**Journal of Modern Applied Statistical Methods**

Volume 15 | Issue 1 Article 38

5-1-2016

# The xgamma Distribution: Statistical Properties and Application

Subhradev Sen
*Malabar Cancer Centre, Thalassery*, s.subhradev@gmail.com

Sudhansu S. Maiti
*Visva-Bharati University, Santiniketan, India*, dssm1@rediffmail.com

N. Chandra
*Pondicherry University, Puducherry, India*, nc.stat@gmail.com

# The xgamma Distribution: Statistical Properties and Application

**Cover Page Footnote**
Footnote: Corresponding author. Acknowledgement: The authors want to thank the Editor of the journal and the anonymous referees for their valuable suggestion and comments that have improved the paper in its revised version. The first author wants to thank Dr. Satheesan Balasubramanian, Director, Malabar Cancer Centre, India, for his kind support and encouragement during the preparation of the manuscript.

Journal of Modern Applied Statistical Methods
May 2016, Vol. 15, No. 1, 774-788.


ISSN 1538 − 9472

# The xgamma Distribution: Statistical Properties and Application

**Subhradev Sen**
Malabar Cancer Centre
Thalassery, Kerala, India

**Sudhansu S. Maiti**
Visva-Bharati University
Santinikentan, India

**N. Chandra**
Pondicherry University
Puducherry, India

A new probability distribution, the xgamma distribution, is proposed and studied. The distribution is generated as a special finite mixture of exponential and gamma distributions and hence the name proposed. Various mathematical, structural, and survival properties of the xgamma distribution are derived, and it is found that in many cases the xgamma has more flexibility than the exponential distribution. To evaluate the comparative behavior, stochastic ordering of the distribution is studied. To estimate the model parameter, the method of moment and the method of maximum likelihood estimation are proposed. A simulation algorithm to generate random samples from the xgamma distribution is indicated along with a simulation study. A real life dataset on the remission times of patients receiving an analgesic is analyzed, and it is found that the xgamma model provides better fit to the data as compared to the exponential model.

*Keywords:* Finite mixture distribution, lifetime distribution, survival properties, maximum likelihood estimation

## Introduction

The exponential and gamma are well known probability distributions used for modeling lifetime data. Both distributions possess some interesting structural properties, for example, exponential distribution possesses memory less and constant hazard rate properties. Moreover, as a special case of the gamma distribution, the exponential distribution can be used in modeling time-to-event data or modeling waiting times. Various extensions of both distributions can be obtained in the literature for describing the uncertainty behind real life phenomena arising in the area of survival modeling (see Johnson, Kotz, & Balakrishnan, 1994; 1995; Lawless, 2002) and reliability engineering (for more details, see Barlow & Proschan, 1981). The introduction of new lifetime distributions or modified lifetime distributions has become a time-honored fashion in statistical and biomedical

*Subhradev Sen is a Lecturer in Biostatistics. Email him at: s.subhradev@gmail.com.*

research. Finite mixture distributions arising from the standard distributions play, in most situations, a better role in modeling real-life phenomena as compared to the standard ones (see Mclachlan & Peel, 2000).

With the advances in technology and science, a wealth of information has allowed the statistician and data modeler to think in a broader way in the process of gathering knowledge. In order to make inferences about the population of interest, statisticians gather and analyze these information keeping, the responsibility of accurate inference. In recent years, it has been observed that many well-known distributions used to model data sets do not offer enough flexibility to provide an adequate fit. It is, therefore, the need of time that guides statisticians to model real-life scenarios by introducing distributions that are more flexible.

Keeping the role of finite mixture distributions (for more details, see Everitt, 1996; Everitt & Howell, 2005) in modeling time-to-event data, a new distribution, namely the xgamma distribution, is introduced in this article. A special mixture of exponential and gamma distributions is considered in order to obtain the form of xgamma and hence the name proposed. The basic structural and survival properties of the xgamma distribution are obtained in the subsequent sections. Maximum Likelihood and method of moments estimators for the parameters of the model are found, as is a simulation study algorithm, and finally a real data application is made to show the superiority of the xgamma distribution over the exponential distribution.

## Methodology & Synthesis

A special finite mixture of exponential and gamma distributions is used to obtain a new probability distribution, called the xgamma distribution. A random variable $X$ is said to have a finite mixture distribution if its probability density function (pdf) f($x$) is of the form

$$\mathrm{f}\left(x\right)=\sum_{i=1}^{k}\pi_i\,\mathrm{f}_i\left(x\right)\ , \tag{1}$$

where each $\mathrm{f}_i(x)$ is a pdf and $\pi_1, \pi_2, \ldots, \pi_k$ denote the mixing proportions that are non-negative and sum to one.

