

11-1-2016

A Generalization of the Weibull Distribution with Applications

Maalee Almheidat

University of Petra, Amman, Jordan, malmheidat@uop.edu.jo

Carl Lee

Central Michigan University, carl.lee@cmich.edu

Felix Famoye

Central Michigan University, felix.famoye@cmich.edu

 Part of the [Applied Statistics Commons](#), [Social and Behavioral Sciences Commons](#), and the [Statistical Theory Commons](#)

Recommended Citation

Almheidat, Maalee; Lee, Carl; and Famoye, Felix (2016) "A Generalization of the Weibull Distribution with Applications," *Journal of Modern Applied Statistical Methods*: Vol. 15 : Iss. 2 , Article 47.

DOI: 10.22237/jmasm/1478004300

A Generalization of the Weibull Distribution with Applications

Maalee Almheidat
University of Petra
Amman, Jordan

Carl Lee
Central Michigan University
Mount Pleasant, MI

Felix Famoye
Central Michigan University
Mount Pleasant, MI

The Lomax-Weibull distribution, a generalization of the Weibull distribution, is characterized by four parameters that describe the shape and scale properties. The distribution is found to be unimodal or bimodal and it can be skewed to the right or left. Results for the non-central moments, limiting behavior, mean deviations, quantile function, and the mode(s) are obtained. The relationships between the parameters and the mean, variance, skewness, and kurtosis are provided. The method of maximum likelihood is proposed for estimating the distribution parameters. The applicability of this distribution to modeling real life data is illustrated by three examples and the results of comparisons to other distributions in modeling the data are also presented.

Keywords: Estimation, moments, quantile function, Shannon's entropy, T -Weibull $\{Y\}$ family

Introduction

The Weibull distribution is a popular distribution for modeling phenomena with monotonic failure rates (Weibull, 1939; 1951). It is used to model lifetime data. However, it cannot capture the behavior of lifetime data sets that exhibit bathtub or upside-down bathtub (unimodal) failure rate, often encountered in reliability and engineering studies. A number of new distributions were developed as generalizations or modifications of the Weibull distribution. Xie and Lai (1995) introduced the additive Weibull model, which was obtained by adding two Weibull survival functions. Mudholkar and Srivastava (1993) proposed the exponentiated Weibull distribution. Xie, Tang, and Goh (2002) studied the modified Weibull extension. Bebbington, Lai, and Zitikis (2007) proposed a flexible Weibull

Dr. Almheidat is an Assistant Professor in the Department of Basic Sciences. Email her at: malmheidat@uop.edu.jo. Dr. Lee is a Professor in the Department of Mathematics. Email him at: carl.lee@cmich.edu. Dr. Famoye is a Professor in the Department of Mathematics. Email him at: felix.famoye@cmich.edu.

distribution and discussed its properties. For a review of some generalized Weibull distributions, one may refer to Lai (2014).

Different methods to generate probability distributions continue to appear. Eugene, Lee, and Famoye (2002) introduced the beta-generated family and some properties of the family were studied by Jones (2004). Many beta-generated distributions were studied (e.g., Eugene et al., 2002; Nadarajah & Kotz, 2004; Famoye, Lee, & Eugene, 2004; Famoye, Lee, & Olumolade, 2005; Nadarajah & Kotz, 2006; Akinsete, Famoye, & Lee, 2008; Barreto-Souza, Santos, & Cordeiro, 2010; Mahmoudi, 2011; Alshawarbeh, Lee, & Famoye, 2012). For a review of beta-generated distributions and other generalizations, see Lee, Famoye, and Alzaatreh (2013).

Alzaatreh, Lee, and Famoye (2013) extended the idea of beta-generated distributions to using any continuous random variable T with probability density function (PDF) $r(t)$ as a generator and developed a new class of distributions called the ' T - X family'. Given a random variable X with cumulative distribution function (CDF) $F(x)$, the CDF of the T - X family of distributions is defined by Alzaatreh, Lee, and Famoye (2013) as

$$G(x) = \int_a^{W(F(x))} r(t) dt \tag{1}$$

where $W(F(x))$ is a monotonic and absolutely continuous function of the CDF $F(x)$. Alzaatreh, Lee, and Famoye (2013) studied in details the case when $W(F(x)) = -\log(1 - F(x))$. Some members of the family have been investigated, including gamma-Pareto distribution (Alzaatreh, Famoye, & Lee, 2012), Weibull-Pareto distribution (Alzaatreh, Famoye, & Lee, 2013), and gamma-normal distribution (Alzaatreh, Famoye, & Lee, 2014a).

Aljarrah, Lee, and Famoye (2014) used the quantile function Q_Y of a random variable Y to define the transformation $W(\cdot)$ in the T - X family in (1) and called it the T - $R\{Y\}$ family. Following the notation proposed by Alzaatreh, Famoye, and Lee (2014b), the CDF of the T - $R\{Y\}$ family, as defined by Aljarrah et al. (2014), is given by

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t) dt = F_T\{Q_Y(F_R(x))\} \tag{2}$$

where $F_T(x)$, $F_R(x)$, and $F_Y(x)$ are, respectively, the CDFs of the random variables T , R , and Y . The PDF corresponding to (2) is

$$f_x(x) = \frac{f_R(x)}{f_Y\{Q_Y(F_R(x))\}} f_T\{Q_Y(F_R(x))\} \quad (3)$$

Almheidat, Famoye, and Lee (2015) used the T - $R\{Y\}$ framework to define and study different approaches to the generalization of the Weibull distribution, the T -Weibull $\{Y\}$ family. The authors defined the T -Weibull $\{Y\}$ family by taking R in (2) to be a Weibull random variable with CDF $F_R(x) = 1 - e^{-(x/\lambda)^k}$ and using the quantile function of the random variable Y , where Y has uniform, exponential, log-logistic, Fréchet, logistic, or extreme value distribution. When Y follows log-logistic distribution with parameters θ and β , the CDF and PDF of the T -Weibull $\{\log\text{-logistic}\}$ (T -Weibull $\{LL\}$) family are, respectively, given by

$$F_x(x) = F_T\left\{\theta\left(\frac{F_R(x)}{1-F_R(x)}\right)^{1/\beta}\right\} \quad (4)$$

$$f_x(x) = \frac{\theta f_R(x)}{\beta F_R^{(\beta-1)/\beta}(x)(1-F_R(x))^{(\beta+1)/\beta}} f_T\left\{\theta\left(\frac{F_R(x)}{1-F_R(x)}\right)^{1/\beta}\right\} \quad (5)$$

Setting $\beta = 1 = \theta$ and taking T in (4) to be a Lomax random variable with CDF $F_T(x) = 1 - (1 + (x/\theta))^{-\alpha}$, Almheidat et al. (2015) defined the Lomax-Weibull $\{LL\}$ distribution (LWD) as an example of T -Weibull $\{LL\}$ family.

The purpose of this study is to investigate the LWD as a generalization of the Weibull distribution and a member of T -Weibull $\{Y\}$ family.

Definition and Some Properties of the LWD

The CDF of the LWD defined in Almheidat et al. (2015) is given by

$$F_x(x) = 1 - \left[1 + \left(e^{(x/\lambda)^k} - 1\right) / \theta\right]^{-\alpha} \quad (6)$$

and the PDF corresponding to (6) is

$$f_x(x) = \frac{k\alpha}{\theta\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{(x/\lambda)^k} \left[1 + \left(e^{(x/\lambda)^k} - 1\right)/\theta\right]^{-(\alpha+1)}, \quad x > 0, \alpha, \theta, k, \lambda > 0 \quad (7)$$

Special cases of the LWD are as follows:

- when $\theta = \alpha = 1$, the LWD reduces to the Weibull distribution with parameters k and λ .
- when $\theta = k = 1$, the LWD reduces to the exponential distribution with mean λ/α .
- when $\alpha = 1/2$, $\theta = 1$, and $k = 2$, the LWD reduces to the Rayleigh distribution with parameter λ .

Lemma 1: (Transformations)

1. If a random variable T follows a Lomax distribution with parameters α and θ , then the random variable $X = \lambda\{\ln(T + 1)\}^{1/k}$ follows the LWD.
2. If a random variable T follows an exponential distribution with mean $1/\alpha$, then the random variable $X = \lambda\{\ln(\theta e^T - \theta + 1)\}^{1/k}$ follows the LWD.
3. If a random variable T follows a standard uniform distribution, then the random variable $X = \lambda\{\ln[\theta(1 - T)^{-1/\alpha} - \theta + 1]\}^{1/k}$ follows the LWD.

Proof: Using the transformation technique, it is easy to show that the random variable X follows the LWD as given in (7).

Hazard Function

The hazard function associated with the LWD in (7) is

$$h_x(x) = \frac{f_x(x)}{1 - F_x(x)} = \frac{k\alpha}{\theta\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{(x/\lambda)^k} \left[1 + \left(e^{(x/\lambda)^k} - 1\right)/\theta\right]^{-1} \quad (8)$$

The following Lemma addresses the limiting behaviors of the hazard function in (8).

A GENERALIZATION OF THE WEIBULL DISTRIBUTION

Lemma 2: The limits of the LWD hazard function as $x \rightarrow 0$ and as $x \rightarrow \infty$ are, respectively, given by

$$\lim_{x \rightarrow 0} h_x(x) = \begin{cases} 0, & k > 1 \\ \frac{\alpha}{\theta\lambda}, & k = 1 \\ \infty, & k < 1 \end{cases}, \quad \lim_{x \rightarrow \infty} h_x(x) = \begin{cases} \infty, & k > 1 \\ \frac{\alpha}{\lambda}, & k = 1 \\ 0, & k < 1 \end{cases} \quad (9)$$

Proof: This result is obtained by taking the limit of the hazard function in (8).

The following theorem is on the limiting behaviors of the PDF in (7).

