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The Application of Legendre Multiwavelet Functions in Image Compression

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Legendre multiwavelets are introduced. These functions can be designed in such a way that the properties of orthogonality, polynomial approximation, and symmetry hold at the same time. In this way, they can be effectively deployed in image compression.

Keywords: Image compression, Legendre multiwavelet, multiwavelets, preprocessing, wavelets

Introduction

In recent years, the wavelets theory has played a significant role in signal processing, especially in image processing. These wavelets are mainly scalar, where there is only one scaling function. However, multiwavelets are based on more than one scaling function. Such growing interests in multiwavelets mainly stem from the following facts: (i) multiwavelets can simultaneously possess orthogonality, symmetry, and a high order of approximation for a given support of the scaling functions (this is not possible for any real valued scalar wavelets); and (ii) multiwavelets have produced promising results in the areas of image compression.

A multiwavelet system can provide perfect reconstruction while preserving length (orthogonality), good performance at the boundaries (via linear-phase symmetry), and a high order of approximation (vanishing moments). Thus, multiwavelets offer the possibility of superior performance for image processing applications, compared with scalar wavelets. In this paper, we use linear Legendre multiwavelets in image compression, and show its usefulness through actual examples.

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Multiwavelets

As with scalar wavelets, multiwavelets are based on the concept of multiresolution analysis (MRA). The only difference is the number of scaling functions used to generate those functions. The standard multiresolution has one scaling function $\phi(t)$, which satisfies the following properties (Aboufadel & Schillicker, 1999; Martin & Bell, 2001, Burrus & Gopinath, 1998; Daubechies, 1992).

- The translates $\phi(t - k)$ are linearly independent and produce a basis for the subspace $\phi_1(t), \dots, \phi_N(t)$.
- The dilates $\phi(2^j t - k)$ generate subspaces $V_j, j \in \mathbb{Z}$, such that $\dots \subset V_{-1} \subset V_0 \subset V_1 \subset \dots \subset V_j \subset \dots \overline{\cup_{j=-\infty}^{\infty} V_j} = L^2(\mathbb{R}), \cap_{j=-\infty}^{\infty} V_j = \{0\}$.
- To obtain the subspace V_1 , it is sufficient to add the family of functions $\psi(t - k)$, W_0 , to V_0 , i.e., $V_1 = V_0 \oplus W_0$.

In multiwavelets, the notion of MRA is used in the same way except that the basis for V_0 is generated by the translates of N different scaling functions $\phi_1(t - k), \phi_2(t - k), \dots, \phi_N(t - k)$. As in the scalar case, the vector $\Phi(t) = [\phi_1(t), \dots, \phi_N(t)]^T$, satisfies the matrix dilation equation

$$\Phi(t) = \sum_k C[k] \Phi(2t - k)$$

where $C[k]$ is an N by N matrix of coefficients. There are also N wavelets $\psi_1(t), \dots, \psi_N(t)$ satisfying the matrix wavelet equation

$$\Psi(t) = \sum_k D[k] \Phi(2t - k)$$

where $\Psi(t) = [\psi_1(t), \dots, \psi_N(t)]^T$ is a vector, and $D[k]$ is an N by N matrix.

The scaling functions $\phi_1(t), \dots, \phi_N(t)$ are in V_0 whose basis is $\{\sqrt{2}\phi_i(2t - k) : 1 \leq i \leq N, k \in \mathbb{Z}\}$. Thus, the scaling function and the multiwavelet functions have to satisfy the two-scale dilation equations

$$\Phi(t) = \sqrt{2} \sum_k H_k \Phi(2t - k) \text{ and } \Psi(t) = \sqrt{2} \sum_k G_k \Phi(2t - k),$$

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where G_k and $H_k \in \ell^2(2)^{N \times N}$ are $N \times N$ matrices of filter coefficients. Moreover, the set of scaling functions $\{\phi_{i,j,k}(t) : 1 \leq i \leq N, k \in \mathbb{Z}\}$ with $\phi_{i,j,k}(t) = 2^{-j/2} \phi_i(t / 2^j - k)$ is a Riesz basis for V_i . We also write

$$\left[\varphi_{1,j,k}(t), \dots, \varphi_{N,j,k}(t) \right]^T = \Phi_{j,k}(t).$$