We have considered $\mathrm{f}_1(x)$ to follow an exponential distribution with parameter $\theta$ and $\mathrm{f}_2(x)$ to follow a gamma distribution with scale parameter $\theta$ and shape parameter 3, i.e. $\mathrm{f}_1(x) \sim \mathrm{Exp}(\theta)$ and $\mathrm{f}_2(x) \sim \mathrm{Gamma}(3, \theta)$ with $\pi_1 = \theta/(1 + \theta)$ and $\pi_2 = 1 - \pi_1$.

***Definition:*** A continuous random variable $X$ is said to follow an xgamma distribution if its pdf is of the form

$$\mathrm{f}\left(x\right)=\frac{\theta^{2}}{\left(1+\theta\right)}\left(1+\frac{\theta}{2}x^{2}\right)\mathrm{e}^{-\theta x},\quad x>0,\theta>0 \tag{2}$$

and is denoted by $X$ ~ xgamma($\theta$). The cumulative density function (cdf) of $X$ is given by

$$\mathrm{F}\left(x\right)=1-\frac{\left(1+\theta+\theta x+\frac{\theta^{2}x^{2}}{2}\right)}{\left(1+\theta\right)}e^{-\theta x},\quad x>0,\theta>0 \tag{3}$$

## Shape

The first derivative of (2) is

$$\frac{d}{dx}\mathrm{f}\left(x\right)=\frac{\theta^{2}}{\left(1+\theta\right)}\left(\theta x-\theta-\frac{\theta^{2}}{2}x^{2}\right)e^{-\theta x}\ .$$

It follows that

i. For $\theta\leq\frac{1}{2}$, $\frac{d}{dx}\mathrm{f}\left(x\right)=0$ implies that $\frac{1+\sqrt{1-2\theta}}{\theta}$ is the unique critical point at which f($x$) is maximized

ii. For $\theta>\frac{1}{2}$, $\frac{d}{dx}\mathrm{f}\left(x\right)\leq 0$, i.e. f($x$) is decreasing in $x$

Figure 1 shows the pdf of the xgamma distribution given in (2) for selected values of $\theta$.

The mode of the xgamma distribution is given by

$$\mathrm{Mode}\left(X\right)=\begin{cases}\frac{1+\sqrt{1-2\theta}}{\theta}, & 0<\theta<\frac{1}{2}\\ 0, & \text{otherwise}\end{cases}$$

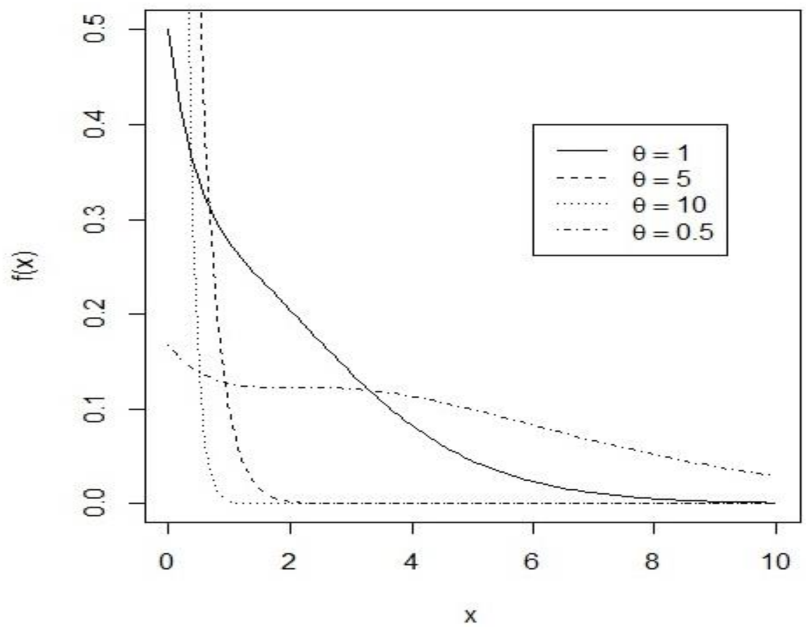


**Figure 1.** Probability density of xgamma($\theta$) for selected values of $\theta$

***Remark:*** It is to be noted that the mode of the exponential distribution is always 0 while the mode of xgamma can be varied as seen above. It is easy to show that if $X \sim$ xgamma($\theta$), then Mode($X$) < Median($X$) < Mean($X$), which also holds good for exponential distribution.