Theorem 1: The limit of the LWD as $x \rightarrow \infty$ is 0 and the limit as $x \rightarrow 0$ is given by

$$\lim_{x \rightarrow 0} f_x(x) = \begin{cases} 0, & k > 1 \\ \frac{\alpha}{\theta\lambda}, & k = 1 \\ \infty, & k < 1 \end{cases} \quad (10)$$

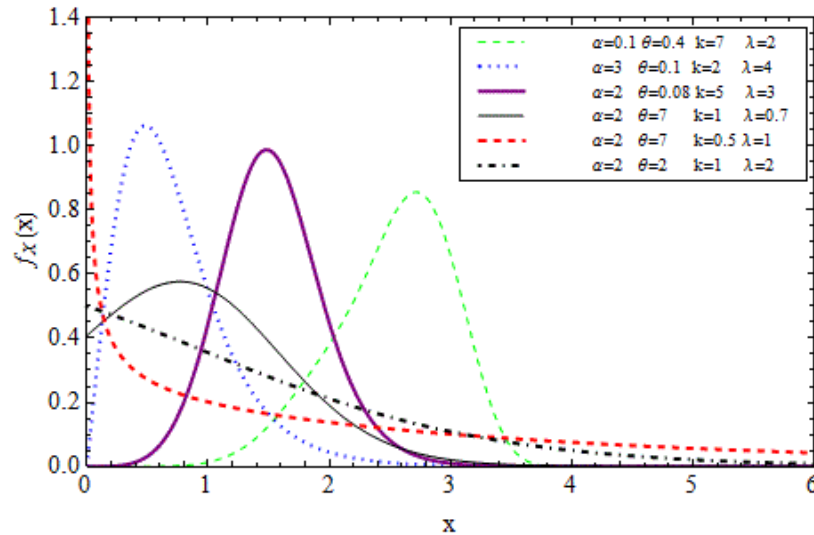


Figure 1. The PDFs of LWD for various values of α , θ , k , and λ

Proof: The $\lim_{x \rightarrow \infty} f_X(x) = 0$. If $k \leq 1$, the result follows from Lemma 2 and the fact that $f_X(x) = h_X(x)(1 - F_X(x))$. If $k > 1$, using L'Hôpital's rule, we have

$$\begin{aligned} \lim_{x \rightarrow \infty} f_X(x) &= \lim_{x \rightarrow \infty} \frac{k\alpha (x/\lambda)^{k-1} e^{(x/\lambda)^k}}{\theta \lambda \left[1 + \left(e^{(x/\lambda)^k} - 1 \right) / \theta \right]^{\alpha+1}} \\ &= \frac{\alpha}{\lambda(\alpha+1)} \lim_{x \rightarrow \infty} \left\{ \frac{k(x/\lambda)^{k-1}}{\left[1 + \left(e^{(x/\lambda)^k} - 1 \right) / \theta \right]^\alpha} + \frac{k-1}{(x/\lambda) \left[1 + \left(e^{(x/\lambda)^k} - 1 \right) / \theta \right]^\alpha} \right\} \\ &= \frac{\alpha}{\lambda(\alpha+1)} \lim_{x \rightarrow \infty} \left\{ \frac{k(x/\lambda)^{k-1}}{\left[1 + \left(e^{(x/\lambda)^k} - 1 \right) / \theta \right]^\alpha} + 0 \right\} \\ &= \frac{\alpha}{\lambda(\alpha+1)} \lim_{x \rightarrow \infty} \left\{ \frac{\theta(k-1)}{\alpha(x/\lambda) e^{(x/\lambda)^k} \left[1 + \left(e^{(x/\lambda)^k} - 1 \right) / \theta \right]^{\alpha-1}} \right\} = 0 \end{aligned}$$

This completes the proof of the limit as $x \rightarrow \infty$. The result in (10) follows directly by taking the limit of the LWD.

In Figures 1 and 2, various graphs of $f_X(x)$ are provided for different values of the parameters. The graphs in Figure 1 indicate that the LWD is unimodal with different shapes such as left-skewed, right-skewed with long right tail, or monotonically decreasing (reversed J- shape). The graphs in Figure 2 show that the LWD can be bimodal with two positive modal points (when $k > 1$) or one positive mode and the other mode at zero (when $k < 1$). The parameters α and k are shape parameters which characterize the skewness, kurtosis, and bimodality of the distribution. However, the parameter λ is a scale parameter and the parameter θ is a shape and scale parameter.

A GENERALIZATION OF THE WEIBULL DISTRIBUTION

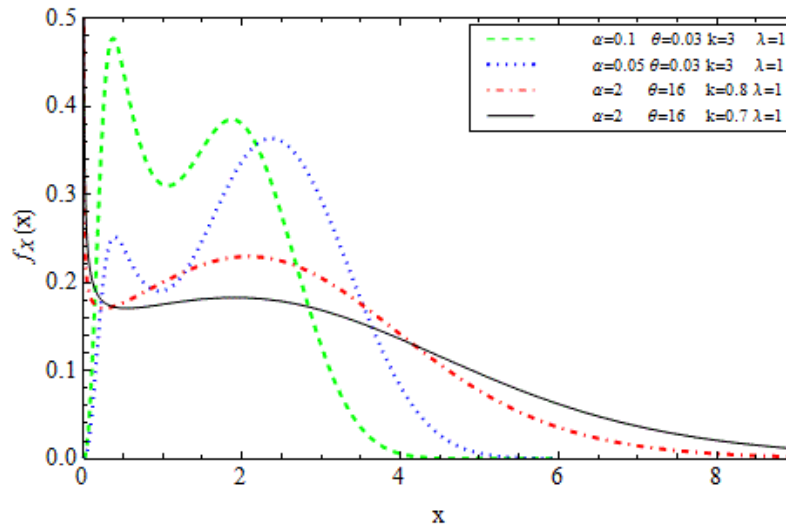


Figure 2. The PDFs of LWD for various values of α , θ , and k when $\lambda = 1$

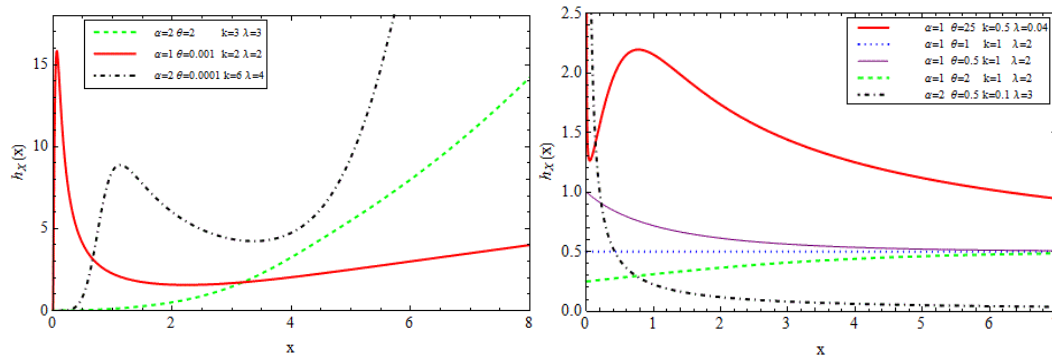


Figure 3. Hazard function of LWD for various values of α , θ , k , and λ

Displayed in Figure 3 are different graphs of the hazard function related to the LWD for various values of α , θ , k and λ . When $k = 1$, the LWD failure rate is either constant (when $\theta = 1$) or first increases (when $\theta > 1$) or decreases (when $\theta < 1$) and then becomes a constant. When $k < 1$, the failure rate of the LWD is either monotonically decreasing or decreasing followed by unimodal (reflected N-shape). When $k > 1$, the failure rate of the LWD is either increasing or unimodal followed by increasing (N-shape). These different failure rate shapes provide more flexibility to the LWD over the Weibull distribution, which has only increasing, decreasing, or constant failure rate.

Quantile Function

The quantile function is commonly used in general statistics (Steinbrecher & Shaw, 2008). Many distributions do not have a closed form quantile function. For the LWD, the quantile function has a closed form as given in the following lemma.

Lemma 3: The quantile function of the LWD is given by

$$Q_x(p) = \lambda \left\{ \ln \left[\theta(1-p)^{-1/\alpha} - \theta + 1 \right] \right\}^{1/k}, \quad 0 < p < 1 \tag{11}$$

Proof: The result follows directly by using part (iii) of Lemma 2 in Almheidat et al. (2015) when the random variable T follows a Lomax distribution.

Using the formula in (11), the quantile function of the LWD is

- an increasing function of λ when α , θ , and k are held fixed.
- a decreasing function of α when θ , λ , and k are held fixed.
- an increasing function of θ when α , k , and λ are held fixed.
- a decreasing (increasing, or constant) function of k , if $\theta < B$ ($\theta > B$, or $\theta = B$), when α , θ , and λ are held fixed, where $B = (e - 1)/[(1 - p)^{-1/\alpha} - 1]$.

The closed form quantile function in (11) makes simulating the LWD random variates straightforward. If U is a uniform random variate on the unit interval (0, 1), then the random variable $X = Q_x(U)$ follows the LWD. Note that the median (M) can be calculated by setting $p = 0.5$ in the quantile function in (11). The median of the LWD is given by $M = Q(0.5) = \lambda \{ \ln[\theta(0.5)^{-1/\alpha} - \theta + 1] \}^{1/k}$.

Mode(s)

From Almheidat et al. (2015), the mode(s) of T -Weibull{LL} family satisfy the implicit equation

A GENERALIZATION OF THE WEIBULL DISTRIBUTION

$$x = \begin{cases} \lambda \left[\frac{k}{(k-1)} \left(\frac{-f'_T(\mathbb{F}_R(x)/\bar{\mathbb{F}}_R(x))}{\bar{\mathbb{F}}_R(x) f_T(\mathbb{F}_R(x)/\bar{\mathbb{F}}_R(x))} - 1 \right) \right]^{-1/k}, & k \neq 1 \\ \lambda \log \left[\frac{1}{\bar{\mathbb{F}}_R^2(x)} \left(\frac{f'_T(\mathbb{F}_R(x)/\bar{\mathbb{F}}_R(x))}{f_T(\mathbb{F}_R(x)/\bar{\mathbb{F}}_R(x))} + 2\bar{\mathbb{F}}_R(x) \right) \right], & k = 1 \end{cases} \quad (12)$$

where $F_R(x)$ and $\bar{F}_R(x)$ are, respectively, the CDF and the survival function of the Weibull distribution. When T is a Lomax random variable, (12) can be simplified to

$$x = \begin{cases} \lambda \left[\frac{k(\alpha + 1 - \theta \bar{F}_R(x) - F_R(x))}{(k-1)(\theta \bar{F}_R(x) + F_R(x))} \right]^{-1/k}, & k \neq 1 \\ \lambda \log \left[\frac{-(\alpha + 1) + 2\theta \bar{F}_R(x) + 2F_R(x)}{\bar{F}_R(x)(\theta \bar{F}_R(x) + F_R(x))} \right], & k = 1 \end{cases} \quad (13)$$