In the case of orthonormal multiscaling functions, $\{\phi_i(t) : 1 \leq i \leq N, k \in \mathbb{Z}\}$ is not just a Riesz basis, but is orthonormal, i.e.,

$$\langle \Phi(t), \Phi(t-k) \rangle = \int \Phi(t) \Phi^T(t-k) dt = I_N \delta_{k,0}, k \in \mathbb{Z}$$

where I_N is the $N \times N$ identity matrix. This implies that

$$\sum_k G_k G_{2l+k}^T = I_r \delta_{0,l}, l \in \mathbb{Z},$$

In scalar wavelets, this means that the sum of squares of low-pass filter coefficients equals unity, and the filter is orthogonal to its even translates.

Linear Legendre Multiwavelets

A pair of linear Legendre scaling functions $\phi_1(x)$ and $\phi_2(x)$ on $[0,1]$ are introduced and depicted in Figure 1.

$$\phi_1(x) = 1, \text{ and } \phi_2(x) = \sqrt{3}(2x - 1).$$

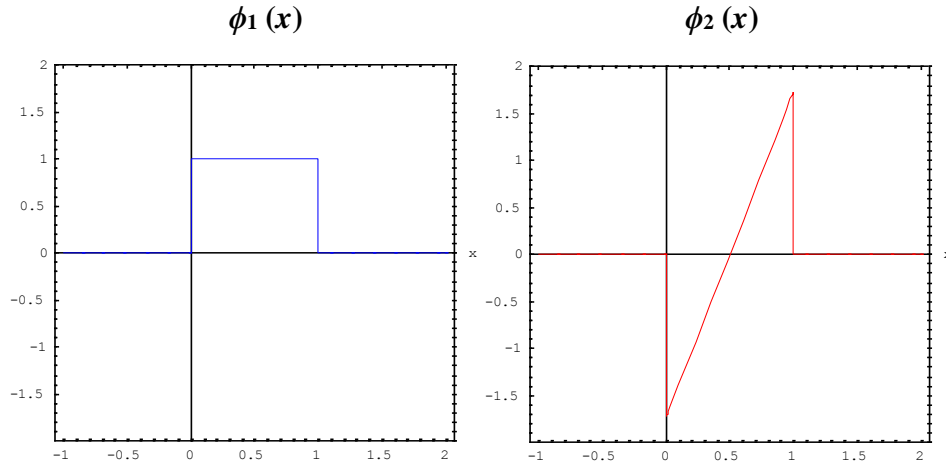


Figure 1. A pair of linear Legendre scaling functions. The left is $\phi_1(x)$ and the right is $\phi_2(x)$.

The integer translates of $\phi_1(x)$ together with the ones of $\phi_2(x)$ span a subspace V_0 . Furthermore, the translates of $1/2$ scaled version of $\phi_1(x)$ and $\phi_2(x)$ that span $V_1 \supset V_0$ are given by

$$\begin{cases} \phi_1(2x-k) = 1 & \text{for } \frac{k}{2} \leq x < \frac{k+1}{2} \\ \phi_2(2x-k) = \sqrt{3}(4x-2k-1) & \text{for } \frac{k}{2} \leq x < \frac{k+1}{2} \end{cases}$$

The two-scale relations for linear Legendre scaling functions should express each of $\phi_1(x)$ and $\phi_2(x)$ in terms of the four scaling functions $\phi_i(2x)$ and $\phi_i(2x-1)$, $i = 1, 2$ (Udea & Lodha, 1995). We propose these relations to be

$$\begin{bmatrix} \phi_1(x) \\ \phi_2(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \phi_1(2x) \\ \phi_2(2x) \\ \phi_1(2x-1) \\ \phi_2(2x-1) \end{bmatrix} \quad (1)$$

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Let $\psi_1(x)$ and $\psi_2(x)$ be wavelets on $[0,1]$ defined based on the linear Legendre scaling functions. By using (1), $\psi_1(x)$ and $\psi_2(x)$ can be expressed in terms of $\phi_i(2x)$ and $\phi_i(2x - 1)$, $i = 1,2$. The conditions of being orthonormal and vanishing moments help reduce the two-scale relation for linear Legendre multiwavelets as follows.