## Moments and Related Measures

The $r^{\text{th}}$ moment about the origin of xgamma distribution is

$$\mu_r' = \mathrm{E}\left(X^r\right) = \frac{r!\left(\theta + r + a_r\right)}{\theta^r\left(1+\theta\right)} ,$$

where $a_r = a_{r-1} + r$ for $r = 1, 2, 3, \ldots$ with $a_0 = 0$ and $a_1 = 2$. In particular,

$$\mu_1' = \frac{\left(\theta+3\right)}{\theta\left(1+\theta\right)} = \mathrm{Mean}\left(X\right) = \mu\left(\text{say}\right)$$

$$\mu_2' = \frac{2\left(\theta+6\right)}{\theta^2\left(1+\theta\right)}, \mu_3' = \frac{6\left(\theta+10\right)}{\theta^3\left(1+\theta\right)}, \mu_4' = \frac{24\left(\theta+15\right)}{\theta^4\left(1+\theta\right)}$$

It is to be noted that, for the exponential distribution with parameter $\theta$, the $r^{\text{th}}$ order moment about origin is

$$\mu_r' = \frac{r!}{\theta^r} \ .$$

The $j^{\text{th}}$ order central moment of the xgamma distribution is $\mu_j = \mathrm{E}\left[\left(X-\mu\right)^j\right] = \sum_{r=0}^{j}\binom{j}{r}\mu_r'\left(-\mu\right)^{j-r}$. In particular,

$$\mu_2 = \frac{\left(\theta^2+8\theta+3\right)}{\theta^2\left(1+\theta\right)^2} = \mathrm{var}\left(X\right) = \sigma^2 \ (\mathrm{say})$$

$$\mu_3 = \frac{2\left(\theta^3+15\theta^2+9\theta+3\right)}{\theta^3\left(1+\theta\right)^3}$$

$$\mu_4 = \frac{3\left(5\theta^4+88\theta^3+310\theta^2+288\theta+177\right)}{\theta^4\left(1+\theta\right)^4}$$

The coefficients of variation ($\gamma$), skewness $\left(\sqrt{\beta_1}\right)$,and kurtosis ($\beta_2$) are

$$\gamma = \frac{\sqrt{\theta^2+8\theta+3}}{\left(\theta+3\right)}$$

$$\sqrt{\beta_1} = \sqrt{\frac{\mu_3^2}{\mu_2^3}} = \frac{2\left(\theta^3+15\theta^2+9\theta+3\right)}{\left(\theta^2+8\theta+3\right)^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\left(5\theta^4+88\theta^3+310\theta^2+288\theta+177\right)}{\left(\theta^2+8\theta+3\right)^2}$$

The coefficients are increasing functions in $\theta$ (see Figure 2 for the graph of $\gamma$ and $\sqrt{\beta_1}$ for varying $\theta$).

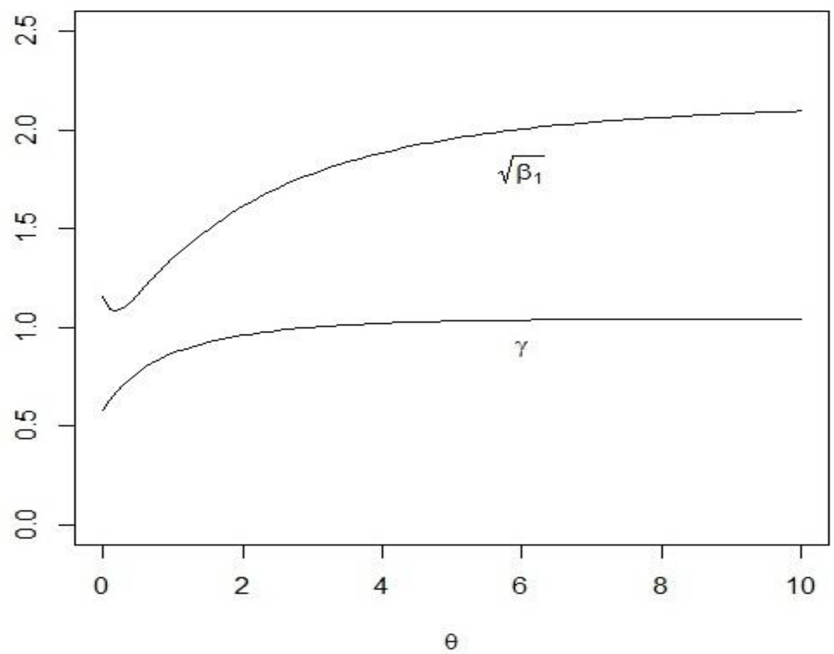


**Figure 2.** Coefficients for variation and skewness

---

***Remark:*** It should be noted that the values of $\gamma$, $\sqrt{\beta_1}$ , and $\beta_2$ for the exponential distribution are 1, 2, and 6, respectively. Hence the xgamma distribution is again more flexible than the exponential distribution.