Thus, the mode(s) of the LWD satisfy (13). Consider the variational behavior with respect to changes in the parameter values. When $k \neq 1$, (13) can be simplified to

$$x = \lambda \left[\frac{(k-1)(1 + (\theta-1)e^{-(x/\lambda)^k})}{k(\alpha - (\theta-1)e^{-(x/\lambda)^k})} \right]^{1/k} \quad (14)$$

Rewriting (14),

$$(x/\lambda)^k = \frac{(k-1)(1 + (\theta-1)e^{-(x/\lambda)^k})}{k(\alpha - (\theta-1)e^{-(x/\lambda)^k})} \quad (15)$$

Setting $u = (x/\lambda)^k$ in (15),

$$u = \frac{(k-1)(1+(\theta-1)e^{-u})}{k(\alpha-(\theta-1)e^{-u})} \quad (16)$$

Both x and u have the same variational behaviors with respect to changes in the parameters α and θ . The first derivatives of u with respect to α and θ are, respectively, given by

$$\frac{\partial u}{\partial \alpha} = \frac{-(k-1)(1+(\theta-1)e^{-u})}{k(\alpha-(\theta-1)e^{-u})^2}, \quad \frac{\partial u}{\partial \theta} = \frac{(k-1)(\alpha+1)e^{-u}}{k(\alpha-(\theta-1)e^{-u})^2} \quad (17)$$

From (17), the mode is a decreasing function of α when $k > 1$ and an increasing function of α when $k < 1$. On the other hand, the mode is an increasing function of θ when $k > 1$ and a decreasing function of θ when $k < 1$. When $k = 1$, (13) can be simplified as

$$x = \lambda \log \left[\frac{-\alpha + 1 + 2(\theta - 1)e^{-x/\lambda}}{e^{-x/\lambda}((\theta - 1)e^{-x/\lambda} + 1)} \right] \quad (18)$$

or, equivalently,

$$e^{x/\lambda} = \frac{-\alpha + 1 + 2(\theta - 1)e^{-x/\lambda}}{e^{-x/\lambda}((\theta - 1)e^{-x/\lambda} + 1)}$$

On simplifying (18),

$$x = \lambda \log [(\theta - 1)/\alpha] \quad (19)$$

Therefore, when $k = 1$, the mode is an increasing function of θ and a decreasing function of α . The mode is an increasing function of the scale parameter λ . However, it is not easy to determine increasing/decreasing behavior of the mode with respect to changes in parameter k .

From Figures 1 and 2, the LWD can be unimodal or bimodal depending on the parameter values. This property gives more flexibility to the LWD over the

A GENERALIZATION OF THE WEIBULL DISTRIBUTION

Weibull distribution, which is only unimodal. The following theorem shows some cases when the LWD is only unimodal.

Theorem 2: The LWD is unimodal whenever (i) $k = 1$ or (ii) $k < 1$ and $\theta \leq 1$.

- i) If $k = 1$, then the mode is at the point $x = 0$ whenever $\theta - 1 \leq \alpha$ and the mode is at the point $x = \lambda \ln[(\theta - 1)/\alpha]$ whenever $\theta - 1 > \alpha$.
- ii) If $k < 1$ and $\theta \leq 1$, the mode is at the point $x = 0$.

Proof: The derivative with respect to x of the PDF in (7) is given by

$$f'_x(x) = \frac{k\alpha}{(\lambda\theta)^2} \left(\frac{x}{\lambda}\right)^{k-2} e^{(x/\lambda)^k} \left[1 + \frac{e^{(x/\lambda)^k} - 1}{\theta}\right]^{-(\alpha+2)} m(x) \quad (20)$$

where

$$m(x) = (k-1) \left[\theta - 1 + e^{(x/\lambda)^k} \right] + k(x/\lambda)^k \left[\theta - 1 - \alpha e^{(x/\lambda)^k} \right] \quad (21)$$

By using (20) when $k \leq 1$, the critical points of $f_x(x)$ are $x = 0$ and $x = x_0$ where $m(x_0) = 0$. Hence, if there is a mode of the LWD, then it will be either at $x = 0$ or at $x = x_0$ where $m(x_0) = 0$. Note that the signal of $f'_x(x)$ is the same as that of $m(x)$.

If $k = 1$, then $m(x) = (\theta - 1) - \alpha e^{(x/\lambda)}$. Equating $m(x)$ to zero and solving for x we get $x = \lambda \log[(\theta - 1)/\alpha]$, the same result we obtained in (19). If $\theta - 1 > \alpha$, then the modal point is at $x = \lambda \log[(\theta - 1)/\alpha]$, otherwise the mode is at $x = 0$. If $k < 1$, it is easy to see that $m(x) < 0$ whenever $\theta \leq 1$, therefore $f'_x(x) < 0$, so $f_x(x)$ is strictly decreasing. From Theorem 1, $\lim_{x \rightarrow 0} f_x(x) = \infty$ and $\lim_{x \rightarrow \infty} f_x(x) = 0$. Thus $f_x(x)$ has a unique mode at $x = 0$.

Graphical displays of the LWD for many combinations of the parameters when $k < 1$ and $\theta > 1$, and when $k > 1$ indicate that the LWD is unimodal or bimodal depending on the parameter values. However, no analytical method has been used to show when the distribution is unimodal or bimodal.

Numerical methods are applied to study the regions of unimodality and bimodality. To study the modes of the LWD, the number of turning points of $f_x(x)$

in (7) is examined, which is equivalent to examining the sign of $f'_x(x)$. This is equivalent to studying the sign of the equation $m(x)$ in (21).

Consider the situation when $k < 1$. Select a fixed value of $k < 1$ ($k = 0.5, 0.7, 0.9$) and allow the values of α and θ to change from 0.001 to 15 at an increment of 0.001 and the values of x to change from 10^{-6} to 30 at an increment of 0.001.

A matrix \mathbf{M}_1 is constructed with two entries $\{0, 2\}$ which indicates the number of turning points of $f_x(x)$. For each combination of α and θ , if the sign of $m(x)$ is negative for all values of x between 10^{-6} and 30, then it is indicated by 0 in the matrix \mathbf{M}_1 . If the sign of $m(x)$ starts as being negative, turns positive, then turns negative, it is indicated by 2 in the matrix \mathbf{M}_1 . This leads to the following two regions: In the first region (the values corresponding to 0 in the matrix \mathbf{M}_1), $f_x(x)$ contains no turning points. This region indicates that the distribution has only one mode, which is at zero (reversed J-shape). In the second region (corresponding to 2 in the matrix \mathbf{M}_1), $f_x(x)$ contains two turning points. This region indicates that the distribution has two modes (one of them at zero). By using the boundary between the two regions, we draw a regression line which is a linear function relating α to θ for each value of k in the set $\{0.5, 0.7, 0.9\}$. The regression lines all have $R^2 = 100\%$.

Shown in Figure 4 is the region when LWD is unimodal or bimodal for different values of k and three PDFs for the bimodal case when k is 0.5, 0.7, and 0.9. Values of $k < 1$, $k = 0.1$ to 0.9 are also considered at an increment of 0.1, and the relationship between α and θ on the boundary points of the bimodality region remains linear.

For the case $k > 1$, a matrix \mathbf{M}_2 is constructed with entries $\{1, 3\}$. If the sign of $m(x)$ starts as being positive then turns negative for x values between 10^{-6} and 30, then it is indicated by 1 in the matrix \mathbf{M}_2 . If the sign of $m(x)$ starts as being positive, turns negative, then turns positive again and finally becomes negative, it is indicated by 3 in the matrix \mathbf{M}_2 .

This leads to the following regions: In the first region (where the values in the matrix \mathbf{M}_2 are 1), $f_x(x)$ contains one turning point. This region indicates that the distribution has only one positive mode. In the second region (where the value in the matrix \mathbf{M}_2 are 3), $f_x(x)$ contains three turning points. This region indicates that the distribution has two positive modes. By using the boundary between the two regions, we draw two regression lines which are non-linear functions relating α to θ for each value of k in the set $\{2, 4, 6\}$. Each regression line has $R^2 = 100\%$.

A GENERALIZATION OF THE WEIBULL DISTRIBUTION

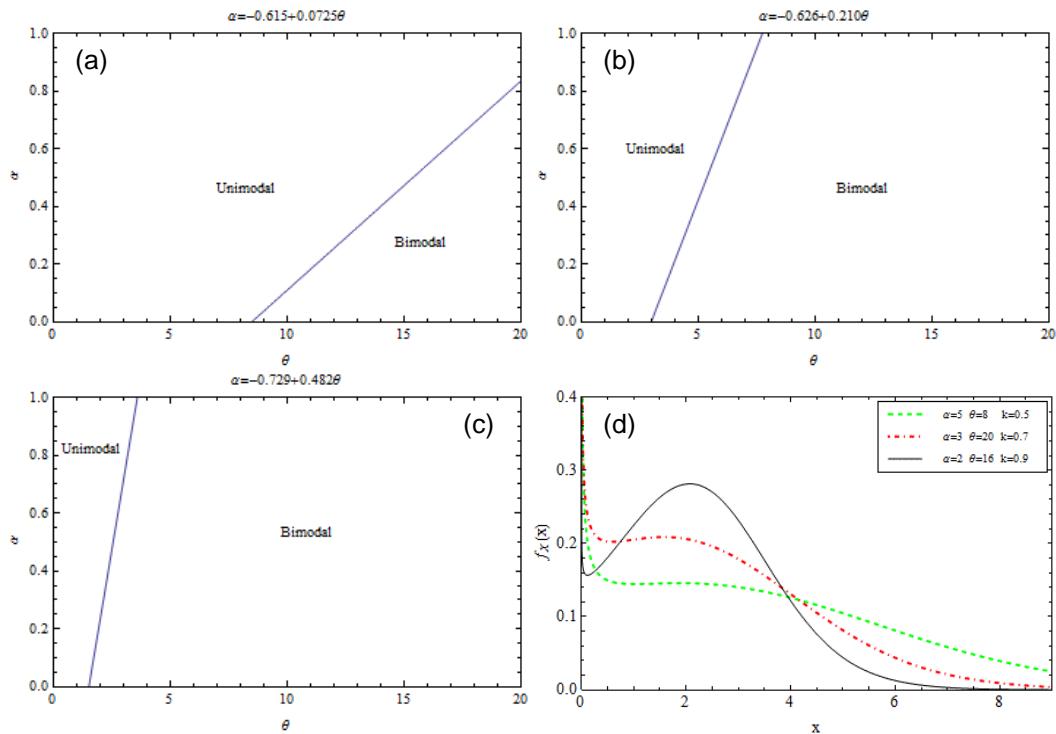


Figure 4. Regions of modality of LWD when $\lambda = 1$ and $k = 0.5$ (a); $k = 0.7$ (b); $k = 0.9$ (c); Some PDFs of LWD when $\lambda = 1$ and $k = \{0.5, 0.7, 0.9\}$ (d)

Shown in Figure 5 are the regions when LWD is unimodal or bimodal and three PDFs for the bimodal case when k is 2, 4 and 6. Note that, from Figures 4 and 5, the bimodal region increases as k increases when $k < 1$ and the bimodal region decreases as k increases when $k > 1$. Notice when k is large ($k > 20$), the region of bimodality does not change with respect to changes in the value of parameter k .

