\lambda = 0.50$	100	0.554	0.026	0.054
$\alpha = 0.50$		0.563	0.038	0.063
$\beta = 2.00$		1.999	0.467	-0.003
$\lambda = 0.50$	200	0.537	0.009	0.037
$\alpha = 0.50$		0.539	0.012	0.039
$\beta = 2.00$		2.040	0.178	0.040

Conclusions

In several applied areas, there is a clear need for more flexible distributions which can be used to capture skewness and kurtosis behavior. A new generalized Burr X-G (GBX-G) family of distributions was introduced, which extends the Burr X-G class of Yousof et al. (2017) by adding one extra shape parameter. Three special models of the new family were defined. Some mathematical properties of the GBX-G family including explicit expressions for the quantile and generating functions, ordinary and incomplete moments, order statistics, and Rényi entropy are provided. The model parameters are estimated by maximum likelihood. Two real data sets were used to demonstrate certain models of the GBX-G family can provide better fits than other models generated from well-known families.

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