## Survival Properties

The hazard rate function or failure rate function for a continuous distribution with pdf f($x$), cdf F($x$), and survival function (sf) S($x$) is defined as

$$\mathrm{h}(x) = \lim_{\Delta x \to 0} \frac{\mathrm{P}(X < x + \Delta x \mid X > x)}{\Delta x} = \frac{\mathrm{f}(x)}{1 - \mathrm{F}(x)} = \frac{\mathrm{f}(x)}{\mathrm{S}(x)} .$$

For the xgamma distribution, the hazard rate function is given by

$$\mathrm{h}(x)=\frac{\theta^2\left(1+\frac{\theta}{2}x^2\right)}{\left(1+\theta+\theta x+\frac{\theta^2x^2}{2}\right)} \tag{4}$$

It is to be noted that

i. $\mathrm{h}(0)=\dfrac{\theta^2}{(1+\theta)}=\mathrm{f}(0)$

ii. h($x$) is an increasing function in $x$ and $\theta$ with $\theta^2/(1+\theta) < \mathrm{h}(x) < \theta$

***Remark:*** For the exponential distribution with parameter $\theta$, h($x$) = $\theta$, and so (4) shows the flexibility of the xgamma distribution over the exponential distribution. Figure 3 shows the hazard rate function of the xgamma distribution for selected values of $\theta$.