$$\begin{bmatrix} \psi_1(x) \\ \psi_2(x) \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \phi_1(2x) \\ \phi_2(2x) \\ \phi_1(2x-1) \\ \phi_2(2x-1) \end{bmatrix} \quad (2)$$

Then, the explicit formulae for $\psi_1(x)$ and $\psi_2(x)$ are

$$\psi_1(x) = \pm \begin{cases} -\sqrt{3}(4x-1), & 0 \leq x < \frac{1}{2} \\ +\sqrt{3}(4x-3), & \frac{1}{2} \leq x < 1 \end{cases}, \quad \text{and} \quad \psi_2(x) = \pm \begin{cases} 6x-1, & 0 \leq x < \frac{1}{2} \\ 6x-5, & \frac{1}{2} \leq x < 1 \end{cases}.$$

Decomposition relations

The matrix equation

$$\begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \psi_1(x) \\ \psi_2(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & -1 & 0 & 1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \phi_1(2x) \\ \phi_2(2x) \\ \phi_1(2x-1) \\ \phi_2(2x-1) \end{bmatrix} \quad (3)$$

which combines (1) and (2) is called the reconstruction relation. Note that $\phi_1(x), \phi_2(x) \in V_0$ and $\psi_1(x), \psi_2(x) \in W_0$. This four bases are expressed in terms of $\phi_1(2x), \phi_2(2x), \phi_1(2x - 1)$ and $\phi_2(2x - 1)$ in V_1 subspace.

The decomposition relation is simply defined as the inverse of the reconstruction relation defined in (3). The square matrix in (3) is orthogonal with constant magnitude. This is because $\{\phi_1(x), \phi_2(x), \psi_1(x), \psi_2(x)\}$ and $\{\phi_1(2x), \phi_2(2x), \phi_1(2x - 1), \phi_2(2x - 1)\}$. Thanks to the property $M^{-1} = M^T$ of any orthogonal matrix, the decomposition relation would be

$$\begin{bmatrix} \varphi_1(2x) \\ \varphi_2(2x) \\ \varphi_2(2x-1) \\ \varphi_2(2x-1) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 & \frac{\sqrt{3}}{2} \\ 1 & \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 1 & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \psi_1(x) \\ \psi_2(x) \end{bmatrix}$$

Preprocessing and sampling

Preprocessing aims to convert the given scalar input signal of length N to a sequence of two-dimensional vectors $v_{0,k}$. This is used in the analysis algorithm as shown in Figure 2. Here, N is assumed to be a multiple of a power of 2, and therefore it even. The signal $\{X_k\}$ is taken to be a function $x(\cdot)$ which is observed at integer time points. If the preprocessing produces N two-dimensional vectors, it is said to be an oversampling scheme. If it produces $N/2$ two-dimensional vectors, the result is a critical sampling. After the wavelet reconstruction step (synthesis), e.g., inverse DMWT, a postfilter is applied. Evidently, prefiltering, wavelet transform, inverse transform, and postfiltering should exactly recover the input signal.

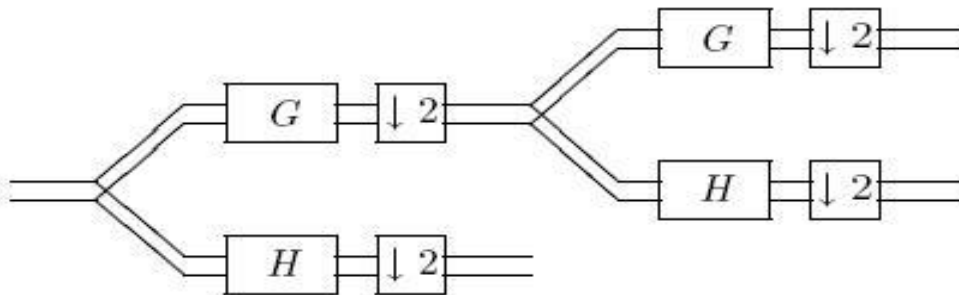


Figure 2. A multiwavelet filter bank which is iterated once.