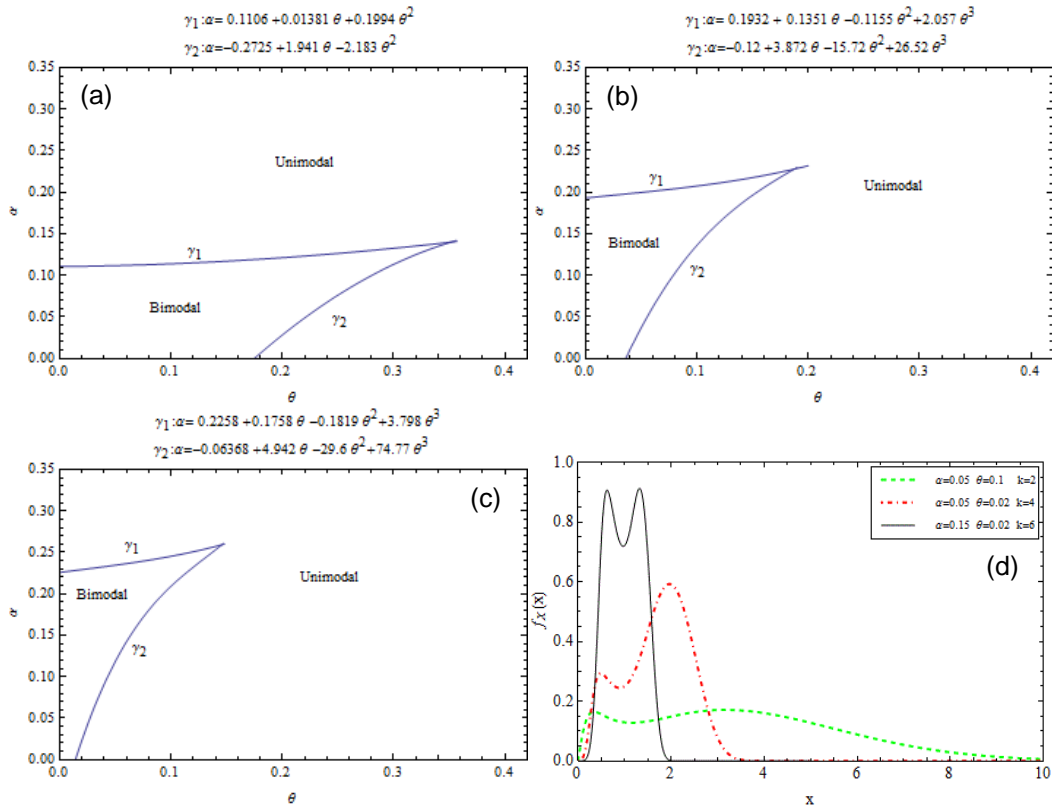


Figure 5. Regions of modality of LWD when $\lambda = 1$ and $k = 2$ (a); $k = 4$ (b); $k = 6$ (c); Some PDFs of LWD when $\lambda = 1$ and $k = \{2, 4, 6\}$ (d)

Moments, Mean Deviations, and Shannon's Entropy

Moments

The n^{th} non-central moment $E(X^n)$ of the LWD can be computed by using an infinite sum as shown in the following theorem:

Theorem 3: The n^{th} non-central moment of the LWD is given by the expression

A GENERALIZATION OF THE WEIBULL DISTRIBUTION

$$E(X^n) = \lambda^n \alpha \sum_{i=0}^{\infty} \frac{(\alpha+1)_i}{i!} \left\{ \frac{1}{\theta^{i+1}} \left[\sum_{j_1=0}^i \binom{i}{j_1} (-1)^{j_1} w_{i,j_1} \right] + (-1)^i \theta^{i+\alpha} \left[\sum_{j_2=0}^{\infty} \frac{(\alpha+i+1)_{j_2} \Gamma_{i,j_2}}{j_2! (\alpha+i+j_2)^{n/k}} \right] \right\} \quad (22)$$

where $\Gamma_{i,j_2} = \Gamma(n/k + 1, (\alpha + i + j_2) \log(\theta + 1))$, $(a)_r = a(a + 1) \dots (a + r - 1)$ is the ascending factorial, $\Gamma(a, x)$ is the incomplete gamma function given in Abramowitz and Stegun (1972) by

$$\Gamma(a, x) = \int_x^{\infty} t^{a-1} e^{-t} dt$$

and

$$w_{i,j_1} = \sum_{m=0}^{\infty} \frac{(j_1 + 1)^m}{m! \binom{n}{k} + m + 1} (\log(\theta + 1))^{(n/k) + m + 1}$$

Proof: By definition,

$$E(X^n) = \int_0^{\infty} x^n f_X(x) dx = \left(\frac{\lambda^{n-1} \alpha k}{\theta} \right) \int_0^{\infty} (x/\lambda)^{n+k-1} e^{(x/\lambda)^k} \left[1 + \frac{e^{(x/\lambda)^k} - 1}{\theta} \right]^{-(\alpha+1)} dx \quad (23)$$

Using the substitution $u = (x/\lambda)^k$, the integral in (23) can be simplified as

$$\begin{aligned}
 E(X^n) &= \frac{\lambda^n \alpha}{\theta} \int_0^\infty u^{n/k} e^u \left[1 + \frac{e^u - 1}{\theta} \right]^{-(\alpha+1)} du \\
 &= \frac{\lambda^n \alpha}{\theta} \left\{ \int_0^{\log(\theta+1)} u^{n/k} e^u \left[1 + \frac{e^u - 1}{\theta} \right]^{-(\alpha+1)} du \right. \\
 &\quad \left. + \theta^{\alpha+1} \int_{\log(\theta+1)}^\infty u^{n/k} e^u (e^u - 1)^{-(\alpha+1)} \left[1 + \frac{\theta}{e^u - 1} \right]^{-(\alpha+1)} du \right\} \\
 &= \frac{\lambda^n \alpha}{\theta} \{I_1 + I_2\}
 \end{aligned} \tag{24}$$

Using the generalized binomial expansion

$$\left[1 + \frac{e^u - 1}{\theta} \right]^{-(\alpha+1)} = \sum_{i=0}^\infty \frac{(-1)^i (\alpha+1)_i}{i!} \left(\frac{e^u - 1}{\theta} \right)^i$$

the integral I_1 in (24) reduces to

$$I_1 = \sum_{i=0}^\infty \frac{(-1)^i (\alpha+1)_i}{i!} \left\{ \frac{1}{\theta^i} \int_0^{\log(\theta+1)} u^{n/k} e^u (e^u - 1)^i du \right\} \tag{25}$$

where $(\alpha+1)_i$ is the ascending factorial. Using the binomial expansion

$$(e^u - 1)^i = \sum_{j_1=0}^i \binom{i}{j_1} (-1)^{i-j_1} e^{j_1 u}$$

equation (25) can be simplified as

$$I_1 = \sum_{i=0}^\infty \frac{(\alpha+1)_i}{i!} \left\{ \left(\frac{1}{\theta^i} \right) \sum_{j_1=0}^i \binom{i}{j_1} (-1)^{j_1} \int_0^{\log(\theta+1)} u^{n/k} e^{(j_1+1)u} du \right\} \tag{26}$$

On using the series representation for the exponential function

A GENERALIZATION OF THE WEIBULL DISTRIBUTION

$$e^{(j_1+1)u} = \sum_{m=0}^{\infty} \frac{((j_1+1)u)^m}{m!}$$

equation (26) becomes

$$\begin{aligned} I_1 &= \sum_{i=0}^{\infty} \frac{(\alpha+1)_i}{i!} \left\{ \left(\frac{1}{\theta^i} \right) \left[\sum_{j_1=0}^i \binom{i}{j_1} (-1)^{j_1} \left(\sum_{m=0}^{\infty} \frac{(j_1+1)^m}{m!} \int_0^{\log(\theta+1)} u^{\frac{n}{k}+m} du \right) \right] \right\} \\ &= \sum_{i=0}^{\infty} \frac{(\alpha+1)_i}{i!} \left\{ \left(\frac{1}{\theta^i} \right) \left[\sum_{j_1=0}^i \binom{i}{j_1} (-1)^{j_1} \sum_{m=0}^{\infty} \frac{(j_1+1)^m}{m! \binom{\frac{n}{k}+m+1}{k}} \log(\theta+1)^{\frac{n}{k}+m+1} \right] \right\} \quad (27) \\ &= \sum_{i=0}^{\infty} \frac{(\alpha+1)_i}{i!} \left\{ \left(\frac{1}{\theta^i} \right) \left[\sum_{j_1=0}^i \binom{i}{j_1} (-1)^{j_1} w_{i,j_1} \right] \right\} \end{aligned}$$

where w_{i,j_1} is as defined after equation (22) in Theorem 3.

By using the generalized binomial expansion

$$\left[1 + \frac{\theta}{e^u - 1} \right]^{-(\alpha+1)} = \sum_{i=0}^{\infty} \frac{(-1)^i (\alpha+1)_i}{i!} \left(\frac{e^u - 1}{\theta} \right)^{-i}$$

the integral I_2 in (24) reduces to

$$I_2 = \sum_{i=0}^{\infty} \frac{(\alpha+1)_i}{i!} \left\{ (-1)^i \theta^{(i+\alpha+1)} \int_{\log(\theta+1)}^{\infty} u^{n/k} e^{-(\alpha+i)u} (1-e^u)^{-(\alpha+i+1)} du \right\} \quad (28)$$

Using the generalized binomial expansion

$$(1 - e^{-u})^{-(\alpha+i+1)} = \sum_{j_2=0}^{\infty} \frac{(\alpha+i+1)_{j_2}}{j_2!} e^{-j_2 u}$$

equation (28) reduces to

$$\begin{aligned}
 I_2 &= \sum_{i=0}^{\infty} \frac{(\alpha+1)_i}{i!} \left\{ (-1)^i \theta^{(i+\alpha+1)} \left[\sum_{j_2=0}^{\infty} \frac{(\alpha+i+1)_{j_2}}{j_2!} \int_{\log(\theta+1)}^{\infty} u^{n/k} e^{-(\alpha+i+j_2)u} du \right] \right\} \\
 &= \sum_{i=0}^{\infty} \frac{(\alpha+1)_i}{i!} \left\{ (-1)^i \theta^{(i+\alpha+1)} \left[\sum_{j_2=0}^{\infty} \left(\frac{(\alpha+i+1)_{j_2} \Gamma_{i,j_2}}{j_2! (\alpha+i+j_2)^{n/k}} \right) \right] \right\}
 \end{aligned} \tag{29}$$

where Γ_{i,j_2} is as defined after equation (22) in Theorem 3. Substituting I_1 given by (27) and I_2 given by (29) into (24) completes the proof of the result in (22).