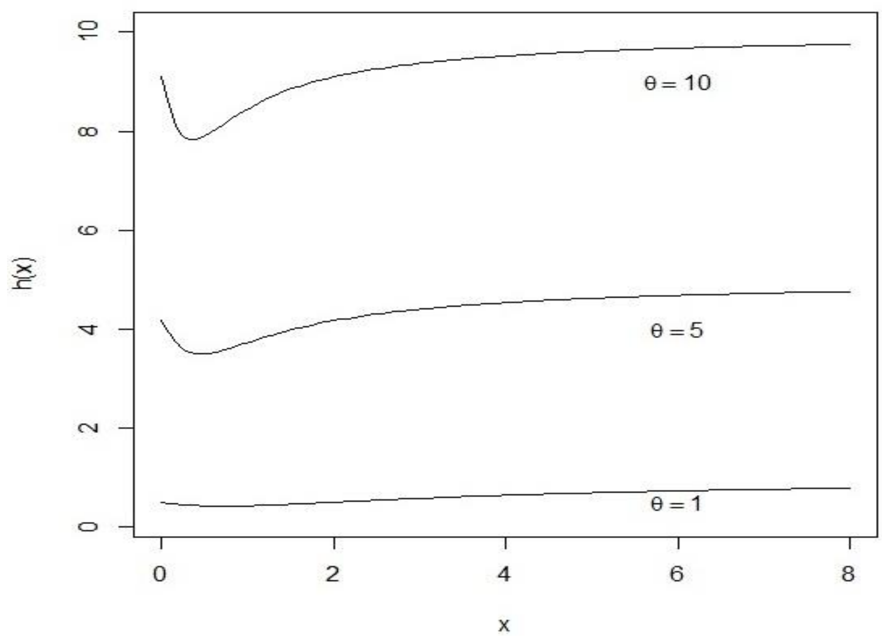


**Figure 3.** Hazard rate function of xgamma($\theta$) for selected values of $\theta$

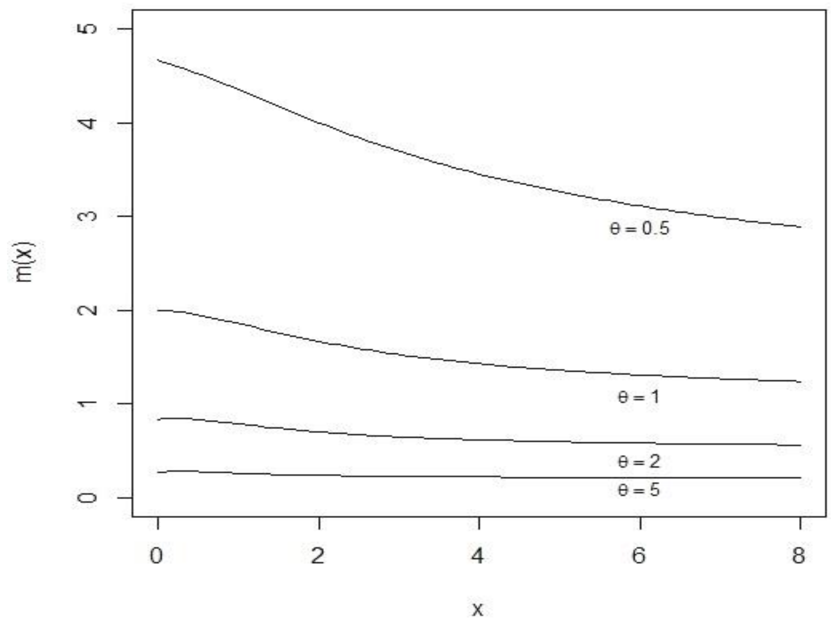


**Figure 4.** Mean residual life function plot of xgamma($\theta$) for selected values of $\theta$

---

For a continuous random variable $X$ with pdf f($x$) and cdf F($x$), the mean residual life (mrl) function is defined as

$$\mathrm{m}(x) = \mathrm{E}(X - x \mid X > x) = \frac{1}{1-\mathrm{F}(x)} \int_x^\infty \left[1 - \mathrm{F}(t)\right] dt \ .$$

For the xgamma distribution, the mrl function (see Figure 4) is given by

$$\begin{aligned} m(x) &= \frac{1}{\left(1+\theta+\theta x+\frac{\theta^2 x^2}{2}\right)\mathrm{e}^{-\theta x}} \int_x^\infty \left(1+\theta+\theta x+\frac{\theta^2 x^2}{2}\right)\mathrm{e}^{-\theta x} \\ &= \frac{1}{\theta} + \frac{(x+\theta x)}{\theta\left(1+\theta+\theta x+\frac{\theta^2 x^2}{2}\right)} \end{aligned} \tag{5}$$

It is to be noted that

i. $\mathrm{m}(0) = \mu = \dfrac{(\theta+3)}{\theta(1+\theta)}$

ii. m($x$) is decreasing in $x$ and θ with $\dfrac{1}{\theta} < \mathrm{m}(x) < \dfrac{(\theta+3)}{\theta(1+\theta)}$

***Remark:*** For the exponential distribution, the mrl function is $1/\theta$ and hence (6) again shows the flexibility of the xgamma distribution over the exponential distribution.

## Stochastic Ordering

For a positive continuous random variable, stochastic ordering is an important tool for judging the comparative behavior. Recall some basic definitions:

A random variable $X$ is said to be smaller than a random variable $Y$ in the

i. stochastic order ($X \leq_{st} Y$) if $F_X(x) \geq F_Y(x)$ for all $x$
ii. hazard rate order ($X \leq_{hr} Y$) if $h_X(x) \geq h_Y(x)$ for all $x$
iii. mean residual life order ($X \leq_{mrl} Y$) if $m_X(x) \leq m_Y(x)$ for all $x$
iv. likelihood ratio order ($X \leq_{lr} Y$) if $f_X(x)/f_Y(x)$ decreases in $x$

The following implications (see Shaked & Shanthikumar, 1994) are well justified:

$$\begin{aligned} &X \leq_{\mathrm{lr}} Y \Rightarrow X \leq_{\mathrm{hr}} Y \Rightarrow X \leq_{\mathrm{mrl}} Y, \\ &X \leq_{\mathrm{hr}} Y \Rightarrow X \leq_{\mathrm{st}} Y \end{aligned} \tag{6}$$

The following theorem shows that the xgamma distributions are ordered with respect to the strongest likelihood ratio ordering.

***Theorem 1.*** Let X ~ xgamma($\theta_1$) and Y ~ xgamma($\theta_2$). If $\theta_1 > \theta_2$ then $X \leq_{lr} Y$ and hence the other ordering in (7).