Repeated Row Preprocessing: Oversampling Scheme

The obvious way to get two input rows from a given signal is to repeat the signal. Two identical rows go into the multifilter bank. It then begins oversampling of the data by a factor of two. Although, the usefulness of oversampling has been proven, it requires more calculations than critical sampling. Furthermore, in data compression applications, one intends to remove redundancy, not to increase it. In the case of one-dimensional signals the repeated row scheme is convenient to be implemented (Strela & Walden, 1998).

Matrix (approximation) Preprocessing: Critical Sampling

Let $x(t)$ belongs to V_0 and is generated by the translates of linear Legendre multiscaling functions so that $x(t) = \sum_k [v_{0,k}^{(0)}\phi_1(t-k) + v_{0,k}^{(1)}\phi_2(t-k)]$.

Suppose that the input sequence samples are made at the half-integers, i.e., $X_{2k} = x(k)$, and $X_{2k+1} = x(k + \frac{1}{2})$.

From Fig. 1 for the linear Legendre multiscaling function, the only nonzero function values at the integers and half-integers are $\phi_1(0)=1, \phi_2(\frac{1}{2})=1$, and $\phi_2(0)=-\sqrt{3}$.

Thus,

$$X_{2k} = v_{0,k}^{(0)} - \sqrt{3}v_{0,k}^{(1)}, \text{ and } X_{2k+1} = v_{0,k}^{(0)}.$$

Hence,

$$v_{0,k}^{(0)} = X_{2k+1}, \tag{4}$$

and

$$v_{0,k}^{(1)} = \frac{1}{\sqrt{3}} X_{2k+1} - \frac{1}{\sqrt{3}} X_{2k} \tag{5}$$

Equations (4) and (5) give the required vector $v_{0,k}$ for $k = 0, \dots, N/2-1$. We can write it in matrix form as follows.

$$\begin{bmatrix} v_{0,k}^{(0)} \\ v_{0,k}^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} X_{2k} \\ X_{2k+1} \end{bmatrix}$$

In fact, approximation prefiltering is a special case of matrix prefiltering,

$$v_{0,k} = \sum_{m=0}^M P_m \begin{bmatrix} X_{2(m+k)} \\ X_{2(m+k)+1} \end{bmatrix}$$

where P_0, P_1, \dots, P_M are 2×2 matrices. Thus, for linear Legendre multiwavelets, we have

$$P_0 = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}, \text{ and } P_i = 0_2, i = 1, \dots, M,$$

where 0_2 is a 2×2 matrix of zeros.

If prefilters are N -dimensional, the matrix preprocessing is represented by $\rho X = V_0$ where now ρ is $N \times N$, X is $N \times 1$, and V_0 is $N \times 1$. Then, we have

$$\begin{bmatrix} P_0 & P_1 & \cdots & P_M & 0_2 & \cdots & \cdots \\ 0_2 & P_0 & P_1 & \cdots & P_M & 0_2 & \cdots \\ & & & & & & \ddots \\ & & & & & & \vdots \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} v_{0,0}^{(0)} \\ v_{0,0}^{(1)} \\ v_{0,1}^{(0)} \\ v_{0,1}^{(1)} \\ \vdots \end{bmatrix},$$

where P_m 's are 2×2 matrices (Strela, Heller, Strang, Topiwala, & Heil, 1999; Strela, 1996; Kim & Li, 2003).

Image Compression Using Legendre Wavelets

The notions stated above were applied to compress a given image. In doing so, critical sampling was deployed on a well-known photo of Barbara. Initially

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preprocess all the rows (see Strela & Walden, 1998), which results in the picture shown in Figure 3.



Figure 3. The result of row preprocessing on Barbara's photo.

Then, preprocess all columns. The amount of data remains unchanged, since we use critical sampling.



Figure 4. Result of column preprocessing on the photo in Figure 3.