Table 1. Mean and variance of LWD for some values of α , θ , and k

θ	α	$k = 0.5$		$k = 1.0$		$k = 7.0$		$k = 10.0$	
		Mean	Var	Mean	Var	Mean	Var	Mean	Var
0.5	0.5	2.9207	59.5720	0.7854	0.8435	0.4894	0.0087	0.4906	0.0044
	0.7	1.3477	13.7330	0.5290	0.3941	0.4628	0.0077	0.4718	0.0040
	1.0	0.5822	2.7659	0.3466	0.1710	0.4364	0.0066	0.4529	0.0036
	5.0	0.0132	0.0013	0.0549	0.0036	0.3392	0.0035	0.3798	0.0022
	7.0	0.0063	0.0003	0.0382	0.0017	0.3226	0.0031	0.3667	0.0020
1.0	0.5	4.0000	80.0000	1.0000	1.0000	0.5164	0.0075	0.5098	0.0037
	0.7	2.0408	20.8240	0.7143	0.5102	0.4922	0.0068	0.4929	0.0035
	1.0	1.0000	5.0000	0.5000	0.2500	0.4677	0.0062	0.4757	0.0032
	5.0	0.0400	0.0080	0.1000	0.0100	0.3716	0.0039	0.4050	0.0023
	7.0	0.0204	0.0021	0.0714	0.0051	0.3542	0.0035	0.3916	0.0022
5.0	0.5	7.8417	149.0700	1.6140	1.3158	0.5704	0.0048	0.5474	0.0023
	0.7	4.8301	49.3640	1.2789	0.7795	0.5518	0.0046	0.5348	0.0022
	1.0	2.9624	16.5910	1.0059	0.4694	0.5327	0.0044	0.5218	0.0022
	5.0	0.3447	0.2425	0.3302	0.0633	0.4493	0.0042	0.4628	0.0023
	7.0	0.2123	0.1009	0.2555	0.0409	0.4319	0.0041	0.4501	0.0023
7.0	0.5	8.9207	167.7400	1.7588	1.3671	0.5800	0.0043	0.5539	0.0020
	0.7	5.6703	57.9240	1.4168	0.8279	0.5624	0.0041	0.5421	0.0020
	1.0	3.6049	20.5970	1.1351	0.5140	0.5445	0.0040	0.5299	0.0020
	5.0	0.4999	0.4265	0.4070	0.0844	0.4649	0.0041	0.4742	0.0022
	7.0	0.3196	0.1925	0.3206	0.0570	0.4480	0.0041	0.4619	0.0023

Given in Table 1 are the mean and the variance of LWD for various combinations of α , θ , and k when $\lambda = 0.5$. Many parameter combinations were used but, to save space, only a few of them are reported in Table 1. For fixed θ and k , the mean is a decreasing function of α . The mean is an increasing function of θ when

A GENERALIZATION OF THE WEIBULL DISTRIBUTION

α and k are fixed. For fixed α and θ , the mean decreases first and then increases as k increases. However, there is no clear pattern for the variance with respect to changes in the parameter values.

The skewness (Sk) and kurtosis (Ku) of LWD are given in Table 2 for some values of α , θ , and k . For fixed α and θ the skewness of LWD decreases as k increases. For fixed values of α and k , the skewness of LWD decreases as θ increases. Note that when $\theta = 1$, at which the LWD reduces to the Weibull distribution with shape parameter k and scale parameter $\lambda\alpha^{-1/k}$, the skewness and the kurtosis do not depend on α . However, there is no clear pattern for the kurtosis with respect to changes in the parameter values.

Table 2. Skewness and kurtosis of LWD for some values of α , θ , and k

θ	α	$k = 0.5$		$k = 1.0$		$k = 7.0$		$k = 10.0$	
		Sk	Ku	Sk	Ku	Sk	Ku	Sk	Ku
0.5	0.5	7.6717	116.0500	2.3412	10.9980	-0.0566	2.6407	-0.2648	2.8196
	0.7	8.1298	130.0700	2.4656	11.9600	-0.0263	2.7001	-0.2391	2.8675
	1.0	8.7698	152.0000	2.5946	13.1370	-0.0146	2.7708	-0.2321	2.9339
	5.0	10.1060	233.5900	2.4961	13.3190	-0.1419	2.8924	-0.3596	3.1300
	7.0	9.4602	206.0800	2.3840	12.3250	-0.1696	2.8902	-0.3853	3.1450
1.0	0.5	6.6188	87.7200	2.0000	9.0000	-0.2541	2.8803	-0.4632	3.1872
	0.7	6.6188	87.7200	2.0000	9.0000	-0.2541	2.8803	-0.4632	3.1872
	1.0	6.6188	87.7200	2.0000	9.0000	-0.2541	2.8803	-0.4632	3.1872
	5.0	6.6188	87.7200	2.0000	9.0000	-0.2541	2.8803	-0.4632	3.1872
	7.0	6.6188	87.7200	2.0000	9.0000	-0.2541	2.8803	-0.4632	3.1872
5.0	0.5	4.8668	49.7400	1.4377	6.5163	-0.6390	3.9785	-0.8628	4.6405
	0.7	4.3383	40.2360	1.2518	5.6651	-0.7372	4.0188	-0.9569	4.7131
	1.0	3.7677	30.7260	1.0804	4.8592	-0.7984	3.9727	-1.0117	4.6712
	5.0	2.9128	16.2010	1.0097	3.9304	-0.6615	3.2898	-0.8568	3.8290
	7.0	3.0655	17.4180	1.0952	4.1609	-0.6009	3.1715	-0.7960	3.6733
7.0	0.5	4.5966	44.8970	1.3539	6.2417	-0.6997	4.2784	-0.9275	5.0251
	0.7	4.0149	35.1060	1.1402	5.3288	-0.8240	4.3688	-1.0481	5.1612
	1.0	3.4034	25.7610	0.9440	4.4858	-0.9047	4.3469	-1.1222	5.1480
	5.0	2.5039	12.4240	0.8373	3.4387	-0.7689	3.5053	-0.9640	4.1107
	7.0	2.6408	13.3050	0.9238	3.6191	-0.6991	3.3404	-0.8931	3.8997

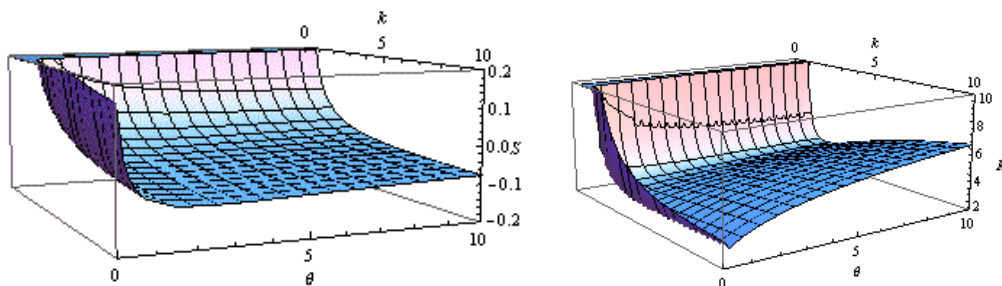


Figure 6. Galton's skewness and Moors' kurtosis for LWD when $\alpha = 0.5$

A measure of skewness and kurtosis, based on the quantile function, is obtained by using Galton's skewness (Galton, 1883) and Moors' kurtosis (Moors, 1988). By using the quantile function defined in (11), Galton's skewness and Moors' kurtosis for LWD, respectively, are given by

$$S = \frac{Q\left(\frac{6}{8}\right) - 2Q\left(\frac{4}{8}\right) + Q\left(\frac{2}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)}, \quad K = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \quad (30)$$

Presented in Figure 6 are three dimensional graphs of Galton's skewness and Moors' kurtosis for the same parameter values as in Table 2. To save space, these values are not reported but are compared with the values in Table 2. The results show similar patterns to those in Table 2.

Mean Deviations

Let X be a random variable with mean μ and median M . The mean deviation from the mean is defined as

$$\begin{aligned} D_{\mu} &= E(|X - \mu|) = \int_{-\infty}^{\infty} |x - \mu| f_X(x) dx \\ &= \int_{-\infty}^{\mu} (\mu - x) f_X(x) dx + \int_{\mu}^{\infty} (x - \mu) f_X(x) dx \\ &= 2\mu F_X(\mu) - 2 \int_{-\infty}^{\mu} x f_X(x) dx \end{aligned} \quad (31)$$

A GENERALIZATION OF THE WEIBULL DISTRIBUTION

where $F_X(\mu) = \int_{-\infty}^{\mu} f_X(x) dx$ can be calculated using (6). Similarly, the mean deviation from the median can be defined as

$$\begin{aligned} D_M &= E(|X - M|) = \int_{-\infty}^{\infty} |x - M| f_X(x) dx \\ &= \int_{-\infty}^M (M - x) f_X(x) dx + \int_M^{\infty} (x - M) f_X(x) dx \\ &= \mu - 2 \int_{-\infty}^M x f_X(x) dx \end{aligned} \quad (32)$$

The integrals $\int_0^{\mu} x f_X(x) dx$ and $\int_0^M x f_X(x) dx$ from (31) and (32), respectively, can be obtained as follows: Let $u = (x/\lambda)^k$. Then

$$\int_0^{\mu} x f_X(x) dx = \frac{\alpha \lambda}{\theta} \int_0^{(\mu/\lambda)^k} u^{1/k} e^u \left[1 + \frac{e^u - 1}{\theta} \right]^{-(\alpha+1)} du \quad (33)$$