***Proof:*** Note that

$$\frac{\mathrm{f}_X(x)}{\mathrm{f}_Y(x)} = \frac{\theta_1^2(1+\theta_2)(2+\theta_1 x^2)}{\theta_2^2(1+\theta_1)(2+\theta_2 x^2)} \mathrm{e}^{(\theta_2-\theta_1)x}, \quad x > 0$$

$$\frac{d}{dx}\left(\frac{\mathrm{f}_X(x)}{\mathrm{f}_Y(x)}\right)=(\theta_2-\theta_1)\frac{\theta_1^2(1+\theta_2)}{\theta_2^2(1+\theta_1)}e^{(\theta_2-\theta_1)x}\left[\frac{(2+\theta_1x^2)}{(2+\theta_2x^2)}-\frac{4x}{(2+\theta_2x^2)^2}\right]<0$$

since $\theta_1 > \theta_2$. Hence $\mathrm{f}_X(x)/\mathrm{f}_Y(x)$ decreases in x and $X \leq_{\mathrm{lr}} Y$. The remaining statement follows from (7) directly.

## Estimation of the Parameter

Given a random sample $X_1, X_2, \ldots, X_n$ of size $n$ from the xgamma distribution in (2), the method of moment (mom) estimator for the parameter $\theta$ of xgamma distribution given in (2) is obtained as follows:

Equate sample mean $\bar{X}=\sum_{i=1}^{n} X_i/n$ with first order moment about origin of (2) which gives the mom estimator of $\theta$ as

$$\hat{\theta}_{\text{mom}}=\frac{-(\bar{X}-1)+\sqrt{(\bar{X}-1)^2+12\bar{X}}}{2\bar{X}}, \quad \bar{X}>0 .$$

The following theorem shows that the mom estimator of $\theta$ in (2) is positively biased:

***Theorem 2.*** The method of moment estimator of the xgamma distribution is positively biased, i.e., $\mathrm{E}(\hat{\theta}_{\text{mom}})-\theta>0$.

***Proof:*** Let $\hat{\theta}_{\text{mom}}=\mathrm{g}(\bar{X})$ and $\mathrm{g}(t)=\frac{-(t-1)+\sqrt{(t-1)^2+12t}}{2t}$. For $t > 0$, $\mathrm{g}''(t) > 0$ and hence g($t$) is strictly convex. Thus by Jensen's inequality, we have $\mathrm{g}\left[\mathrm{E}(\bar{X})\right]<\mathrm{E}\left[\mathrm{g}(\bar{X})\right]$. Now since $\mathrm{g}\left[\mathrm{E}(\bar{X})\right]=\mathrm{g}(\mu)=\mathrm{g}\left((\theta+3)/\theta(1+\theta)\right)=\theta$, we obtain $\mathrm{E}(\hat{\theta}_{\text{mom}})-\theta>0$. Hence the proof.

It should be noted that the sample raw moments are unbiased and consistent estimators of the corresponding population raw moments. They are also asymptotically normally distributed (CAN estimators) by virtue of the central limit theorem. Thus the mom estimator, $\hat{\theta}_{\text{mom}}$, of $\theta$ for xgamma distribution is consistent.

Let **x** = ($x_1$, $x_2$,…, $x_n$) be $n$ observations on a random sample $X_1$, $X_2$,…, $X_n$ of size $n$ drawn from the xgamma distribution in (2). The maximum likelihood estimator (mle) of $\theta$ for given **x** is as follows:

The likelihood function is given by

$$\mathrm{L}\left(\theta \mid x_1, x_2, \ldots, x_n\right) = \mathrm{L}\left(\theta \mid x\right) = \prod_{i=1}^{n} \frac{\theta^2}{\left(1+\theta\right)}\left(1+\frac{\theta}{2}x_i^2\right)\mathrm{e}^{-\theta x_i} .$$

The log-likelihood function is given by

$$\log_{\mathrm{e}} \mathrm{L}\left(\theta \mid \mathbf{x}\right) = 2n\log\theta - n\log\left(1+\theta\right) + \sum_{i=1}^{n}\log\left(1+\frac{\theta}{2}x_i^2\right) - \theta\left(\sum_{i=1}^{n} x_i\right) . \quad (7)$$

The log-likelihood equation corresponding to (7) becomes

$$\mathrm{l}\left(\theta \mid \mathbf{x}\right) = \frac{d}{d\theta}\log_e \mathrm{L}\left(\theta \mid x\right) = 0 ,$$

where l($\theta$ | **x**) is given by

$$\sum_{i=1}^{n} \frac{\frac{x_i^2}{2}}{\left(1+\frac{\theta}{2}x_i^2\right)} + \frac{2n}{\theta} - \frac{n}{\left(1+\theta\right)} - \sum_{i=1}^{n} x_i$$