At the next step, perform the 2D-wavelet cascade. For linear Legendre multiwavelets, the multiwavelet filter bank is obtained as follow. From (3), we have

$$H_1 = \begin{bmatrix} 1 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, H_2 = \begin{bmatrix} 1 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, G_1 = \begin{bmatrix} 0 & -1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, \text{ and } G_2 = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix},$$

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where $H_1, H_2, G_1,$ and G_2 are matrix filters. Using the ideas of matrix block multiplication (Van Fleet, 2000), the above filters can be combined into the matrix below.

$$\begin{bmatrix} \ddots & \ddots & \ddots & & & & & & & & \\ & 0_2 & H_1 & H_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & & \\ \cdots & 0_2 & 0_2 & 0_2 & H_1 & H_2 & 0_2 & 0_2 & 0_2 & \cdots & \\ & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & H_1 & H_2 & 0_2 & & \\ \hline & 0_2 & G_1 & G_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & & \\ \cdots & 0_2 & 0_2 & 0_2 & G_1 & G_2 & 0_2 & 0_2 & 0_2 & \cdots & \\ & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & G_1 & G_2 & 0_2 & & \\ & & & & & & & \ddots & \ddots & \ddots & \end{bmatrix},$$

where 0_2 is a 2×2 zero matrix. The two-channel matrix filter bank operates on two input data streams, filters them into four output data streams, and then, downsamples each of the results by a factor of two. After one step of the 2D multiwavelet cascade algorithm for linear Legendre with approximation preprocessing, the photo of Figure 5 appears.



Figure 5. The result of one-step 2D multiwavelet cascade algorithm with linear Legendre wavelets and approximation preprocessing.

The pattern of image subbands is shown in [Figure 6](#) (Sudhakar & Jayaraman, 2005). By comparing [Figure 5](#) with [Figure 6](#), it is seen that almost all information is in the block of G_1G_1 . This reflects the fact that, for linear Legendre, one scaling function is symmetric while the other is anti-symmetric.

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H_2G_1	H_2G_2	H_2H_1	H_2H_2
H_1G_1	H_1G_2	H_1H_1	H_1H_2
G_2G_1	G_2G_2	G_2H_1	G_2H_2
G_1G_1	G_1G_2	G_1H_1	G_1H_2

Figure 6. The pattern of image subbands.

Figure 7 compares the cumulative energy in the original image with the one resulted from one-step Legendre multiwavelet transform.

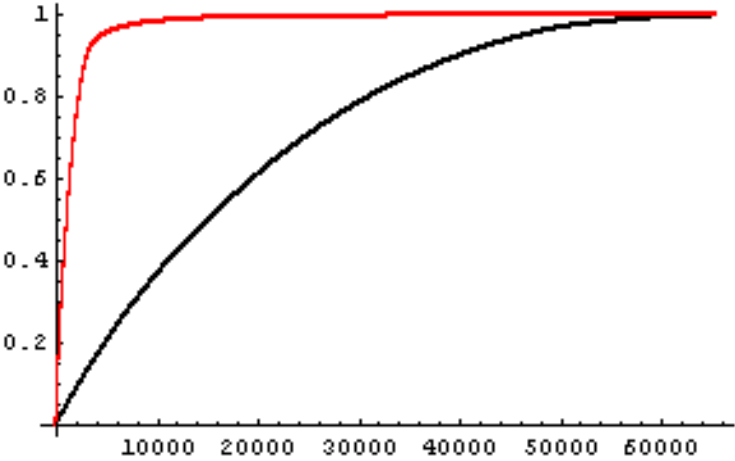


Figure 7. The cumulative energy of the original image and the one produced by the one-step Legendre multiwavelet transform.

The next step is to quantize the components of the Legendre multiwavelet transform. Any values of the transform that are small in modulus will be

converted to 0. The modified transform is then be converted to Huffman codes and the resulting bit stream is encoded. The produced file is markedly smaller than the original one.



Original Barbara image



Compression 64:1, PSNR 29.1



Compression 32:1, PSNR 30.31



Compression 16:1, PSNR 34.8

Figure 8. Some implementations of linear Legendre multiwavelet compression on Barbara's image.

Conclusion

The technique of using Legendre multiwavelets in image compression, as deployed in this paper, shows great advantages compared to the wavelet-based and even other multiwavelet-based methods. This is due to the fact that in a linear Legendre multiwavelet one scaling function is symmetric while the other one is anti-symmetric. In this way, it concentrates well on the image and excludes a tiny amount of details. Linear Legendre multiwavelets were used, but, the higher order Legendre multiwavelets can be employed (Rahbar, 2004).

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