If $\frac{e^{(\mu/\lambda)^k} - 1}{\theta} \leq 1$, using a similar approach as in Theorem 3, (33) reduces to

$$\begin{aligned} &\int_0^{\mu} x f_X(x) dx \\ &= \alpha \lambda \sum_{i=0}^{\infty} \frac{(\alpha+1)_i}{i! \theta^{i+1}} \left\{ \left[\frac{1}{\theta^{i+1}} \sum_{j_1=0}^i \binom{i}{j_1} (-1)^{j_1} \sum_{m=0}^{\infty} \left(\frac{(j_1+1)^m}{m!} \right) \right. \right. \\ &\quad \left. \left. \times \int_0^{(\mu/\lambda)^k} u^{(1/k)+m} du \right) \right] \right\} \quad (34) \\ &= \alpha \lambda \sum_{i=0}^{\infty} \frac{(\alpha+1)_i}{i! \theta^{i+1}} \left\{ \sum_{j_1=0}^i \binom{i}{j_1} (-1)^{j_1} \sum_{m=0}^{\infty} \left[\frac{(j_1+1)^m}{m! \left(\frac{1}{k} + m + 1 \right)} \left(\frac{\mu}{\lambda} \right)^{k+mk+1} \right] \right\} \end{aligned}$$

If $\frac{e^{(\mu/\lambda)^k} - 1}{\theta} > 1$, then

$$\begin{aligned}
 \int_0^\mu x f_X(x) dx &= \frac{\alpha\lambda}{\theta} \int_0^{(\mu/\lambda)^k} u^{1/k} e^u \left[1 + \frac{e^u - 1}{\theta} \right]^{-(\alpha+1)} du \\
 &= \frac{\alpha\lambda}{\theta} \int_0^{\log(\theta+1)} u^{1/k} e^u \left[1 + \frac{e^u - 1}{\theta} \right]^{-(\alpha+1)} du \\
 &\quad + \frac{\alpha\lambda}{\theta} \int_{\log(\theta+1)}^{(\mu/\lambda)^k} u^{1/k} e^u \left[1 + \frac{e^u - 1}{\theta} \right]^{-(\alpha+1)} du \\
 &= I_1^* + I_2^*
 \end{aligned} \tag{35}$$

Again, using the approach in Theorem 3, the integrals I_1^* and I_2^* can be simplified as

$$\begin{aligned}
 I_1^* &= \alpha\lambda \sum_{i=0}^{\infty} \frac{(\alpha+1)_i}{i! \theta^{i+1}} \left\{ \sum_{j_1=0}^i \binom{i}{j_1} (-1)^{j_1} \left[\sum_{m=0}^{\infty} \frac{(j_1+1)^m}{m! \left(\frac{1}{k} + m + 1\right)} [\log(\theta+1)]^{\frac{1}{k} + m + 1} \right] \right\} \\
 I_2^* &= \alpha\lambda \sum_{i=0}^{\infty} \frac{(\alpha+1)_i}{i!} \left\{ (-1)^i \theta^{(i+\alpha)} \left[\sum_{j_2=0}^{\infty} \frac{(\alpha+i+1)_{j_2}}{j_2!} \int_{\log(\theta+1)}^{(\mu/\lambda)^k} u^{1/k} e^{-\alpha+i+j_2} du \right] \right\} \\
 &= \alpha\lambda \sum_{i=0}^{\infty} \frac{(\alpha+1)_i}{i!} \left\{ (-1)^i \theta^{(i+\alpha)} \left[\sum_{j_2=0}^{\infty} \frac{(\alpha+i+1)_{j_2}}{j_2! (\alpha+i+j_2)} \Gamma_{i,j_2}^* \right] \right\}
 \end{aligned}$$

where

$$\Gamma_{i,j_2}^* = \Gamma\left(\frac{1}{k} + 1, (\alpha+i+j_2) \log(\theta+1)\right) - \Gamma\left(\frac{1}{k} + 1, (\alpha+i+j_2) \left(\frac{\mu}{k}\right)^k\right)$$

The integral $\int_0^M x f_X(x) dx$ can be obtained in a similar fashion.

Shannon's Entropy

The entropy of a random variable X is a measure of variation of uncertainty. Shannon (1948) defined the entropy of a random variable X with PDF $g(x)$ to be

$\eta_x = E[-\ln g(X)]$. Entropy has various applications in many fields including science, engineering, and economics. Using Theorem 2 in Almheidat et al. (2015), the Shannon's entropy of LWD is given by

$$\eta_x = \log\left(\frac{\lambda}{k}\right) - \frac{k-1}{k} E\left\{\log[\log(T+1)]\right\} - \left(\frac{1}{\lambda^k}\right) \mu'_k + \eta_T \quad (36)$$

where μ'_k is the k^{th} non-central moment of the LWD and $\eta_T = \ln(\theta/\alpha) + (1/\alpha) + 1$ is the Shannon's entropy of the Lomax random variable. Thus, from (36), the Shannon's entropy of LWD can be simplified as

$$\eta_x = \ln\left(\frac{\theta\lambda}{\alpha k}\right) + \frac{1}{\alpha} + 1 - \left(\frac{1}{\lambda^k}\right) \mu'_k - \frac{k-1}{k} I(\alpha, \theta) \quad (37)$$

where

$$I(\alpha, \theta) = \frac{\alpha}{\theta} \int_0^\infty \ln[\ln(1+t)] \left(1 + \frac{t}{\theta}\right)^{-(\alpha+1)} dt$$

Parameter Estimation

Let X_1, X_2, \dots, X_n be a random sample from LWD with parameters α, θ, k , and λ . The log-likelihood function $\ell = \ell(\alpha, \theta, k, \lambda)$ for the PDF in (7) is given by

$$\begin{aligned} \ell = n(\log k + \log \alpha - \log \theta - \log \lambda) \\ + \sum_{j=1}^n \left\{ (k-1) \log\left(\frac{x_j}{\lambda}\right) + \left(\frac{x_j}{\lambda}\right)^k - (\alpha+1) \log\left[1 + \frac{e^{(x_j/\lambda)^k} - 1}{\theta}\right] \right\} \end{aligned} \quad (38)$$

On taking the first partial derivatives of the log-likelihood function in (38) with respect to the parameters α, θ, k , and λ ,

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \sum_{j=1}^n \log\left[1 + \frac{e^{(x_j/\lambda)^k} - 1}{\theta}\right] \quad (39)$$

$$\frac{\partial \ell}{\partial \theta} = \frac{-n}{\theta} + \frac{(\alpha+1)}{\theta} \sum_{j=1}^n \left[1 + \theta \left(e^{(x_j/\lambda)^k} - 1 \right)^{-1} \right]^{-1} \quad (40)$$

$$\frac{\partial \ell}{\partial k} = \frac{n}{k} + \sum_{j=1}^n \left\{ 1 + \left[1 - (\alpha+1) e^{(x_j/\lambda)^k} \left(\theta + e^{(x_j/\lambda)^k} - 1 \right)^{-1} \right] \left(\frac{x_j}{\lambda} \right)^k \right\} \log \left(\frac{x_j}{\lambda} \right) \quad (41)$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{-nk}{\lambda} + \frac{k}{\lambda} \sum_{j=1}^n \left\{ (\alpha+1) e^{(x_j/\lambda)^k} \left(\theta + e^{(x_j/\lambda)^k} - 1 \right)^{-1} - 1 \right\} \left(\frac{x_j}{\lambda} \right)^k \quad (42)$$

By setting (39) to (42) equal to zero and solving them simultaneously, obtain $\hat{\alpha}, \hat{\theta}, \hat{k}$, and $\hat{\lambda}$, the maximum likelihood estimates (MLEs) for the parameters α, θ, k , and λ , are respectively obtained. The computations are done using the NLMIXED procedure in SAS. In this procedure the initial estimates of α, θ, k , and λ can be obtained as follows: First, assume that the sample data (x_1, x_2, \dots, x_n) is from a Weibull distribution. The parameter estimates given in Johnson, Kotz, and Balakrishnan (1994, pp. 635-643) are used for k and λ as the initial estimates, which are

$$k_0 = \frac{\pi}{s_w \sqrt{6}}, \quad \lambda_0 = \exp \left(\bar{w} + \frac{\gamma}{k_0} \right)$$

where $w_i = \log(x_i)$, \bar{w} and s_w are respectively the mean and the standard deviation of w random sample, and $\gamma = -\Gamma(1) \approx 0.57722$ is the Euler's constant. By using Lemma 1, the sample data (x_1, x_2, \dots, x_n) can be transformed to a data set from Lomax distribution by using

$$y_i = \exp \left(\frac{x}{\lambda_0} \right)^{k_0} - 1$$

The initial estimates for α and θ are the moment estimates of α and θ from the Lomax distribution and they are given by

$$\alpha_0 = \frac{2v_y}{v_y - \bar{y}^2}, \quad \theta_0 = \bar{y}(\alpha_0 - 1)$$

where \bar{y} and v_y are, respectively, the mean and the variance of (y_1, y_2, \dots, y_n) .

Applications

Three applications of the LWD using real life data sets are considered. Each of the three data sets exhibits right skewed, left skewed, or bimodal distribution shape. In these applications, the maximum likelihood estimates of the parameters of the fitted distributions are obtained. The LWD is compared with other distributions based on the maximized log-likelihood, the Kolmogorov-Smirnov (K-S) test along with the corresponding p -value, and Akaike Information Criterion (AIC). In addition, the histogram of the data and the PDFs of the fitted models are presented for graphical illustration of the goodness of fit.

Wheaton River Data

The data set in Table 3, from Choulakian and Stephens (2001), is the exceedances of flood peaks (in m^3/s) of Wheaton River, Yukon Territory, Canada. The data consists of 72 exceedances for the years 1958-1984, rounded to one decimal place. It is a right-skewed data (skewness = 1.5 and kurtosis = 3.19) with a long right tail.

The data set was analyzed using several distributions. Akinsete et al. (2008) used this data set as an application of beta-Pareto distribution (BPD). Alshawarbeh et al. (2012) fitted the data set to beta-Cauchy distribution (BCD). It was also used by Al-Aqtash, Famoye, and Lee (2014) to illustrate the flexibility of Gumbel-Weibull distribution (GWD) to fit different data sets. We fit the LWD to the data set. The MLEs and the goodness of fit statistics are presented in Table 4. The results for BPD, BCD and GWD are taken from Al-Aqtash et al. (2014).