To obtain the mle of $\theta$, $\hat{\theta}_{\text{mle}}$ (say), we can maximize (8) directly with respect to $\theta$ or we can solve the non-linear equation l($\theta$ | **x**) = 0. Note that $\hat{\theta}_{\text{mle}}$ cannot be solved analytically; numerical iteration techniques, such as the Newton-Raphson algorithm, are thus adopted to solve the log-likelihood equation for which (8) is maximized. The initial solution for such an iteration can be taken as:

$$\theta_0 = \frac{n}{\sum_{i=1}^{n} x_i} .$$

Using this initial solution, we have

$$\theta^{(i)} = \theta^{(i-1)} - \frac{\mathbf{1}\left(\theta^{(i-1)} \mid \mathbf{x}\right)}{\mathbf{1}'\left(\theta^{(i-1)} \mid \mathbf{x}\right)}$$

for the $i^{\text{th}}$ iteration. We choose $\theta^{(i)}$ such that $\theta^{(i)} \cong \theta^{(i-1)}$.

***Remark.*** The method of moment and maximum likelihood estimators of the exponential distribution is $1/\bar{X}$ , which is also biased and consistent.

## Simulation Study

The inversion method for generating random data from the xgamma distribution fails because the equation $F(x) = u$, where $u$ is an observation from the uniform distribution on (0, 1), cannot be explicitly solved in $x$. However, we can use the fact that the xgamma distribution is a special mixture of the exponential($\theta$) and gamma(3, $\theta$) distributions.

To generate random data $X_i$, $i = 1, 2,\ldots, n$, from the xgamma distribution with parameter $\theta$, we can use the following algorithm:

1. Generate $U_i \sim$ uniform(0, 1), $i = 1, 2,\ldots, n$
2. Generate $V_i \sim$ exponential($\theta$), $i = 1, 2,\ldots, n$
3. Generate $W_i \sim$ gamma(3, $\theta$), $i = 1, 2,\ldots, n$
4. If $U_i \leq \theta/(1 + \theta)$, then set $X_i = V_i$. Otherwise, set $X_i = W_i$

A Monte Carlo simulation study was carried out considering $N = 10000$ times for selected values of $n$ and $\theta$. Samples of sizes 20, 40, and 100 were considered and values of $\theta$ were taken as 0.1, 0.5, 1.0, 1.5, 3, and 6 .The following two measures were computed:

i. Average bias of the simulated estimates $\hat{\theta}_i$ , $i = 1, 2,\ldots N$: $\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta\right)$

ii. Average Mean Square Error (MSE) of the simulated estimates $\hat{\theta}_i$ , $i = 1, 2,\ldots N$: $\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta\right)^2$

**Table 1.** Average bias and MSE of the estimator $\hat{\theta}$

| *n* | 20 | | 40 | | 100 | |
|---|---|---|---|---|---|---|
| **theta** | **Bias** | **MSE** | **Bias** | **MSE** | **Bias** | **MSE** |
| 0.1 | 0.00193 | 0.00020 | 0.00078 | 0.00009 | 0.00034 | 0.00004 |
| | -0.06375 | 0.00409 | -0.06420 | 0.00414 | -0.06438 | 0.00415 |
| 0.5 | 0.01182 | 0.00595 | 0.00539 | 0.00275 | 0.00199 | 0.00106 |
| | -0.27887 | 0.07934 | -0.28258 | 0.08057 | -0.28455 | 0.08125 |
| 1.0 | 0.02655 | 0.02750 | 0.01347 | 0.01290 | 0.00517 | 0.00499 |
| | -0.48093 | 0.24223 | -0.49040 | 0.24558 | -0.49631 | 0.24828 |
| 1.5 | 0.04411 | 0.07234 | 0.02804 | 0.03395 | 0.00871 | 0.01221 |
| | -0.63116 | 0.43490 | -0.64478 | 0.43260 | -0.65975 | 0.44133 |
| 3.0 | 0.12181 | 0.36497 | 0.05765 | 0.15694 | 0.02204 | 0.05985 |
| | -0.88938 | 1.04281 | -0.94784 | 1.00708 | -0.98001 | 1.00190 |
| 6.0 | 0.27864 | 1.75155 | 0.14144 | 0.78423 | 0.05511 | 0.28684 |
| | -1.06299 | 2.59379 | -1.19641 | 2.09340 | -1.28001 | 1.88262 |

The result of the simulation study has been tabulated in Table 1. In Table 1, for each selected value of θ, the corresponding values relating to the xgamma distribution have been presented in first row and that relating to exponential distribution in second row.

***Remarks.*** i) Table 1 shows that the bias is positive in the case of the xgamma distribution (as shown in the Theorem 2). Table 1 also shows that bias and MSE decreases as $n$ increases and increases when $\theta$ increases. ii) In terms of bias and MSE of the estimates of $\theta$, the xgamma distribution shows more flexibility as compared to the exponential distribution.

## Application

In this section, a real data set is used to show that the xgamma distribution can be a better model than one based on the exponential distribution. The data on relief times (in hours) of 20 patients receiving an analgesic (cf. Gross & Clark, 1975) is used. Both the xgamma and exponential distributions are fitted to this data set. The method of maximum likelihood is used. Maximum likelihood estimates for both the cases are calculated for the data. The required numerical evaluations are carried out using R 3.1.1 software. Table 2 provides the maximum likelihood estimates with corresponding standard errors of the model parameters. The model selection is carried out using the log-likelihood value, AIC (Akaike information criterion), the AICc (consistent Akaike information criteria) and the BIC (Bayesian information criterion):

$$\text{AIC} = -2\log_e L + 2k, \quad \text{AICc} = \text{AIC} + \frac{2k(k+1)}{n-k-1}, \quad \text{BIC} = k\log_e(n) - 2\log_e L$$

where $\log_e L$ denotes the log-likelihood function evaluated at the maximum likelihood estimates, $k$ is the number of parameters, and $n$ is the sample size.

From Table 2, it is clear that the values of the AIC, AICc and BIC are smaller for the xgamma distribution compared with those values of the exponential model, so the new distribution seems to be a very competitive model to these data. It follows that the xgamma distribution provides the better fit to the data.

**Table 2.** Maximum likelihood estimates and model selection statistics

| Distributions | Maximum Likelihood Estimates | Standard Error | Log-likelihood | AIC | AICc | BIC |
|---|---|---|---|---|---|---|
| Exponential | $\hat{\theta}_{mle}$ = 0.52632 | 0.11769 | -32.83708 | 67.67416 | 67.89638 | 68.66989 |
| xgamma | $\hat{\theta}_{mle}$ = 1.10747 | 0.16943 | -31.50824 | 65.01649 | 65.23871 | 66.01221 |

## Conclusion

The xgamma distribution, a special finite mixture of exponential and gamma distributions, was derived. Various mathematical and structural properties of the distribution were studied including the shape, moments, measures of skewness, and kurtosis. Important survival properties like the hazard rate and mean residual life functions were derived and discussed. Stochastic ordering and a simulation algorithm were also proposed. Added flexibility over the exponential distribution was observed with regard to certain important properties of the xgamma distribution. The maximum likelihood method and method of moments were proposed for the parameter estimation. In order to demonstrate the applicability of the xgamma distribution, a simulation study was shown. Moreover, the distribution was fitted to a real data set and compared with the exponential distribution. Results show that the xgamma distribution provides an adequate fit for the data set. The maximum likelihood functions may be further studied under different types of censoring mechanism for future applications of the xgamma model. Bayesian estimation of the parameter of xgamma distribution may further be considered with suitable prior and risk function.

## References

Barlow, R. E., & Proschan, F. (1981). *Statistical theory of reliability and life testing: Probability models*. Silver Spring, MD: To Begin With.

Everitt, B. S. (1996). An introduction to finite mixture distributions. *Statistical Methods in Medical Research, 5*(2), 107-127. doi: 10.1177/096228029600500202

Everitt, B. S., & Howell, B. (Eds.). (2005). *Encyclopedia of statistics in behavioral science*. Hoboken, NJ: John Wiley & Sons.

Gross, A. J., & Clark, V. A. (1975). *Survival distributions: Reliability applications in the biomedical sciences*. New York, NY: John Wiley & Sons.

Johnson, N. L., Kotz, S., & Balakrishnan, N. (1994). *Continuous univariate distributions* (Vol. 1) (2nd ed.). New York, NY: Wiley.

Johnson, N. L., Kotz, S., & Balakrishnan, N. (1995). *Continuous univariate distributions* (2nd ed.) (Vol. 2). New York, NY: Wiley.

Lawless, J. F. (2003). *Statistical models and methods for lifetime data* (2nd ed.). New York, NY: John Wiley & Sons.

McLachlan, G., & Peel, G. (2000). *Finite mixture models*. New York, NY: John Wiley & Sons.

Shaked, M., & Shanthikumar, J. G. (1994). *Stochastic orders and their applications*. New York, NY: Academic Press.