The goodness of fit statistics indicate that the BCD, GWD, and LWD provide good fit based on the p -value of K-S statistic. But the LWD seems to provide the best fit among these distributions in Table 4, since it has the smallest AIC and K-S statistics and the largest log-likelihood value. The LWD seems to be very competitive to other distributions in fitting the data. This suggests that LWD fits highly right-skewed data with a long tail very well. Figure 7 contains the histogram of the data with the fitted distribution and supports the results in Table 4.

Table 3. Exceedances of the Wheaton River data

1.7	2.2	14.4	1.1	0.4	20.6	5.3	0.7	1.9	13.0	12.0	9.3
1.4	18.7	8.5	25.5	11.6	14.1	22.1	1.1	2.5	14.4	1.7	37.6
0.6	2.2	39.0	0.3	15.0	11.0	7.3	22.9	1.7	0.1	1.1	0.6
9.0	1.7	7.0	20.1	0.4	2.8	14.1	9.9	10.4	10.7	30.0	3.6
5.6	30.8	13.3	4.2	25.5	3.4	11.9	21.5	27.6	36.4	2.7	64.0
1.5	2.5	27.4	1.0	27.1	20.2	16.8	5.3	9.7	27.5	2.5	27.0

Table 4. MLEs for Wheaton River data (standard errors in parentheses)

Distribution	BPD	BCD	GWD	LWD
Parameter estimates	$\hat{\alpha} = 7.6954$	$\hat{\alpha} = 317.0256$	$\hat{\mu} = -0.6548$	$\hat{\alpha} = 0.1449$
	$\hat{b} = 85.75$	(312.5864)	(1.1214)	(0.0472)
	$\hat{\theta} = 0.1$	$\hat{b} = 1.4584$	$\hat{\sigma} = 3.3672$	$\hat{\theta} = 0.03124$
	$\hat{k} = 0.0208$	(0.4899)	(0.7295)	(0.0383)
		$\hat{\theta} = -0.0482$	$\hat{\alpha} = 1.4848$	$\hat{k} = 1.6396$
		(1.2301)	(0.3665)	(0.2842)
		$\hat{\lambda} = 0.09617$	$\hat{\lambda} = 8.0323$	$\hat{\lambda} = 6.3766$
		(0.0688)	(2.8206)	(2.1724)
Log Likelihood	-272.1280	-260.4813	-247.8373	-247.4916
AIC	552.256	528.952	503.700	503.000
K-S	0.1625	0.1219	0.0662	0.0587
(p-value)	(0.0446)	(0.2350)	(0.9101)	(0.9652)

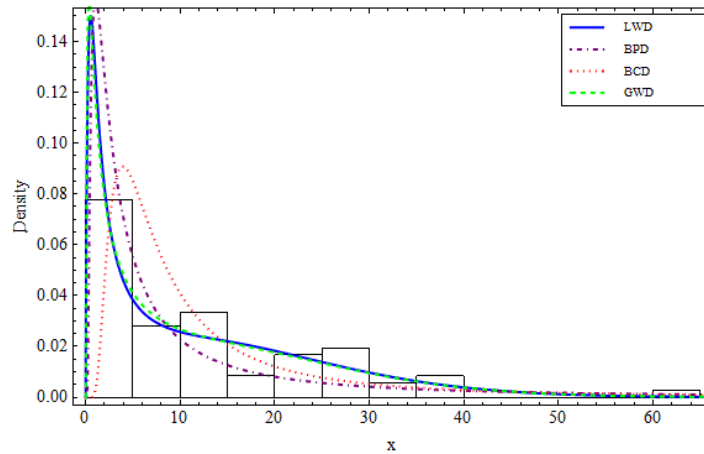


Figure 7. The histogram and the PDFs of the Wheaton River data

A GENERALIZATION OF THE WEIBULL DISTRIBUTION

Strengths of 1.5cm Glass Fibers Data

The second application represents fitting the LWD to the strength of 1.5 cm glass data set given in Table 5. The data set is “sample 1” of Smith and Naylor (1987) and deals with the breaking strength of 63 glass fibers of length 1.5 cm, originally obtained by workers at the UK National Physical Laboratory.

Barreto-Souza et al. (2010) applied the beta generalized exponential distribution (BGED) to fit the data and Barreto-Souza, Cordeiro, and Simas (2011) fitted beta Fréchet distribution (BFD) to the data. Recently, Alzaghal, Famoye, and Lee (2013) used the data in an application of the exponentiated Weibull-exponential distribution (EWED).

Table 5. Strength of 1.5 cm glass fibers data

0.55	0.74	0.77	0.81	0.84	0.93
1.04	1.11	1.13	1.24	1.25	1.27
1.28	1.29	1.30	1.36	1.39	1.42
1.48	1.48	1.49	1.49	1.50	1.50
1.51	1.52	1.53	1.54	1.55	1.55
1.58	1.59	1.60	1.61	1.61	1.61
1.61	1.62	1.62	1.63	1.64	1.66
1.66	1.66	1.67	1.68	1.68	1.69
2.00	2.01	2.24	1.76	1.76	1.77
1.70	1.70	1.73	1.84	1.84	1.89
1.78	1.81	1.82			

Table 6. MLEs for the strength of 1.5 cm glass fibers data (standard errors in parentheses)

Distribution	BFD	BGE	EWED	LWD
Parameter estimates	$\hat{\alpha} = 0.396$ (0.174)	$\hat{\alpha} = 0.4125$ (0.3020)	$\hat{\alpha} = 23.614$ (3.954)	$\hat{\alpha} = 1.1907$ (0.7232)
	$\hat{b} = 225.720$ (164.476)	$\hat{b} = 93.4655$ (120.0850)	$\hat{\gamma} = 7.249$ (0.994)	$\hat{\theta} = 21.9641$ (9.4167)
	$\hat{\lambda} = 1.302$ (0.270)	$\hat{\alpha} = 22.6124$ (21.925)	$\hat{c} = 0.0033$ (0.0030)	$\hat{k} = 2.9842$ (1.2329)
	$\hat{\sigma} = 6.863$ (1.992)	$\hat{\lambda} = 0.9227$ (0.5010)		$\hat{\lambda} = 1.0889$ (0.3105)
Log Likelihood	-19.5900	-15.5995	-14.3300	-11.9905
AIC	47.200	39.199	34.700	32.000
K-S	0.2140	0.1673	0.1370	0.1013
(p-value)	(0.0060)	(0.0588)	(0.1950)	(0.5373)

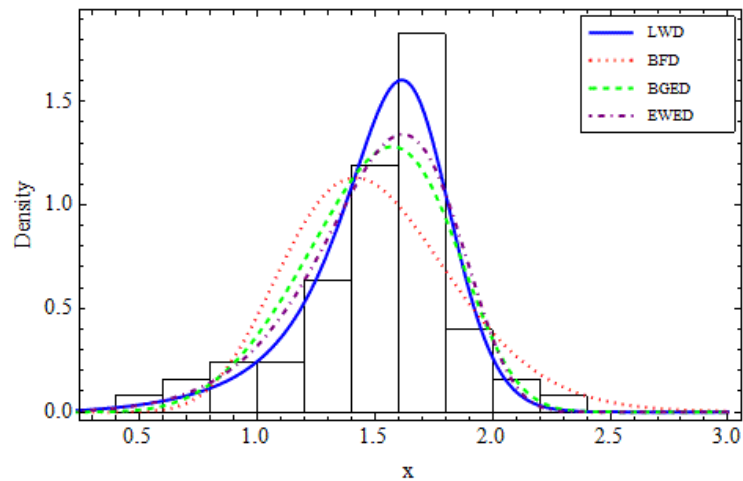


Figure 8. The histogram and the PDFs for the glass fibers data

The LWD is fitted to the data and the estimation results and goodness of fit statistics are presented in Table 6. From Table 6, the BGE, EWED, and LWD provide an adequate fit to the data with the LWD providing the best fit among all distributions in Table 6 based on every criterion. The distribution of the data is skewed to the left (skewness = -0.95 and kurtosis = 1.10). This suggests that the LWD performs well in modeling left skewed data. Contained in Figure 8 are the histogram of the data and the PDFs of the fitted distributions.

Australian Athletes Data

In this example, a data set reported by Cook and Weisberg (1994) about Australian Athletes is considered. It contains 13 variables on 102 male and 100 female athletes collected at the Australian Institute of Sport. Jamalizadeh, Arabpour, and Balakrishnan (2011) used the heights for the 100 female athletes and the hemoglobin concentration levels for the 202 athletes to illustrate the application of a generalized skew two-piece skew-normal distribution. Choudhury and Abdul Matin (2011) also used percentage of the hemoglobin blood cell for the male athletes to illustrate the application of an extended skew generalized normal distribution. In this example we consider the percentage of body fat (%Bfat) variable for the 202 athletes.

A GENERALIZATION OF THE WEIBULL DISTRIBUTION

Table 7. MLEs for the %Bfat data (standard error in parentheses)

Distribution	WD	BND	LND	LWD
Parameter estimates	$\hat{k} = 2.354$ (0.125)	$\hat{\alpha} = 0.1896$ (0.0549)	$\hat{\lambda} = 0.3000$ (0.0235)	$\hat{\alpha} = 0.2650$ (0.0448)
	$\hat{\lambda} = 15.313$ (0.4852)	$\hat{\beta} = 0.2513$ (0.0241)	$\hat{\mu} = 14.632$ (0.369)	$\hat{\theta} = 0.0065$ (0.0062)
		$\hat{\mu} = 15.289$ (1.286)	$\hat{\sigma} = 2.5330$ (0.0682)	$\hat{k} = 5.626$ (0.635)
		$\hat{\sigma} = 2.495$ (0.165)		$\hat{\lambda} = 18.538$ (1.136)
Log Likelihood	-642.416	-649.471	-644.047	-623.427
AIC	1288.8	1306.9	1294.1	1254.9
K-S	0.1091	0.1425	0.1599	0.0468
(p-value)	(0.0163)	(5.4400×10 ⁻⁴)	(6.4700×10 ⁻⁵)	(0.7676)

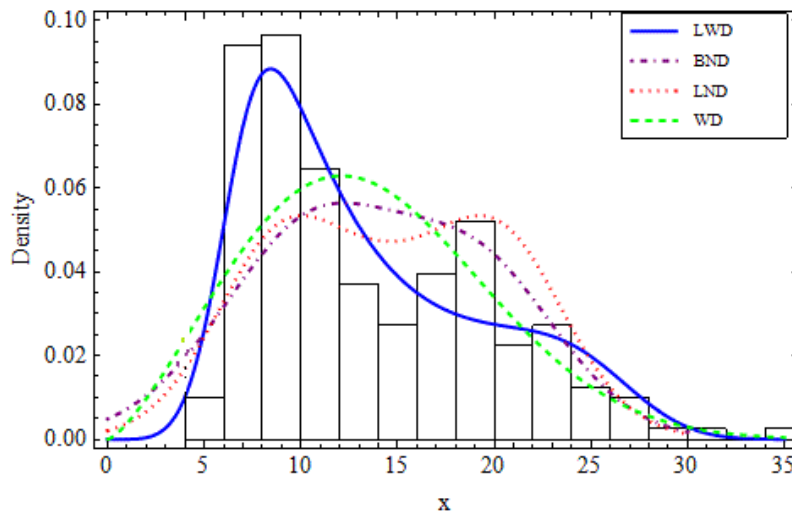


Figure 9. The histogram and the PDFs for %Bfat data

The LWD, the beta-normal distribution (BND) defined by Eugene et al. (2002), the logistic-normal{logistic} distribution (LND) defined by Alzaatreh et al. (2014b), and the Weibull distribution (WD) are applied to fit the data set. Table 7 contains the estimates, standard errors of the estimates, log-likelihood values, AIC, K-S test statistic, and the corresponding p -values.

The histogram and the densities of the fitted distributions are provided in Figure 9. From Figure 9, the distribution of this data appeared to be bimodal and skewed to the right (skewness = 0.759, kurtosis = 2.827).

From Table 7, LWD has the smallest AIC and K-S statistics and the largest log-likelihood value, which indicates that LWD seems to be superior to the other distributions in fitting the data. Even though the BND has the ability to fit bimodal data, it could not capture the bimodality property in fitting the data. On the other hand, the LND capture the bimodality property but with poor fit to the data. This application suggests that LWD has the ability to adequately fit bimodal data.

Conclusion

A four-parameter LWD was proposed as an extension of the Weibull distribution and a member of T -Weibull $\{Y\}$ family defined by Almheidat et al. (2015). The LWD is found to be unimodal or bimodal and reduces to some existing distributions that are known in the literature. Various properties of the LWD are investigated, including the hazard function, the quantile function, and the regions of unimodality and bimodality. Expressions for the moments, the Shannon's entropy, and the mean deviations are derived. The parameters are estimated by the method of maximum likelihood.

The LWD is fitted to three real data sets to illustrate the application of the distribution. The first data set is the exceedances of flood peaks of Wheaton River, the second is the strength of 1.5 cm glass fibers, and the third is the percentage of the body fat of 202 Australian Athletes. In fitting these data sets, different distributions are compared with the LWD based on goodness of fit statistics. The two most competitive distributions to the LWD are the GWD (used in the flood data set) and the EWED (used in the glass fibers data set). The results show that the LWD outperformed these two distributions in fitting both data sets. LWD has an advantage over several other distributions due to the flexibility of this distribution and its ability to model different shapes in real life data sets, including unimodal and bimodal cases.

References

Abramowitz, M., & Stegun, I. (1972). *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. New York: Dover Publications.

A GENERALIZATION OF THE WEIBULL DISTRIBUTION

- Akinsete, A., Famoye, F., & Lee, C. (2008). The beta-Pareto distribution. *Statistics*, 42(6), 547-563. doi: 10.1080/02331880801983876
- Al-Aqtash, R., Famoye, F., & Lee, C. (2014). On generating a new family of distributions using the logit function. *Journal of Probability and Statistical Science*, 13(1), 135-152.
- Aljarrah, M. A., Lee, C., & Famoye, F. (2014). On generating T - X family of distributions using quantile functions. *Journal of Statistical Distributions and Applications*, 1(2), 1-17. doi: 10.1186/2195-5832-1-2
- Almheidat, M., Famoye, F., & Lee, C. (2015). Some generalized families of Weibull distribution: Properties and applications. *International Journal of Statistics and Probability*, 4(3), 18-35. doi: 10.5539/ijsp.v4n3p18
- Alshawarbeh, E., Lee, C., & Famoye, F. (2012). The beta-Cauchy distribution. *Journal of Probability and Statistical Science*, 10(1), 41-57.
- Alzaatreh, A., Famoye, F., & Lee, C. (2012). Gamma-Pareto distribution and its applications. *Journal of Modern Applied Statistical Methods*, 11(1), 78-94. Retrieved from <http://digitalcommons.wayne.edu/jmasm/vol11/iss1/7/>
- Alzaatreh, A., Famoye, F., & Lee, C. (2013). Weibull-Pareto distribution and its applications. *Communications in Statistics – Theory and Methods*, 42(9), 1673-1691.
- Alzaatreh, A., Lee, C., & Famoye, F. (2013). A new method for generating families of continuous distributions. *METRON*, 71(1), 63-79. doi: 10.1007/s40300-013-0007-y
- Alzaatreh, A., Famoye, F., & Lee, C. (2014a). The gamma-normal distribution: Properties and applications. *Journal of Computational Statistics & Data Analysis*, 69(1), 67-80. doi: 10.1016/j.csda.2013.07.035
- Alzaatreh, A., Famoye, F., & Lee, C. (2014b). T -normal family of distributions: A new approach to generalize the normal distribution. *Journal of Statistical Distributions and Applications*, 1(16), 1-18. doi: 10.1186/2195-5832-1-16
- Alzaghal, A., Famoye, F., & Lee, C. (2013). Exponentiated T - X family of distributions with some applications. *International Journal of Statistics and Probability*, 2(3), 31-49. doi: 10.5539/ijsp.v2n3p31
- Barreto-Souza, W., Cordeiro, G. M., & Simas, A. B. (2011). Some results for beta Fréchet distribution. *Communication in Statistics – Theory and Methods*, 40(5), 798-811. doi: 10.1080/03610920903366149

- Barreto-Souza, W., Santos, A. H. S., & Cordeiro, G. M. (2010). The beta generalized exponential distribution. *Journal of Statistical Computation and Simulation*, 80(2), 159-172. doi: 10.1080/00949650802552402
- Bebbington, M., Lai, C.-L., & Zitikis, R. (2007). A flexible Weibull extension. *Reliability Engineering & System Safety*, 92(6), 719-726. doi: 10.1016/j.res.2006.03.004
- Choudhury, K., & Abdul Matin, M. (2011). Extended skew generalized normal distribution. *METRON*, 69(3), 265-278. doi: 10.1007/BF03263561
- Choulakian, V., & Stephens, M. A. (2001). Goodness-of-fit tests for the generalized Pareto distribution. *Technometrics*, 43(4), 478-484. doi: 10.1198/00401700152672573
- Cook, R. D., & Weisberg, S. (1994). *An introduction to regression graphics*. New York: John Wiley and Sons, Inc.
- Eugene, N., Lee, C., & Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics – Theory and Methods*, 31(4), 497-512. doi: 10.1081/STA-120003130
- Famoye, F., Lee, C., & Eugene, N. (2004). Beta-normal distribution: Bimodality properties and applications. *Journal of Modern Applied Statistical Methods*, 3(1), 85-103. Retrieved from <http://digitalcommons.wayne.edu/jmasm/vol3/iss1/10/>
- Famoye, F., Lee, C., & Olumolade, O. (2005). The beta-Weibull distribution. *Journal of Statistical Theory and Applications*, 4(2), 121-136.
- Galton, F. (1883). *Inquiries into human faculty and its development*. London: Macmillan and Co.
- Jamalizadeh, A., Arabpour, A. R., & Balakrishnan, N. (2011). A generalized skew two-piece skew-normal distribution. *Statistical Papers*, 52(2), 431-446. doi: 10.1007/s00362-009-0240-x
- Johnson, N. L., Kotz, S., & Balakrishnan, N. (1994). *Continuous univariate distributions* (Vol. 1) (2nd ed.). New York: John Wiley and Sons, Inc.
- Jones, M. C. (2004). Families of distributions arising from distributions of order statistics. *Test*, 13(1), 1-43. doi: 10.1007/BF02602999
- Lai, C. D. (2014). *Generalized Weibull distributions*. Berlin: Springer.
- Lee, C., Famoye, F., & Alzaatreh, A. (2013). Methods for generating families of univariate continuous distributions in the recent decades. *Wiley Interdisciplinary Reviews: Computational Statistics*, 5(3), 219-238. doi: 10.1002/wics.1255

A GENERALIZATION OF THE WEIBULL DISTRIBUTION

- Mahmoudi, E. (2011). The beta generalized Pareto distribution with application to lifetime data. *Mathematics and Computers in Simulation*, 81(11), 2414-2430. doi: 10.1016/j.matcom.2011.03.006
- Moors, J. J. (1988). A quantile alternative for kurtosis. *Journal of the Royal Statistical Society. Series D (The Statistician)*, 37(1), 25-32. doi: 10.2307/2348376
- Mudholkar, G. S., & Srivastava, D. K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. *IEEE Transactions on Reliability*, 42(2), 299-302. doi: 10.1109/24.229504
- Nadarajah, S., & Kotz, S. (2004). The beta Gumbel distribution. *Mathematical Problems in Engineering*, 2004(4), 323-332. doi: 10.1155/S1024123X04403068
- Nadarajah, S., & Kotz, S. (2006). The beta exponential distribution. *Reliability Engineering & System Safety*, 91(6), 689-697. doi: 10.1016/j.res.2005.05.008
- Shannon, C. E. (1948). A mathematical theory of communication. *Bell System Technical Journal*, 27, 379-432.
- Smith, R. L., & Naylor, J. C. (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. *Journal of the Royal Statistical Society. Series C (Applied Statistics)*, 36(3), 358-369. doi: 10.2307/2347795
- Steinbrecher, G., & Shaw, W. T. (2008). Quantile mechanics. *European Journal of Applied Mathematics*, 19(2), 87-112. doi: 10.1017/S0956792508007341
- Weibull, W. (1939). Statistical theory of the strength of materials. *Ingenjörsk Vetenskaps Akademiens, Handlingar*, 151, 1-45.
- Weibull, W. (1951). Statistical distribution functions of wide applicability. *Journal of Applied Mechanics*, 18(3), 293-297.
- Xie, M., & Lai, C. (1995). Reliability analysis using an additive Weibull model with bathtub-shaped failure rate function. *Reliability Engineering & System Safety*, 52(1), 87-93. doi: 10.1016/0951-8320(95)00149-2
- Xie, M., Tang, Y., & Goh, T. (2002). A modified Weibull extension with bathtub failure rate function. *Reliability Engineering & System Safety*, 76(3), 279-285. doi: 10.1016/S0951-8320(02)00022-4